

# Informal Overview Symbolic Computation (Quantifier Elimination)

Hoon Hong  
[hong@math.ncsu.edu](mailto:hong@math.ncsu.edu)

These slides were given during a joint lecture for two summer schools (SC2 and VTSA) held at MPI in 2017.

Many participants were from satisfiability checking and software verification. Some were from symbolic computation.

Thus

- Style : Informal / Selective
- Credit: Emphasis on the contributions of the participants from symbolic computation

# Symbolic Computation

“Sociological Overview”

Symbolic:

# Symbolic: Countable sets

# Symbolic: Countable sets

- Finite fields

# Symbolic: Countable sets

- Finite fields
- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc

# Symbolic: Countable sets

- Finite fields
- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc

# Symbolic: Countable sets

- Finite fields
- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc
- Vector, Matrices, Tensors, etc

# Symbolic: Countable sets

- Finite fields
- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc
- Vector, Matrices, Tensors, etc
- Logical formulas

# Symbolic: Countable sets

- Finite fields
- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc
- Vector, Matrices, Tensors, etc
- Logical formulas
- .....

# Computation: Simplifications

# Computation: Simplifications

$$x^2 + 2x + 1$$



$$(x + 1)^2$$

Compute

# Computation: Simplifications

$$x^2 + 2x + 1 - x - 1 = 0$$



$$(x + 1)^2$$



$$x = -1$$

Compute

Solve

# Computation: Simplifications

$$x^2 + 2x + 1 \quad x + 1 = 0$$

$$(x + 1)^2$$



$$x = -1$$



$$\forall x \ x^2 \geq 0$$

*true*



Compute

Solve

Prove

# Computation: Simplifications

$$x^2 + 2x + 1 \quad x + 1 = 0$$

$$(x + 1)^2$$

Compute

$$x = -1$$

Solve

$$\forall x \ x^2 \geq 0$$

*true*

Prove

*Object*

*Object*

Simplify

# Main Achievements during 50 yrs

# Main Achievements during 50 yrs

- Computing
  - Polynomial Multiplication, Factorization, GCD, etc
    - Subresultant theory
    - Homomorphism and Lifting
    - ...

# Main Achievements during 50 yrs

- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...

- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...

# Main Achievements during 50 yrs

- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...

- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...

- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...

# Main Achievements during 50 yrs

- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...



- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...

- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...

# Main Achievements during 50 yrs

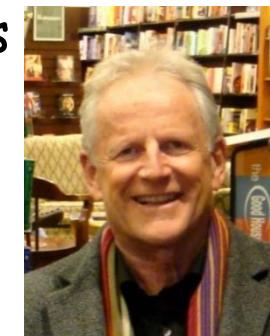
- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...



- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...



- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...

# Main Achievements during 50 yrs

- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...



- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...



- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...

# Main Achievements during 50 yrs

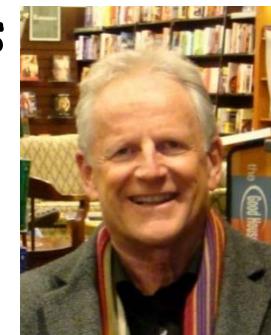
- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...



- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...



- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...



# Main Achievements during 50 yrs

- **Computing**

- Polynomial Multiplication, Factorization, GCD, etc
  - Subresultant theory
  - Homomorphism and Lifting
  - ...



- **Solving**

- Algebraic, Differential, Difference constraints
  - Groebner Basis
  - Characteristic Sets, Regular chains
  - Resultants
  - Differential/Difference Galois theory
  - ...



- **Proving**

- First order over reals
  - CAD
  - Block QE (Deformation and Optimization)
  - Characteristic Sets
  - ...



# Main Conferences

# Main Conferences

## Symbolic and Algebraic Computation (ISSAC)

The poster features a blue background with a landscape image of a coastal town at the bottom. The title "ISSAC 04" is prominently displayed in large white letters. Below it, the subtitle "International Symposium on Symbolic and Algebraic Computation" is written. The logo of the University of Cantabria is on the left. The poster includes sections for "Topics", "Important Dates", and "Sponsors".

**Topics:**

- Algorithmic mathematics: algebraic, symbolic and symbolic-numeric algorithms, simplification, function manipulation, equations, summation, integration, ODE/PDE, linear algebra, number theory, group and geometric computing
- Computer Science: theoretical and practical problems in symbolic computation, systems, problem solving environments, user interfaces, software, libraries, parallel/distributed computing and programming languages for symbolic computation, concrete analysis, benchmarking, theoretical and practical complexity of computer algebra algorithms, automatic differentiation, code generation, mathematical data structures and exchange protocols
- Applications: problem treatments using algebraic, symbolic or symbolic-numeric computation in an essential or a novel way, engineering, economics and finance, physical and biological sciences, computer science, logic, mathematics, statistics, education

**Important Dates:**

- Deadline for Submissions: January 7, 2004
- Notification of Acceptance: March 3, 2004
- Camera-ready copy received: April 16, 2004

<http://www.risc.uni-linz.ac.at/issac2004/>  
issac2004@risc.uni-linz.ac.at

**Sponsors:**

- UC (University of Cantabria)
- COM (Computer Mathematics)
- Maplesoft

# Main Conferences

## Symbolic and Algebraic Computation (ISSAC)



Started in:  
1966 (under different name)

# Main Conferences

## Symbolic and Algebraic Computation (ISSAC)



**Started in:**  
1966 (under different name)

**Emphasis:**  
Foundations and theories

# Main Conferences

## Applications of Computer Algebra (ACA)

**ACA-2004**



**INVITED SPEAKERS:**

- GERALD FARIN (ASU)
- HOON HONG (NCSU)

**PROCEEDINGS OF ACCEPTED FULL PAPERS WILL BE PUBLISHED AND DISTRIBUTED**

- A SPECIAL ISSUE OF JOURNAL OF SYMBOLIC COMPUTATION FOR SELECTED PAPERS
- FULL AUTOMATIC SYSTEM TO HANDLE THE CONFERENCE

**TENTH INTERNATIONAL CONFERENCE ON APPLICATIONS OF COMPUTER ALGEBRA**

**July 21-23, 2004**  
**The Hilton Hotel, Beaumont, Texas**

Hosted by the Department of Computer Science, Lamar University & the International Scientific Committee for Applications of Computer Algebra

The ACA conference series is dedicated to reporting serious applications of symbolic computation (a.k.a. computer algebra) theories and tools for mathematics, logic, science, engineering and education.

**GENERAL CHAIR: QUOC-NAM TRAN, LAMAR UNIVERSITY**  
ORGANIZING COMMITTEE: LAWRENCE OSBORNE (LAMAR) AND STANLY STEINBERG (NEW MEXICO)  
PROGRAMMING CHAIR: QUOC-NAM TRAN (LAMAR) AND VLADIMIR GEYER (DUKE)  
SCIENTIFIC ADVISORY BOARD: JAVIER ARNAU (CANTABRIA)  
BERNARD KUTTER (AUSTRIA)  
VICTOR EDNIPAL (HUSKIE)  
JOHN D. COOPER (UNIVERSITY OF SOUTHERN ILLINOIS)  
STANLY STEINBERG (USA)  
LAWRENCE OSBORNE (NEW MEXICO)  
MICHAEL VANNES (USA)

FRED KATZ (Johns Hopkins)  
JAKOBIS CAI MING (GERMANY)  
HICHIADE LIGIA (CZECH)  
MICHAEL E. WALKER (CANADA)  
HOON HONG (NCSU)  
QUOC-NAM TRAN (LAMAR)  
TONY CHAMPA (LAMAR)

The conference will cover (but not be limited to) the following topics: Computer Algebra Systems, Symbolic Number Theory, Computer Algebra in Economics and Finance, Computer Algebra, Computational Aspects of Computer Algebra, Geometric Design and Geometric Modeling, Theory and Applications of Groebner Bases, Computer Algebra in Robotics, Computer Algebra in Education, Computer Algebra and Theorem Provers, Non Standard Applications, Theory and Applications of Quantifier Elimination, Theory and Applications of Resultants, Theory and Applications of Differential Equations, Theory and Applications of Computer Mathematics, Theory and Applications of Computational Sciences, Types and Specification, Applications to Chemistry, Physics and Mathematics, Industrial and Engineering Applications of Computer Algebra, General and other Applications of Computer Algebra

LOCAL ARRANGEMENTS : JINSUNGWOO CHU (CO-CHAIR), VALENTIN ADREBBY (CO-CHAIR), JENNIFER FOWLER, KYOYO KON, CHUN-SOO LI

Phone: +49 361 966-722  
Fax: +49 361 966-7222  
E-mail: iasc@mathematik.tu-dresden.de  
URL: <http://mathematik.tu-dresden.de/~iasc/>

Design and Copyright by O.N. Tran, 12/05/2004

 **LAMAR UNIVERSITY**

# Main Conferences

## Applications of Computer Algebra (ACA)

**ACA-2004**

- . INVITED SPEAKERS:
  - . GERALD FARIN (ASU)
  - . HOON HONG (NCSU)
- . PROCEEDINGS OF ACCEPTED FULL PAPERS WILL BE PUBLISHED AND DISTRIBUTED
- . A SPECIAL ISSUE OF JOURNAL OF SYMBOLIC COMPUTATION FOR SELECTED PAPERS
- . FULL AUTOMATIC SYSTEM TO HANDLE THE CONFERENCE



**TENTH INTERNATIONAL CONFERENCE ON APPLICATIONS OF COMPUTER ALGEBRA**  
July 21-23, 2004  
The Hilton Hotel, Beaumont, Texas

Hosted by the Department of Computer Science, Lamar University & the International Scientific Committee for Applications of Computer Algebra

The ACA conference series is dedicated to reporting serious applications of symbolic computation (a.k.a. computer algebra) theories and tools for mathematics, logic, science, engineering and education.

GENERAL CHAIR: QUOC-NAM TRAN, LAMAR UNIVERSITY  
ORGANIZING COMMITTEE: LAWRENCE OSBORNE (LAMAR) AND STANLY STEINBERG (NEW MEXICO)  
PROGRAM CHAIR: QUOC-NAM TRAN (LAMAR) AND VLADIMIR GEHTY (DURB)  
SCIENTIFIC COMMITTEE: JAVIER ARAYA (CHILE)  
ERNSTE KUTTLER (AUSTRIA)  
VICTOR EDNIPAL (HUNGARY)  
JAN VILMOS GOMBOS (HUNGARY)  
STANLY STEINBERG (USA)  
LAWRENCE OSBORNE (USA)  
NOMOLAY NABIEV (RUSSIA)  
MICHAEL WYNN (USA)

THE CONFERENCE WILL COVER (but not be limited to) the following topics: Computer Algebra Systems, Symbolic Number Theory, Computer Algebra in Economics and Finance, Computer Algebra, Computational Aspects of Computer Algebra, Geometric Design and Geometric Modeling, Theory and Applications of Groebner Bases, Computer Algebra in Robotics, Computer Algebra in Education, Computer Algebra and Theorem Provers, Non Standard Applications, Theory and Applications of Quantifier Elimination, Theory and Applications of Resultants, Theory and Applications of Differential Equations, Theory and Applications of Difference Equations, Theory and Applications to Computational Sciences, Types and Specification, Applications to Chemistry, Physics and Mathematics, Industrial and Engineering Applications of Computer Algebra, General and other Applications of Computer Algebra

LOCAL ARRANGEMENTS : JUNSWOO CHU (CO-CHAIR), VALENTIN ADRESY (CO-CHAIR), JENNIFER FOWLER, KYOO KON, CHUN-SOON LI

Phone: (409) 884-7222  
Fax: (409) 884-7223  
E-mail: [aca2004@lamar.edu](mailto:aca2004@lamar.edu), <http://math.lamar.edu/aca2004>

Design and Copyright by Q.N. Tran, 12/05/2003

**LAMAR UNIVERSITY**

Started in:  
1995

# Main Conferences

## Applications of Computer Algebra (ACA)



**ACA-2004**

- INVITED SPEAKERS:
  - GERALD FARIN (ASU)
  - HOON HONG (NCSU)
- PROCEEDINGS OF ACCEPTED FULL PAPERS WILL BE PUBLISHED AND DISTRIBUTED
- A SPECIAL ISSUE OF JOURNAL OF SYMBOLIC COMPUTATION FOR SELECTED PAPERS
- FULL AUTOMATIC SYSTEM TO HANDLE THE CONFERENCE

**TENTH INTERNATIONAL CONFERENCE ON APPLICATIONS OF COMPUTER ALGEBRA**

July 21-23, 2004  
The Hilton Hotel, Beaumont, Texas

Hosted by the Department of Computer Science, Lamar University & the International Scientific Committee for Applications of Computer Algebra

The ACA conference series is dedicated to reporting serious applications of symbolic computation (a.k.a. computer algebra) theories and tools for mathematics, logic, science, engineering and education.

GENERAL CHAIR: QUOC-NAM TRAN, LAMAR UNIVERSITY  
ORGANIZING COMMITTEE: LAWRENCE OSBORNE (LAMAR) AND STANLY STEINBERG (NEW MEXICO)  
PROGRAM CHAIR: QUOC-NAM TRAN (LAMAR) AND VLADIMIR GORIĆ (DUNA)  
SCIENTIFIC COMMITTEE: ALEXANDER ARAYA (CHILE)  
BERNARD KUTTERER (AUSTRIA)  
VICTOR EDNIPAL (HUNGARY)  
JOHN D. COOPER (CANADA)  
STANLY STEINBERG (USA)  
LAWRENCE OSBORNE (USA)  
MICHAEL WILLETT (USA)  
JAKOBUS CAI MII (GERMANY)  
HICHIRE LIKA (CZECH REP)  
JAN VYACHESLAVOV (CZECH REP)  
QUOC-NAM TRAN (USA)  
TONY CHAMPA (USA)

The conference will cover (but not be limited to) the following topics: Computer Algebra Systems, Symbolic Number Theory, Computer Algebra in Economics and Finance, Computer Algebra, Computational Aspects of Computer Algebra, Geometric Design and Geometric Modeling, Theory and Applications of Groebner Bases, Computer Algebra in Robotics, Computer Algebra in Education, Computer Algebra and Theorem Provers, Non Standard Applications, Theory and Applications of Quantifier Elimination, Theory and Applications of Resultants, Theory and Applications of Differential Equations, Theory and Applications of Difference Equations, Theory and Applications to Computational Sciences, Types and Specification, Applications to Chemistry, Physics and Mathematics, Industrial and Engineering Applications of Computer Algebra, General and other Applications of Computer Algebra

LOCAL ARRANGEMENTS : JIN-SUNG CHU (CO-CHAIR), VALENTIN ADRESY (CO-CHAIR), JENNIFER FOWLER, KYO-KWON CHUNG, SONG LI

Phone: +401 566-7222  
Fax: +401 566-7223  
E-mail: [aca2004@lamar.edu](mailto:aca2004@lamar.edu), <http://math.lamar.edu/~lamar/aca2004>

Design and Copyright by Q.N. Tran, 12/05/2003

LAMAR UNIVERSITY

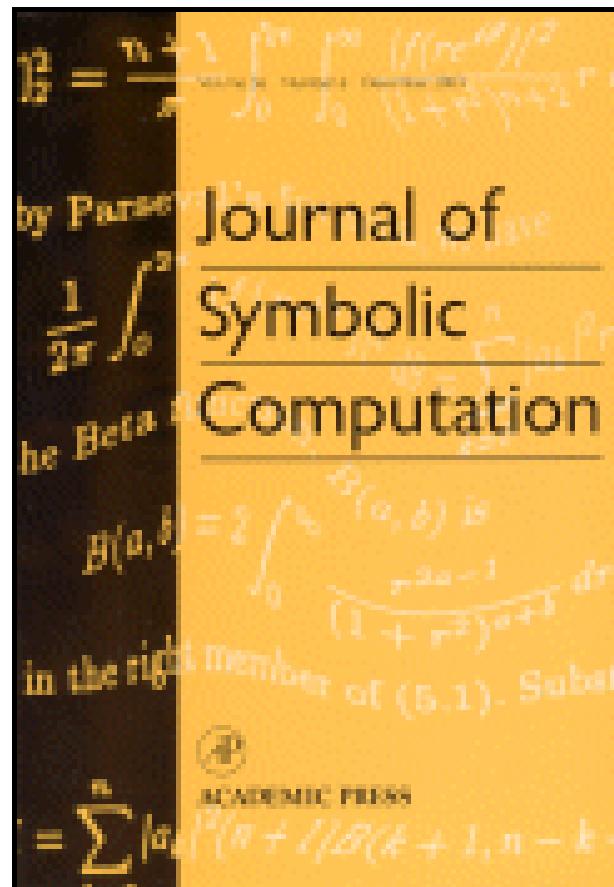
Started in:  
1995

Emphasis:  
Applications

# Main Journals

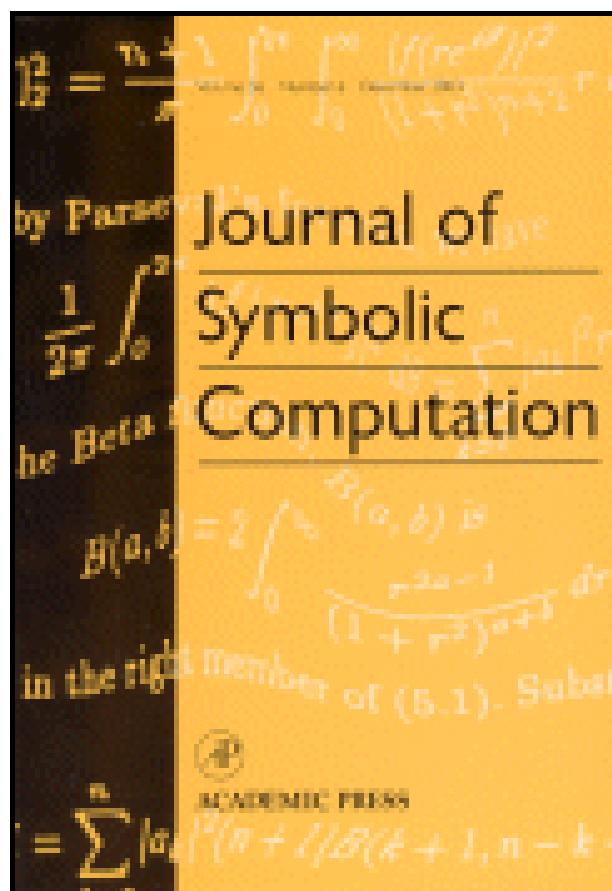
# Main Journals

## Journal of Symbolic Computation (JSC)



# Main Journals

## Journal of Symbolic Computation (JSC)



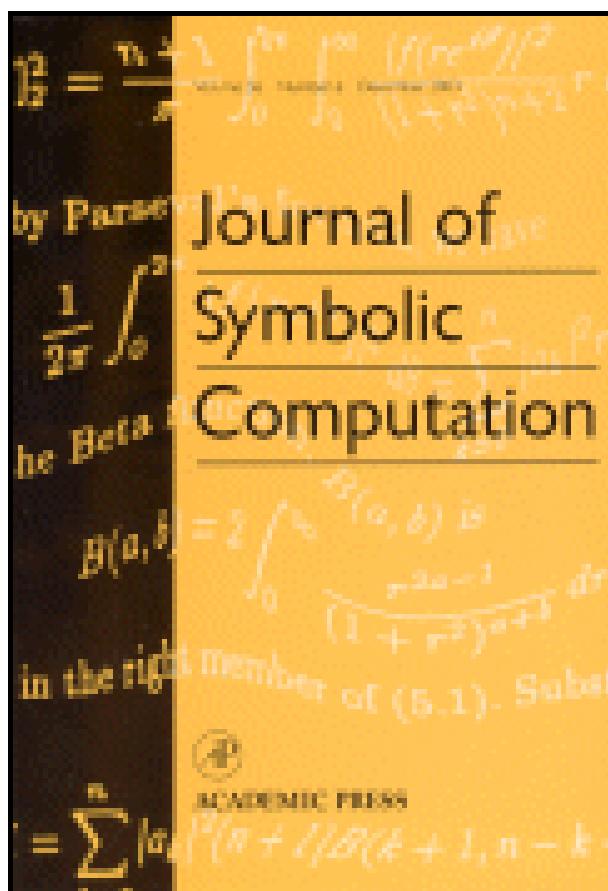
Editor-in-Chief

Hoon Hong



# Main Journals

## Journal of Symbolic Computation (JSC)



**Editor-in-Chief**

Hoon Hong

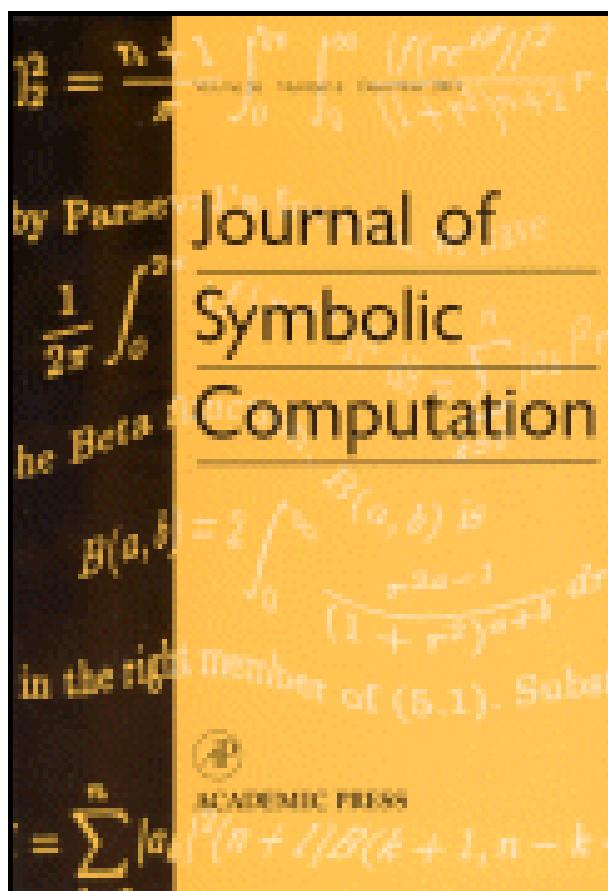


**Editorial Board**

46 international experts

# Main Journals

## Journal of Symbolic Computation (JSC)



**Editor-in-Chief**

Hoon Hong



**Editorial Board**

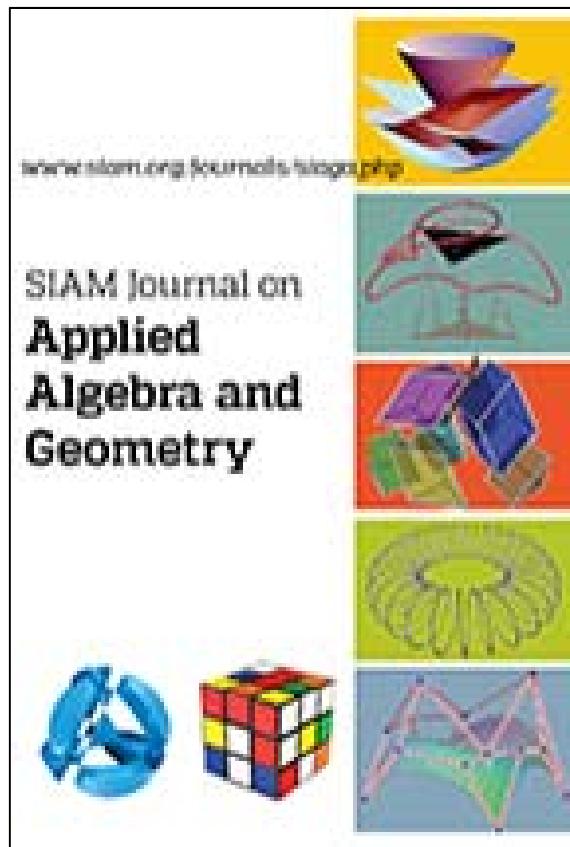
46 international experts

**Emphasis**

Foundations and Theories

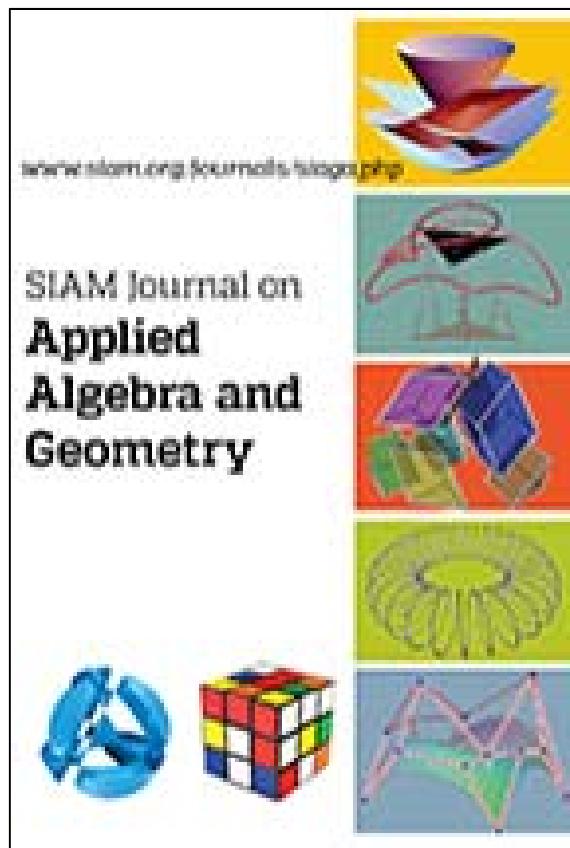
# Main Journals

## Applied Algebra and Geometry



# Main Journals

## Applied Algebra and Geometry

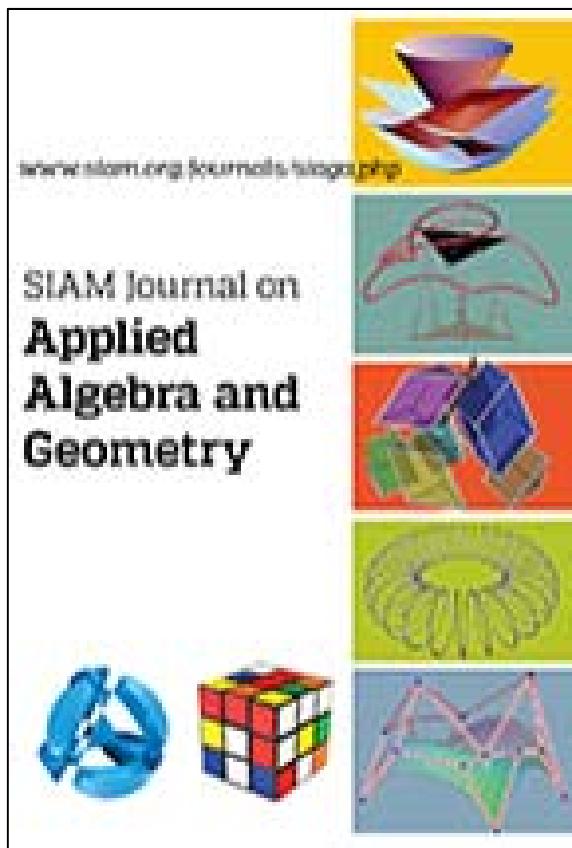


Editor-in-Chief  
Bernd Sturmfelds



# Main Journals

## Applied Algebra and Geometry



**Editor-in-Chief**

Bernd Sturmfelds

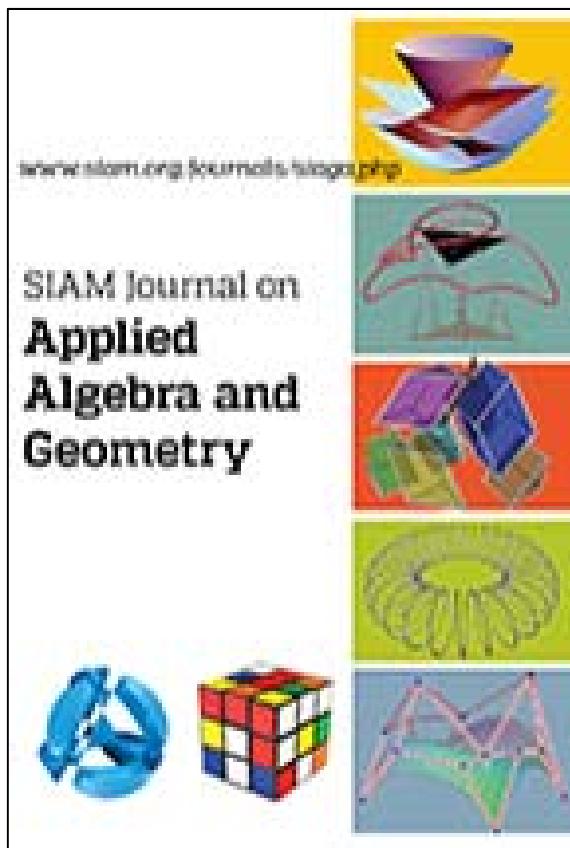
**Editorial Board**

30 international experts



# Main Journals

## Applied Algebra and Geometry



**Editor-in-Chief**

Bernd Sturmfelds

**Editorial Board**

30 international experts

**Emphasis**

Applications

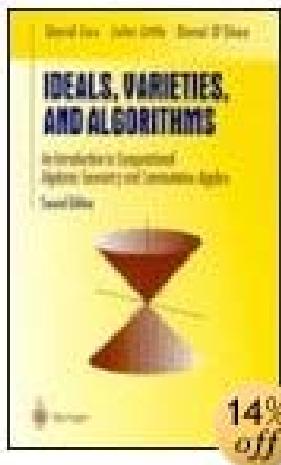


# Textbooks

# Textbooks

About 50 texts

- Undergraduate
- Graduate

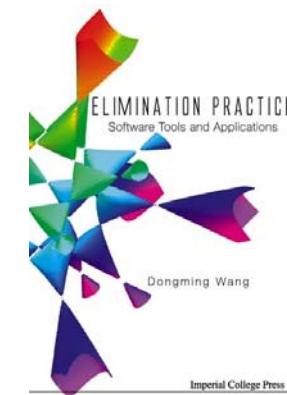


Graduate Texts  
in Mathematics

David Cox  
John Little  
Donal O'Shea

Using  
Algebraic  
Geometry

Springer



# Main Professional Society

# Main Professional Society

## ACM-SIGSAM

Special Interest Group on  
Symbolic and Algebraic Manipulation

- Starting in 1966
- Sponsors ISSAC and Best Paper Awards.



# Main Professional Society

**ACM-SIGSAM**

Special Interest Group on  
Symbolic and Algebraic Manipulation

- Starting in 1966
- Sponsors ISSAC and Best Paper Awards.



# Main Professional Society

**ACM-SIGSAM**

Special Interest Group on  
Symbolic and Algebraic Manipulation

- Starting in 1966
- Sponsors ISSAC and Best Paper Awards.



# Commercial Software Systems

# Commercial Software Systems

Maple

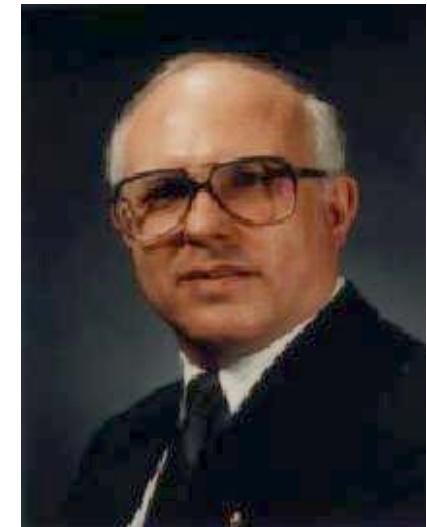


# Commercial Software Systems

## Maple



- 1983 Started as a research project at the university of Waterloo at Canada by Keith .O. Geddes, et al.



# Commercial Software Systems

## Maple



- 1983 Started as a research project at the university of Waterloo at Canada by Keith .O. Geddes, et al.
- 1987 Turned into a Waterloo Maple Inc (MapleSoft).



# Commercial Software Systems

## Maple



- 1983 Started as a research project at the university of Waterloo at Canada by Keith .O. Geddes, et al.
- 1987 Turned into a Waterloo Maple Inc (MapleSoft).
- Makes the employees to read JSC and attend ISSAC/ACA meetings.

# Commercial Software Systems

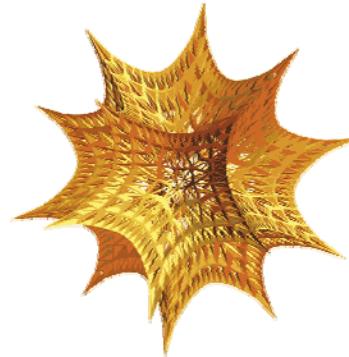
## Maple



- 1983 Started as a research project at the university of Waterloo at Canada by Keith .O. Geddes, et al.
- 1987 Turned into a Waterloo Maple Inc (MapleSoft).
- Makes the employees to read JSC and attend ISSAC/ACA meetings.
- Sponsors various meetings: ISSAC/ACA.

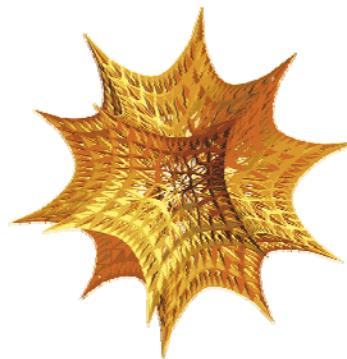
# Commercial Software Systems

Mathematica

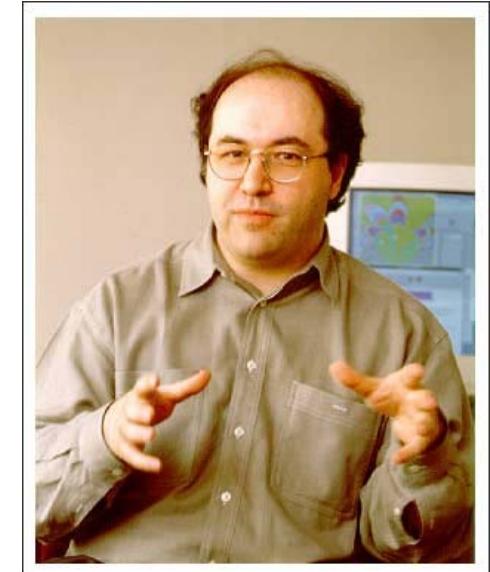


# Commercial Software Systems

Mathematica

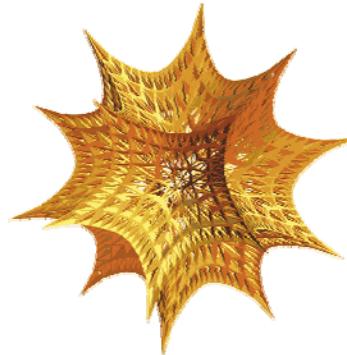


- 1981 Started as a research project by Stephen Wolfram, et al.

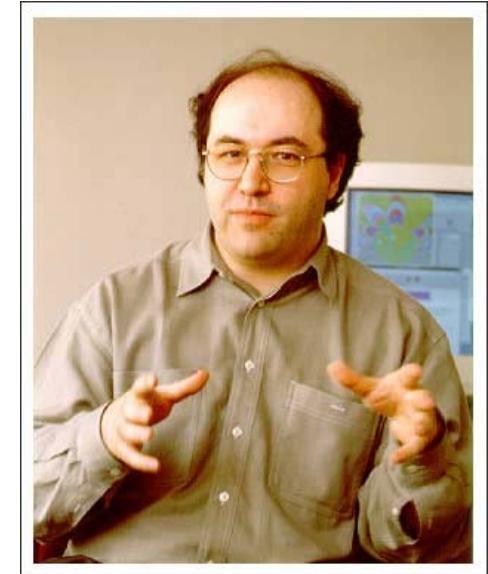


# Commercial Software Systems

Mathematica

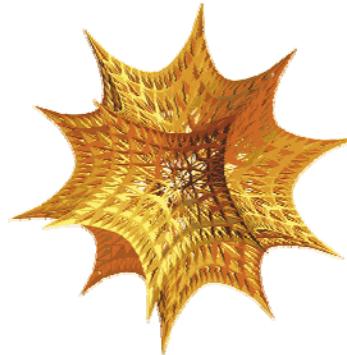


- 1981 Started as a research project by Stephen Wolfram, et al.
- Often attended the ISSAC-kind conference to discuss ideas.

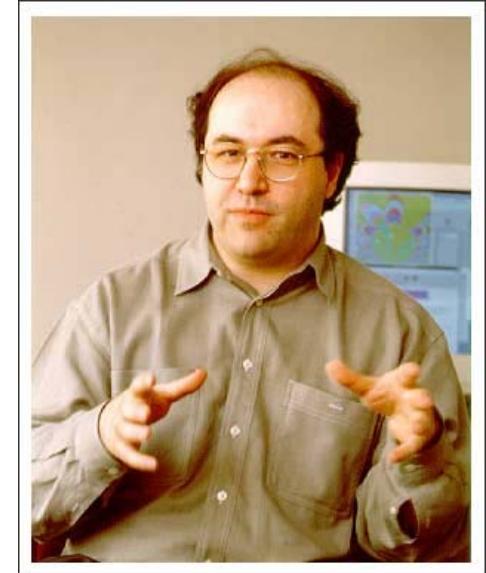


# Commercial Software Systems

## Mathematica

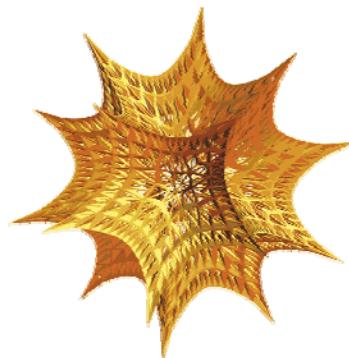


- 1981 Started as a research project by Stephen Wolfram, et al.
- Often attended the ISSAC-kind conference to discuss ideas.
- 1988 Turned into Wolfram Research Inc

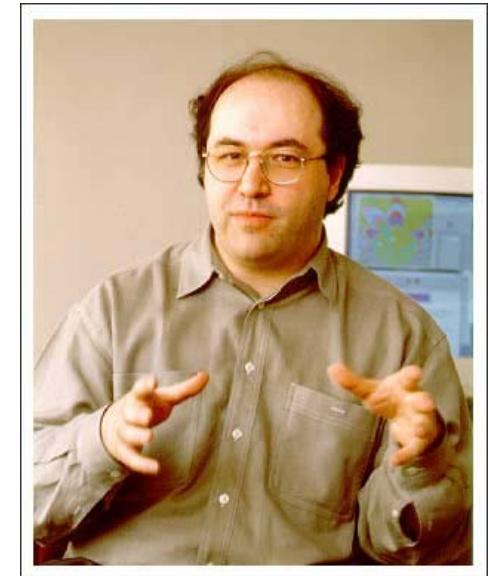


# Commercial Software Systems

## Mathematica



- 1981 Started as a research project by Stephen Wolfram, et al.
- Often attended the ISSAC-kind conference to discuss ideas.
- 1988 Turned into Wolfram Research Inc
- Organizes major conferences for users



# Academic Software Systems

[https://en.wikipedia.org/wiki/List\\_of\\_computer\\_algebra\\_systems](https://en.wikipedia.org/wiki/List_of_computer_algebra_systems)

The screenshot shows a Microsoft Edge browser window with the URL [https://en.wikipedia.org/wiki/List\\_of\\_computer\\_algebra\\_systems](https://en.wikipedia.org/wiki/List_of_computer_algebra_systems) in the address bar. The page title is "List of computer algebra systems". The sidebar on the left contains links such as Main page, Contents, Featured content, Current events, Random article, Donate to Wikipedia, Wikipedia store, Interaction, Help, About Wikipedia, Community portal, Recent changes, Contact page, Tools, What links here, Related changes, Upload file, Special pages, Permanent link, Page information, and a language selector. The main content area includes a "Contents" sidebar with sections like General, Functionality, Operating system support, Graphing calculators, See also, References, and External links. Below this is a "General" section with a table comparing various computer algebra systems based on criteria like System, Creator, Development started, First public release, Latest stable version, Latest stable release date, Cost (USD), License, and Notes. The table shows that the first system listed, Mathematica, was developed in 1993 and has a cost of \$1000 USD. The browser's taskbar at the bottom shows other open tabs and icons.

System	Creator	Development started	First public release	Latest stable version	Latest stable release date	Cost (USD)	License	Notes
Mathematica	Wolfram Research	1993	1993 and	2023	2023	\$1000	Proprietary	General purpose CAS.

# Academic Software Systems

[https://en.wikipedia.org/wiki/List\\_of\\_computer\\_algebra\\_systems](https://en.wikipedia.org/wiki/List_of_computer_algebra_systems)

The screenshot shows a Microsoft Edge browser window displaying the Wikipedia page for a "List of computer algebra systems". The main content is a table titled "General" comparing different computer algebra systems based on various criteria.

**Table Headers:**

- System
- Creator
- Development started
- First public release
- Latest stable version
- Latest stable release date
- Cost (USD)
- License
- Notes

**Table Data:**

Axiom	Richard Jenks	1977	1993 and 2002<sup>[7]</sup>		August 2014<sup>[8]</sup>	Free	modified BSD license	General purpose CAS. Continuous Release using Docker Containers
Cadabra	Kasper Peeters	2001	2007	2.1.4	April 14, 2017	Free	GNU GPL	CAS for tensor field theory
CoCoA-4	The CoCoA Team	1987	1995	4.7.5	2009	Free for non-commercial use	own license	Specialized CAS for commutative algebra
CoCoA-5	John Abbott, Anna M. Bigatti, Giovanni Lagorio	2000	2011	5.2.0	May 2, 2017	Free	GNU GPL	Specialized CAS for commutative algebra
Derive	Soft Warehouse	1979	1988	6.1	November 2007	Discontinued	Proprietary	CAS designed for pocket calculators; it was discontinued in 2007
DataMelt (DMelt)	jWork.ORG (Sergei Chekanov)	2005	2015	1.8	June 24, 2017	\$0 for academic usage, commercial license unknown	Proprietary	Java-based. Runs on the Java platform. Supports Python, Ruby, Groovy, Java and Octave.
Erable (aka ALGB)	Bernard Parisse, Mika Heiskanen, Claude-Nicolas Flechter	1993	1993	4.20060919	April 21, 2009	Free	LGPL	CAS designed for Hewlett-Packard scientific graphing calculators of the HP 48/49/40/50 series; discontinued in 2009

# Academic Software Systems

[https://en.wikipedia.org/wiki/List\\_of\\_computer\\_algebra\\_systems](https://en.wikipedia.org/wiki/List_of_computer_algebra_systems)

W List of computer algebra x

Secure | https://en.wikipedia.org/wiki/List\_of\_computer\_algebra\_systems

General [ edit ]

System	Creator	Development started	First public release	Latest stable version	Latest stable release	Notes
Axiom	Richard Jenks	1977	1993 and 2002 <sup>[7]</sup>		August 2017	General purpose CAS. Continuous Release using Docker Containers
Cadabra	Kasper Peeters	2001	2007	2.1.4	April 1, 2017	CAS for tensor field theory
CoCoA-4	The CoCoA Team	1987	1998	4.7.5	2017	Specialized CAS for commutative algebra
CoCoA-5	John Abbott, Anna M. Bigatti, Giovanni Lagorio	2000	2011	5.2.0	May 2, 2017	Specialized CAS for commutative algebra
Derive	Soft Warehouse	1979	1988	6.1	November 2016	CAS designed for pocket calculators; it was discontinued in 2007
DataMelt (DMelt)	jWork.ORG (Sergei Chekanov)	2005	2015	1.8	June 2017	Java-based. Runs on the Java platform. Supports Python, Ruby, Groovy, Java and Octave.
Erasable (aka ALGB)	Bernard Parisse, Mika Heiskanen, Claude-Nicolas Flechter	1993	1993	4.20060919	April 2017	CAS designed for Hewlett-Packard scientific graphing calculators of the HP 48/49/40/50 series; discontinued in 2009

Tools

- What links here
- Related changes
- Upload file
- Special pages
- Permanent link
- Page information
- Wikidata item
- Cite this page

Print/export

- Create a book
- Download as PDF
- Printable version

Languages

- Español
- 日本語
- 中文

Edit links

11:30 AM 11:30 AM 11:30 AM 11:30 AM 11:30 AM 11:30 AM 11:31 AM

# Quantifier Elimination

“Sociological Overview”

# Problem : Quantifier Elimination

# Problem : Quantifier Elimination

Input:

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:  $\forall x \quad x^2 + 3x + 1 > 0$

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:  $\forall x \quad x^2 + 3x + 1 > 0$

Output: *False*

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:  $\forall x \quad x^2 + 3x + 1 > 0$

Output: *False*

Input:

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:  $\forall x \quad x^2 + 3x + 1 > 0$

Output: *False*

Input:  $\forall x \quad x^2 + bx + 1 > 0$

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \quad x^2 + 1 > 0$

Output: *True*

Input:  $\forall x \quad x^2 + 3x + 1 > 0$

Output: *False*

Input:  $\forall x \quad x^2 + bx + 1 > 0$

Output:  $-2 < b < 2$

# Problem : Quantifier Elimination

# Problem : Quantifier Elimination

Input:

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \exists y \quad x^2 + xy + b > 0$

$\wedge$

$x + ay^2 + b \leq 0$

Output:

# Problem : Quantifier Elimination

Input:  $\forall x \exists y \quad x^2 + xy + b > 0$

$\wedge$

$x + ay^2 + b \leq 0$

Output:  $a < 0 \wedge b > 0$

# Problem : Quantifier Elimination

# Problem : Quantifier Elimination

Input:

Output:

# Problem : Quantifier Elimination

**Input:**  $\exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$

$$X = \{a, b\}$$

$$Y = \{c_1, s_1, c_2, s_2\}$$

$$F = \{c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1\}$$

$$G = \{g\}$$

$$\begin{aligned} g = & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\ & 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\ & 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4a^6b^2c_1^3c_2^2 + 8a^5b^3s_1s_2c_1^3 + \\ & 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a^4b^4c_1^4c_2 - 24a^4b^4c_1^3c_2^2 - \\ & 24a^4b^4c_1^2c_2^3 - 8a^4b^4c_1c_2^4 + 16a^3b^5s_1s_2c_1^2c_2 + 12a^3b^5s_1s_2c_1c_2^2 + \\ & 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 - 4ab^7s_1s_2c_2^3 + a^8c_1^4 + \\ & 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^4 - 12a^6b^2c_1^3c_2 - 12a^6b^2c_1^2c_2^2 - 4a^5b^3s_1s_2c_1^2 - \\ & 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + 4a^4b^4c_2^4 - 4a^4b^2c_1^4c_2^2 - \\ & 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^5s_1s_2c_1^2c_2^2 - 12a^2b^6c_1^2c_2^2 - \\ & 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12ab^7s_1s_2c_2^2 + b^8c_2^4 - 4a^8c_1^3 - \\ & 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 12a^6b^2c_1^2c_2^2 + 20a^6b^2c_1c_2^2 - 16a^5b^3s_1s_2c_1 - \\ & 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1^3 + 8a^4b^4c_1^3 + 12a^4b^4c_1^2c_2 + 12a^4b^4c_1c_2^2 + \\ & 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\ & 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20a^2b^6c_1^2c_2 + 12a^2b^6c_1c_2^2 + \\ & 16a^2b^6c_2^3 + 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2^3 - \\ & 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2^2 - 2a^6c_1^4 + \\ & 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2^2 - \\ & 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2 + 4a^3b^5s_1s_2c_1^2 + 16a^3b^5s_1s_2c_1c_2 + \\ & 4a^3b^5s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1^2c_2^2 - 4a^2b^4c_1c_2^3 + \\ & 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6c_2^4 - 4a^8c_1 - 16a^6b^2c_1 + \\ & 8a^6c_1^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1^2c_2 - 16a^4b^2c_1c_2^2 - \\ & 4a^3b^5s_1s_2c_1 - 4a^3b^5s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1^2c_2 - 8a^2b^4c_1c_2^2 + \\ & 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_3^3 + a^8 + 8a^6b^2 - 12a^6c_1^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\ & 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a^2b^6 + 6a^2b^4c_1^2 + 12a^2b^4c_1c_2 - \\ & 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2^2 + b^4c_2^4 + 8a^6c_1 + 4a^4b^2c_1 - \\ & 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\ & 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\ & 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\ & 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2 \end{aligned}$$

**Output:**

# Problem : Quantifier Elimination

**Input:**  $\exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$

$$X = \{a, b\}$$

$$Y = \{c_1, s_1, c_2, s_2\}$$

$$F = \{c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1\}$$

$$G = \{g\}$$

$$\begin{aligned} g = & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\ & 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\ & 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4a^6b^2c_1^3c_2^2 + 8a^5b^3s_1s_2c_1^3 + \\ & 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a^4b^4c_1^4c_2 - 24a^4b^4c_1^3c_2^2 - \\ & 24a^4b^4c_1^2c_2^3 - 8a^4b^4c_1c_2^4 + 16a^3b^5s_1s_2c_1^2c_2 + 12a^3b^5s_1s_2c_1c_2^2 + \\ & 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 - 4ab^7s_1s_2c_2^3 + a^8c_1^4 + \\ & 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^4 - 12a^6b^2c_1^3c_2 - 12a^6b^2c_1^2c_2^2 - 4a^5b^3s_1s_2c_1^2 - \\ & 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + 4a^4b^4c_2^4 - 4a^4b^2c_1^4c_2^2 - \\ & 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^5s_1s_2c_1^2c_2^2 - 12a^2b^6c_1^2c_2^2 - \\ & 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12ab^7s_1s_2c_2^2 + b^8c_2^4 - 4a^8c_1^3 - \\ & 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 12a^6b^2c_1^2c_2^2 + 20a^6b^2c_1c_2^2 - 16a^5b^3s_1s_2c_1 - \\ & 4a^5b^3s_1s_2c_2 + 4a^5b^3s_1s_2c_1^3 + 8a^4b^4c_1^3 + 12a^4b^4c_1^2c_2 + 12a^4b^4c_1c_2^2 + \\ & 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\ & 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20a^2b^6c_1^2c_2 + 12a^2b^6c_1c_2^2 + \\ & 16a^2b^6c_2^3 + 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2^3 - \\ & 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2^2 - 2a^6c_1^4 + \\ & 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2^2 - \\ & 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2 + 4a^3b^5s_1s_2c_1^2 + 16a^3b^5s_1s_2c_1c_2 + \\ & 4a^3b^5s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1^2c_2^2 - 4a^2b^4c_1c_2^3 + \\ & 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6c_2^4 - 4a^8c_1 - 16a^6b^2c_1 + \\ & 8a^6c_1^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1^2c_2 - 16a^4b^2c_1c_2^2 - \\ & 4a^3b^5s_1s_2c_1 - 4a^3b^5s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1^2c_2 - 8a^2b^4c_1c_2^2 + \\ & 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2^3 + a^8 + 8a^6b^2 - 12a^6c_1^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\ & 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a^2b^6 + 6a^2b^4c_1^2 + 12a^2b^4c_1c_2 - \\ & 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2^2 + b^4c_2^4 + 8a^6c_1 + 4a^4b^2c_1 - \\ & 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\ & 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\ & 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\ & 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2 \end{aligned}$$

**Output:**  $a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$

# Problem : Quantifier Elimination

# Problem : Quantifier Elimination

Input:

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

*Algebraic expressions*

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

∧   ∨   ¬   ⇒   ⇐   ⇔

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

∧   ∨   ¬   ⇒   ⇐   ⇔

∀   ∃

Output:

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

∧   ∨   ¬   ⇒   ⇐   ⇔

∀   ∃

Output: *Expression*

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

∧   ∨   ¬   ⇒   ⇐   ⇔

∀   ∃

Output: *Expression*

equivalent

# Problem : Quantifier Elimination

Input: *Expression made of*

Algebraic expressions

=   ≠   >   <   ≥   ≤

∧   ∨   ¬   ⇒   ⇐   ⇔

∀   ∃

Output: *Expression*

equivalent

without   ∀   ∃

# Motivation

# Motivation

## *Foundation of Mathematics*

- Hilbert's Program (1900)

# Motivation

## *Foundation of Mathematics*

- Hilbert's Program (1900)

## *Applications in Science and Engineering*

- Stability analysis of PDE and Finite differences
- Robust control system design
- Reachability analysis
- Parametric optimization
- Hybrid system analysis
- Parameter estimation
- Robot motion planning
- Computer vision
- Dynamic geometric constraint solving
- Education software for real analysis
- ....

# Geometric Reasoning

# Geometric Reasoning

A. Dolzman, ADG (2000), (*Edited by X. Gao, D. Wang and L. Yang*)

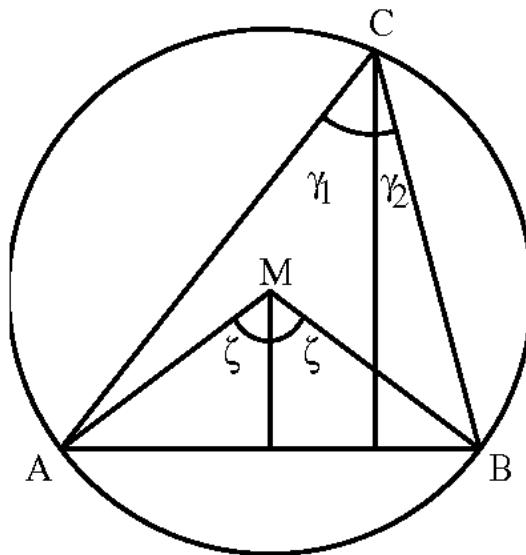
# Geometric Reasoning

A. Dolzman, ADG (2000), (*Edited by X. Gao, D. Wang and L. Yang*)



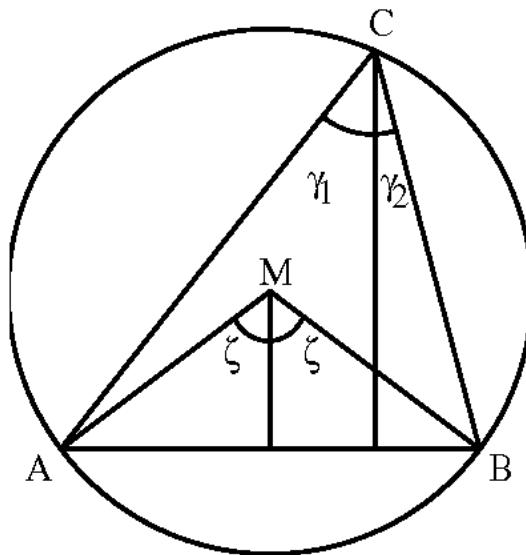
# Geometric Reasoning

A. Dolzman, ADG (2000), (Edited by X. Gao, D.Wang and L.Yang)



# Geometric Reasoning

A. Dolzman, ADG (2000), (Edited by X. Gao, D.Wang and L.Yang)



$$\begin{aligned} \forall x \forall t_1 \forall t_2 \forall t \forall b (c^2 = a^2 + b^2 \wedge c^2 = x_0^2 + (y_0 - b)^2 \wedge \\ y_0 t_1 = a + x_0 \wedge y_0 t_2 = a - x_0 \wedge (1 - t_1 t_2)t = t_1 + t_2 \longrightarrow bt = a) \end{aligned}$$

# Nonlinear Control of Aircraft

# Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)

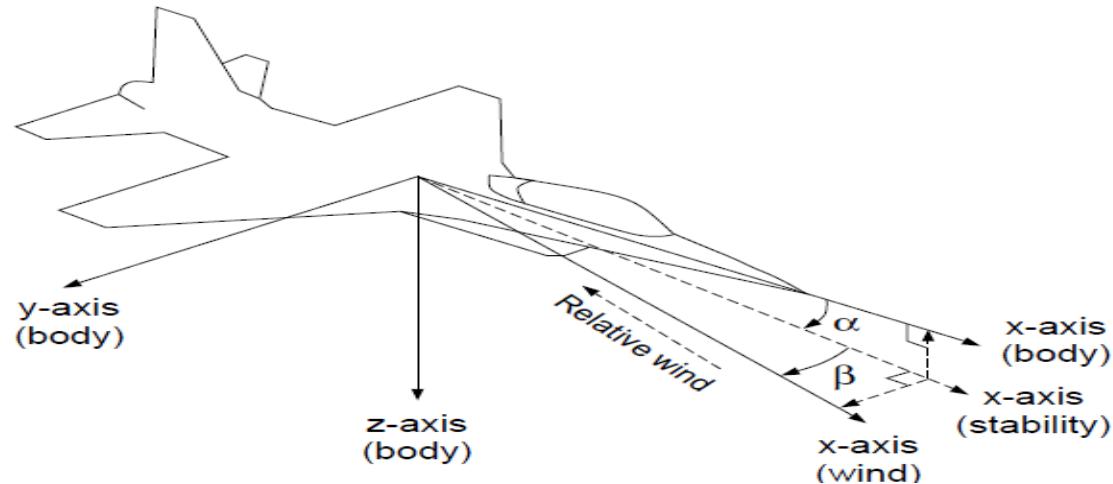
# Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)



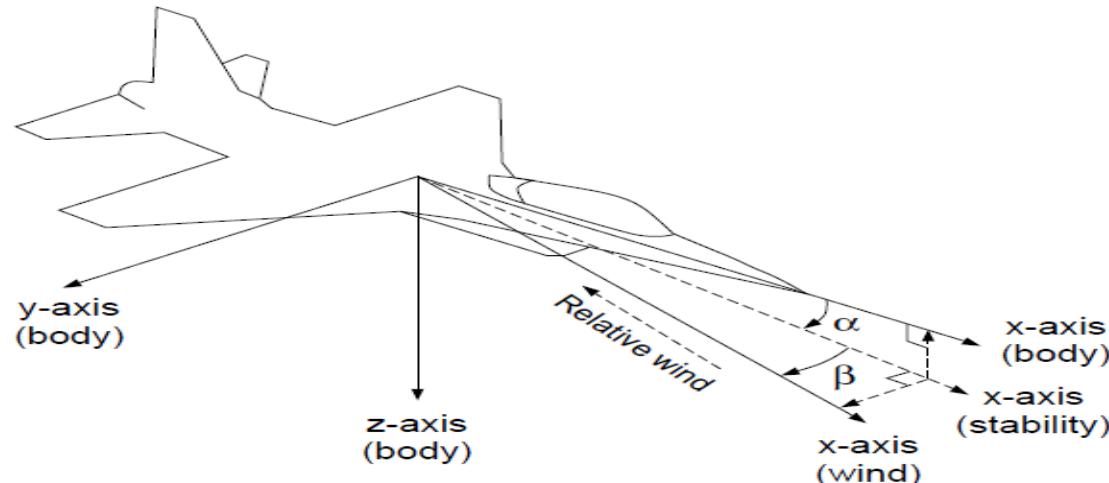
# Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)



# Nonlinear Control of Aircraft

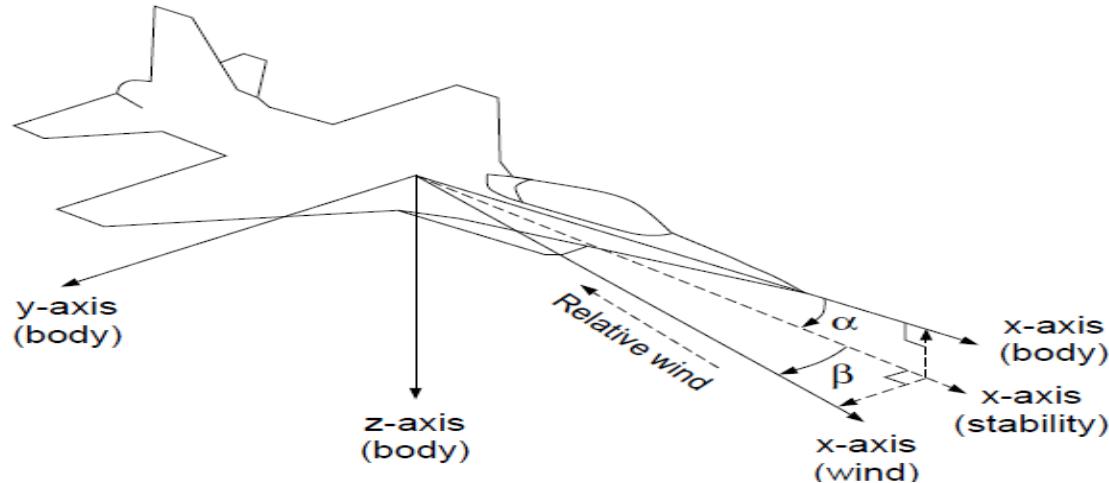
M. Jirstrand *J. Symbolic Computation* (1997)



$$\exists u_1 \exists u_2 \exists u_3 [C_L = 0 \wedge C_M = 0 \wedge C_N = 0 \wedge u_i^2 \leq 1, i = 1, 2, 3]$$

# Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)



$$\exists u_1 \exists u_2 \exists u_3 [C_L = 0 \wedge C_M = 0 \wedge C_N = 0 \wedge u_i^2 \leq 1, i = 1, 2, 3]$$

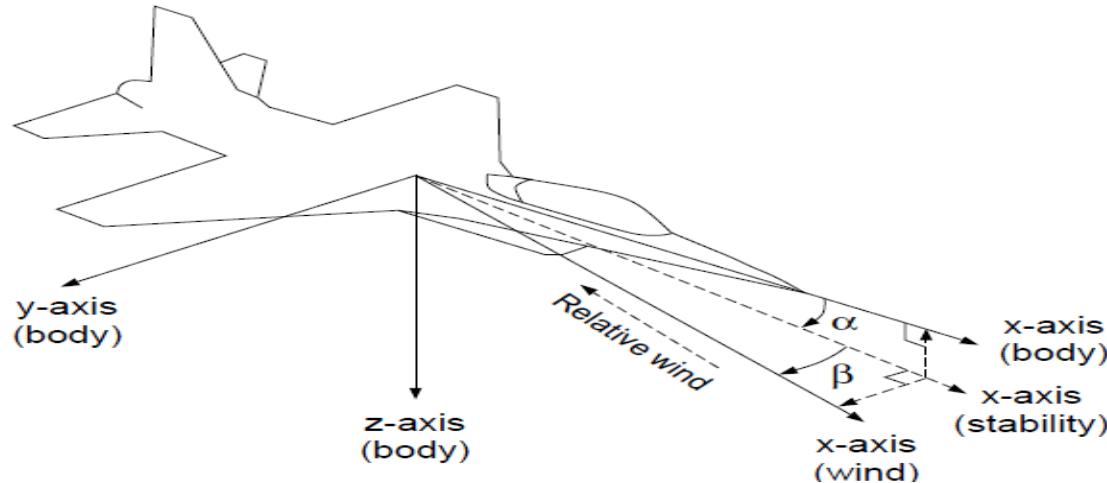
$$\begin{aligned} C_L(x_1, x_2, u_1, u_3) = & -38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3 \\ & + u_1(-52 - 2x_1 + 114x_1^2 - 79x_1^3 + 7x_2^2 + 14x_1x_2^2) \\ & + u_3(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 - 13x_1x_2^2 \\ & + 20x_1^2x_2^2 + 11x_2^4) \end{aligned}$$

$$\begin{aligned} C_M(x_1, u_2) = & -12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 + 17u_2^2x_1 \\ & - 202x_1^2 + 81u_2x_1^2 + 139x_1^3 \end{aligned}$$

$$\begin{aligned} C_N(x_1, x_2, u_1, u_3) = & 139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 - 38x_2^3 + 185x_1x_2^3 \\ & + u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2) \\ & + u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 + 63x_1x_2^2 - 54x_1^4 \\ & - 69x_1^2x_2^2 - 26x_2^4) \end{aligned}$$

# Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)

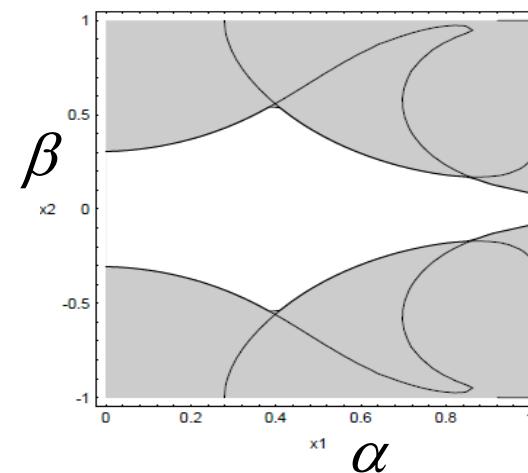


$$\exists u_1 \exists u_2 \exists u_3 [C_L = 0 \wedge C_M = 0 \wedge C_N = 0 \wedge u_i^2 \leq 1, i = 1, 2, 3]$$

$$\begin{aligned} C_L(x_1, x_2, u_1, u_3) = & -38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3 \\ & + u_1(-52 - 2x_1 + 114x_1^2 - 79x_1^3 + 7x_2^2 + 14x_1x_2^2) \\ & + u_3(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 - 13x_1x_2^2 \\ & + 20x_1^2x_2^2 + 11x_2^4) \end{aligned}$$

$$\begin{aligned} C_M(x_1, u_2) = & -12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 + 17u_2^2x_1 \\ & - 202x_1^2 + 81u_2x_1^2 + 139x_1^3 \end{aligned}$$

$$\begin{aligned} C_N(x_1, x_2, u_1, u_3) = & 139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 - 38x_2^3 + 185x_1x_2^3 \\ & + u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2) \\ & + u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 + 63x_1x_2^2 - 54x_1^4 \\ & - 69x_1^2x_2^2 - 26x_2^4) \end{aligned}$$



# Design of HDD Swing-arm

.

# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu

# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu



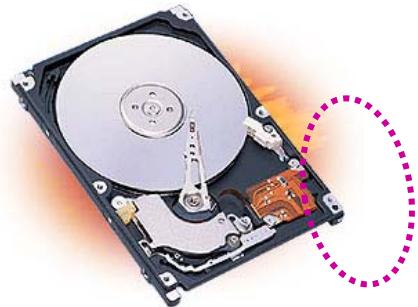
# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu



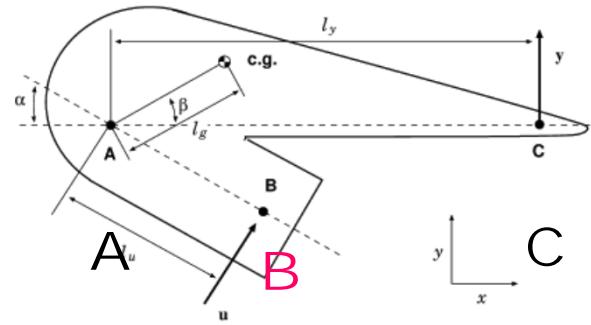
# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu



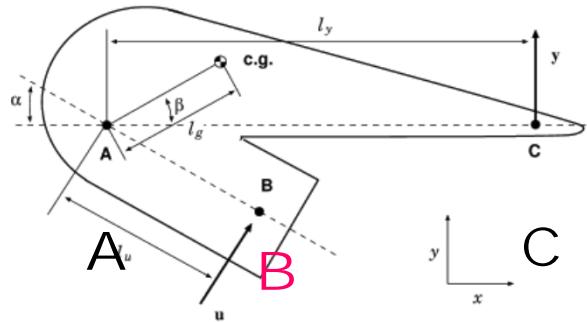
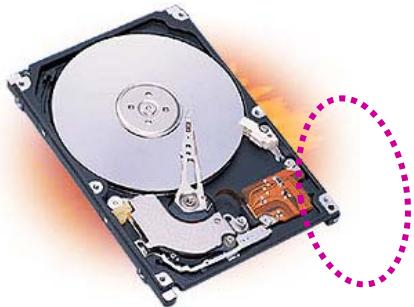
# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu



# Design of HDD Swing-arm

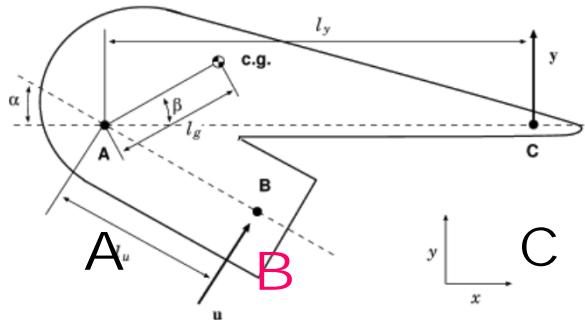
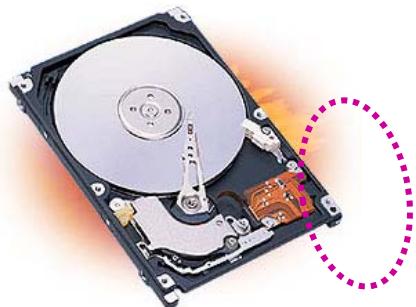
H.Anai, S.Hara (2000) Fujitsu



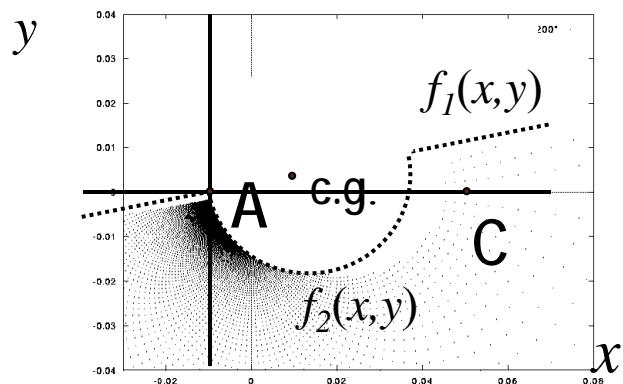
Where should the actuator **B** be located so that the arm satisfies “**stability**” and “**positive realness (PR)**” ?

# Design of HDD Swing-arm

H.Anai, S.Hara (2000) Fujitsu



Where should the actuator **B** be located so that the arm satisfies “**stability**” and “**positive realness (PR)**” ?



# Optimal Numerical Algorithm

# Optimal Numerical Algorithm

M. Erascu, H. Hong, Reliable Computing (2013), JSC (2016)

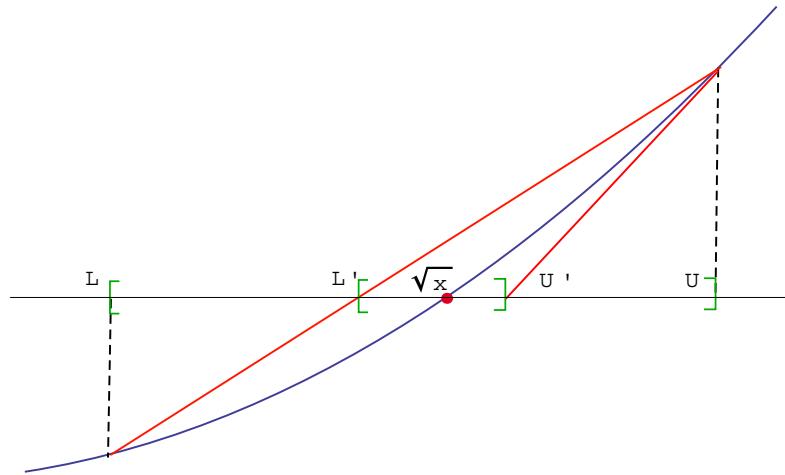
# Optimal Numerical Algorithm

M. Erascu, H. Hong, Reliable Computing (2013), JSC (2016)



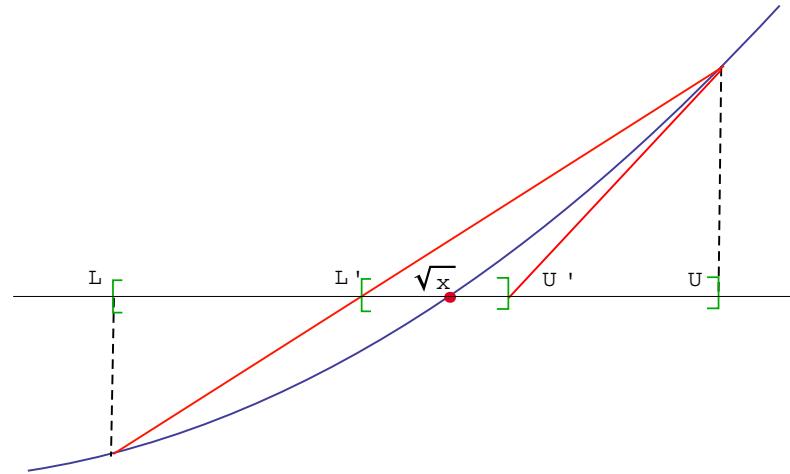
# Optimal Numerical Algorithm

M. Erascu, H. Hong, Reliable Computing (2013), JSC (2016)



# Optimal Numerical Algorithm

M. Erascu, H. Hong, Reliable Computing (2013), JSC (2016)



$$L' = L + \frac{x + p_0 L^2 + p_1 L U + p_2 U^2}{p_3 L + p_4 U} \quad U' = U + \frac{x + q_0 U^2 + q_1 U L + q_2 L^2}{q_3 U + q_4 L}$$

$$\text{Correctness}(p, q) : \iff \forall_{L, U, x} 0 < L \leq \sqrt{x} \leq U \implies 0 < L' \leq \sqrt{x} \leq U'$$

$$\text{Termination}(p, q) : \iff \exists_{c \in (0, 1)} \forall_{L, U, x} 0 < L \leq \sqrt{x} \leq U \implies 0 \leq U' - L' \leq c(U - L)$$

# Stability of Numerical PDE

## McCormack Scheme

# Stability of Numerical PDE McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

# Stability of Numerical PDE McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)



# Stability of Numerical PDE McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

$$u(x, y, t + 2\Delta_t) \approx Mu(x, y, t).$$



# Stability of Numerical PDE McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

$$u(x, y, t + 2\Delta_t) \approx Mu(x, y, t).$$



$$G_{++} = I + a(T_x - I) + b(T_y - I),$$

$$G_{--} = I + a(I - T_x^{-1}) + b(I - T_y^{-1}),$$

$$G_{-+} = I + a(I - T_x^{-1}) + b(T_y - I),$$

$$G_{+-} = I + a(T_x - I) + b(I - T_y^{-1}),$$

$$M_1 = \frac{1}{2}(I + G_{++}G_{--}),$$

$$M_2 = \frac{1}{2}(I + G_{-+}G_{+-}),$$

$$M = M_2M_1.$$

# Stability of Numerical PDE

## McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

$$u(x, y, t + 2\Delta_t) \approx Mu(x, y, t). \quad \exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$$

$$G_{++} = I + a(T_x - I) + b(T_y - I),$$

$$G_{--} = I + a(I - T_x^{-1}) + b(I - T_y^{-1})$$

$$G_{-+} = I + a(I - T_x^{-1}) + b(T_y - I)$$

$$G_{+-} = I + a(T_x - I) + b(I - T_y^{-1})$$

$$M_1 = \frac{1}{2}(I + G_{++}G_{--}),$$

$$M_2 = \frac{1}{2}(I + G_{-+}G_{+-}),$$

$$M = M_2M_1.$$

$$X = \{a, b\}$$

$$Y = \{c_1, s_1, c_2, s_2\}$$

$$F = \{c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1\}$$

$$G = \{g\}$$

$$\begin{aligned} g = & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\ & 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\ & 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4a^6b^2c_1^3c_2^2 + 8a^5b^3s_1s_2c_1^3 + \\ & 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a^4b^4c_1^4c_2 - 24a^4b^4c_1^3c_2^2 - \\ & 24a^4b^4c_1^2c_2^3 - 8a^4b^4c_1c_2^4 + 16a^3b^5s_1s_2c_1^2c_2 + 12a^3b^5s_1s_2c_1c_2^2 + \\ & 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 - 4ab^7s_1s_2c_2^3 + a^8c_1^4 + \\ & 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^3c_2 - 12a^6b^2c_1^2c_2^2 - 4a^5b^8s_1s_2c_1^2 - \\ & 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + 4a^4b^4c_2^4 - 4a^4b^2c_1^4c_2^2 - \\ & 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^5s_1s_2c_1^2c_2^2 - 12a^2b^6c_1^2c_2^2 - \\ & 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12ab^7s_1s_2c_2^2 + b^8c_2^4 - 4a^8c_1^3 - \\ & 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 20a^6b^2c_1c_2^2 - 16a^5b^3s_1s_2c_1^2 - \\ & 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1^3 + 8a^4b^4c_1^3 + 12a^4b^4c_1^2c_2 + 12a^4b^4c_1c_2^2 + \\ & 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 + \\ & 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20a^2b^6c_1^2c_2 + 12a^2b^6c_1c_2^2 + \\ & 16a^2b^6c_2^3 - 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2^3 - \\ & 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2^2 - 2a^6c_1^4 + \\ & 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2^2 - \\ & 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2c_1^2 + 16a^3b^3s_1s_2c_1c_2 + \\ & 4a^3b^3s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1^2c_2^2 - 4a^2b^4c_1c_2^3 + \\ & 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6c_2^4 - 4a^8c_1 - 16a^6b^2c_1 + \\ & 8a^6c_1^3 + 12a^5bs_1s_2c_1^2 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1^2c_2 - 16a^4b^2c_1c_2^2 - \\ & 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1^2c_2 - 8a^2b^4c_1c_2^2 + \\ & 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2^3 + a^8 + 8a^6b^2 - 12a^6c_1^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\ & 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a^2b^6 + 6a^2b^4c_1^2 + 12a^2b^4c_1c_2 - \\ & 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2^2 + b^4c_2^4 + 8a^6c_1 + 4a^4b^2c_1 - \\ & 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\ & 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\ & 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\ & 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2 \end{aligned}$$

$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$$



# Stability of Numerical PDE

## McCormack Scheme

H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

$$u(x, y, t + 2\Delta_t) \approx Mu(x, y, t). \quad \exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$$

$$G_{++} = I + a(T_x - I) + b(T_y - I),$$

$$G_{--} = I + a(I - T_x^{-1}) + b(I - T_y^{-1})$$

$$G_{-+} = I + a(I - T_x^{-1}) + b(T_y - I)$$

$$G_{+-} = I + a(T_x - I) + b(I - T_y^{-1})$$

$$M_1 = \frac{1}{2}(I + G_{++}G_{--}),$$

$$M_2 = \frac{1}{2}(I + G_{-+}G_{+-}),$$

$$M = M_2M_1.$$

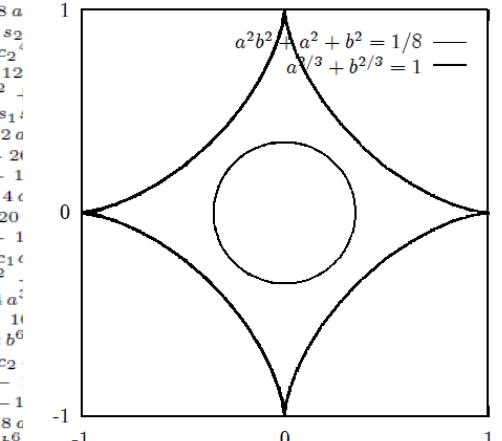
$$X = \{a, b\}$$

$$Y = \{c_1, s_1, c_2, s_2\}$$

$$F = \{c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1\}$$

$$G = \{g\}$$

$$\begin{aligned} g = & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\ & 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\ & 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4 \\ & 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a \\ & 24a^4b^4c_1^2c_2^3 - 8a^4b^6c_1c_2^4 + 16a^3b^5s_1s_2 \\ & 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 \\ & 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^3c_2 - 12 \\ & 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + \\ & 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^5s_1s_2 \\ & 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12a \\ & 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 12a^6b^2c_2^2 + 20 \\ & 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1^3 + 8a^4b^4c_1^3 + 1 \\ & 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4c \\ & 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20 \\ & 16a^2b^6c_2^3 + 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 1 \\ & 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 \\ & 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 \\ & 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2 + 4a^4 \\ & 4a^3b^3s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 1 \\ & 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6 \\ & 8a^6c_1^3 + 12a^5bs_1s_2c_1^2 - 12a^4b^4c_1 - 12a^4b^4c_2 \\ & 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 \\ & 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2^3 + a^8 + 8a^6b^2 - 1 \\ & 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a \\ & 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6 \\ & 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4 \\ & 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\ & 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\ & 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2 \end{aligned}$$



$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$$

# Stability Analysis

## Multi-stable model of Cell Differentiation

# Stability Analysis

## Multi-stable model of Cell Differentiation

H, Hong, X.Tang, B.Xia (2013)

# Stability Analysis

## Multi-stable model of Cell Differentiation

H, Hong, X.Tang, B.Xia (2013)



# Stability Analysis

## Multi-stable model of Cell Differentiation

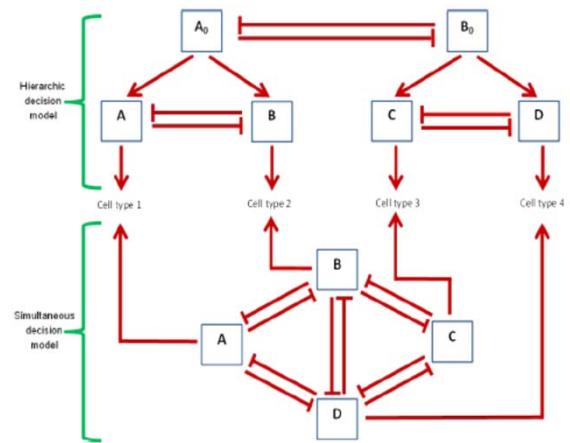
H, Hong, X.Tang, B.Xia (2013)



# Stability Analysis

## Multi-stable model of Cell Differentiation

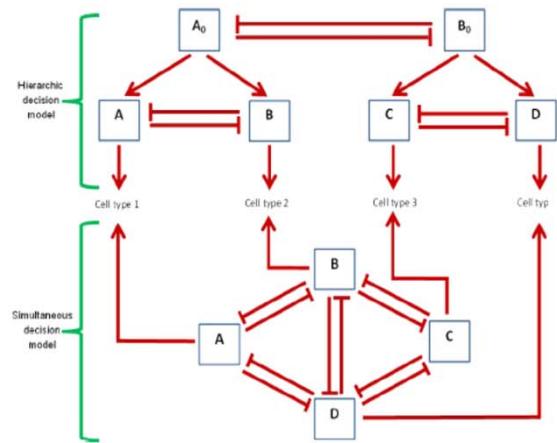
H, Hong, X.Tang, B. Xia (2013)



# Stability Analysis

## Multi-stable model of Cell Differentiation

H. Hong, X.Tang, B.Xia (2013)

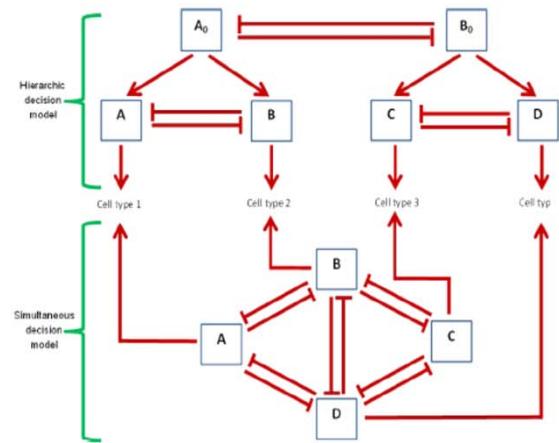


$$\frac{dx_k}{dt} = -x_k + \sigma \frac{1}{1 + \sum_{m=1}^n x_m^c - x_k^c}$$

# Stability Analysis

## Multi-stable model of Cell Differentiation

H, Hong, X.Tang, B.Xia (2013)



$$\frac{dx_k}{dt} = -x_k + \sigma \frac{1}{1 + \sum_{m=1}^n x_m^c - x_k^c}$$

$$\exists \mathbf{x} \left( \mathbf{f}(\sigma, \mathbf{x}) = 0 \wedge \det(J_{\mathbf{f}}(\sigma, \mathbf{x})) \cdot \prod_{k=1}^n \Delta_k(\sigma, \mathbf{x}) = 0 \right)$$

# Hopf Bifurcation

Gene regulatory network

Circadian clock of green alga

# Hopf Bifurcation

## Gene regulatory network

### Circadian clock of green alga

T. Sturm, et al (2013)

# Hopf Bifurcation

## Gene regulatory network

### Circadian clock of green alga

T. Sturm, et al (2013)

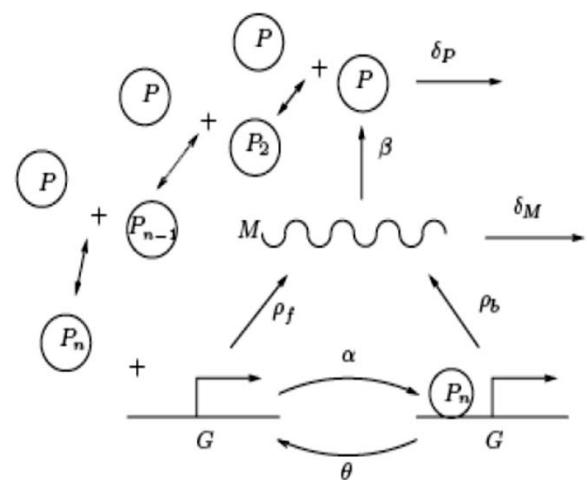


# Hopf Bifurcation

## Gene regulatory network

### Circadian clock of green alga

T. Sturm, et al (2013)



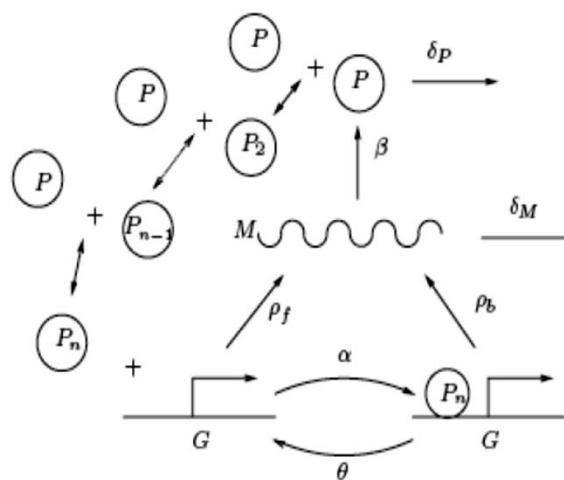
# Hopf Bifurcation

## Gene regulatory network

### Circadian clock of green alga



T. Sturm, et al (2013)



$$\begin{aligned}
 & \exists v_2 \exists v_1 \exists v_3 (0 < v_1 \wedge 0 < v_3 \wedge 0 < v_2 \wedge \theta \gamma_0 - v_1 - v_1 v_3^9 = 0 \wedge \\
 & \lambda_1 v_1 + \gamma_0 \mu - v_2 = 0 \wedge 9\alpha \gamma_0 - v_1 - v_1 v_3^9 + \delta v_2 - v_3 = 0 \wedge \\
 & 0 < \theta \delta + \theta v_3^9 \delta + 9\lambda_1 \theta v_1 v_3^8 \delta \wedge \\
 & 162\theta v_3^{17} \alpha v_1 + 162\theta \alpha v_1 v_3^8 + 162\alpha v_1 v_3^8 \delta + \theta + 2\theta v_3^9 \delta + \theta^2 v_3^{18} \delta \\
 & + \theta v_3^9 \theta \delta + 81\alpha v_1 v_3^8 \theta \delta + 81\alpha v_1 v_3^{17} \theta \delta + \theta^2 + \theta \delta^2 + \theta^2 \delta + \theta^2 \\
 & + 2\theta^2 v_3^9 + \theta^2 v_3^{18} + 6561\alpha^2 v_1^2 v_3^{16} + 2\theta^2 v_3^9 \delta + \theta + 81\alpha v_1 v_3^8 \\
 & + \theta v_3^9 \delta^2 - 9\lambda_1 \theta v_1 v_3^8 \delta = 0 \wedge \\
 & 0 < \theta \wedge 0 < \gamma_0 \wedge 0 < \mu \wedge 0 < \delta \wedge 0 < \alpha)
 \end{aligned}$$

# Motivation (Recap)

# Motivation (Recap)

- Arise as a fundamental question in the logical foundation of mathematics

# Motivation (Recap)

- Arise as a fundamental question in the logical foundation of mathematics
- Numerous problems from science and engineering can be reduced to QE problems.

# History/State of the Art

# History/State of the Art

- ~1930: Tarski

$$2^{2^{2 \cdot \cdot n}}$$



# History/State of the Art

- ~1930: Tarski

$$2^{2^{2^{\cdot\cdot}n}}$$



- ~1975: Collins

$$2^{2^n}$$

# History/State of the Art

- ~1930: Tarski

$$2^{2^{2^{\cdot\cdot}n}}$$



- ~1975: Collins

$$2^{2^n}$$



- ~1990: Canny, Grigorev, Renegar, Roy, Renegar, ...

$2^n$  for no alternation of quantifiers

# History/State of the Art

- ~1930: Tarski

$$2^{2^{2^{\cdot\cdot n}}}$$



- ~1975: Collins

$$2^{2^n}$$



- ~1990: Canny, Grigorev, Renegar, Roy, Renegar, ...

$2^n$  for no alternation of quantifiers

- ----- : Arnon, McCallum, Hong, Brown, Strezbonski, Xia, Chen, Kosta, England, ...

Efficient algorithms for moderate inputs

# History/State of the Art

- ~1930: Tarski

$$2^{2^{2^{\cdot\cdot n}}}$$



- ~1975: Collins

$$2^{2^n}$$



- ~1990: Canny, Grigorev, Renegar, Roy, Renegar, ...

$2^n$  for no alternation of quantifiers

- ----- : Arnon, McCallum, Hong, Brown, Strezbonski, Xia, Chen, Kosta, England, ...

Efficient algorithms for moderate inputs

- ----- : Weispfenning, Sturm, Dolzman, Gonzalez-Vega, Anai, Hong, Safey-Eldin, Moura, Jovanovic, Abraham,.....

Efficient algorithms for special inputs

# Software Packages

# Software Packages

## General Inputs

# Software Packages

- ❑ General Inputs

- QEPCAD (in C)

# Software Packages

- ❑ General Inputs
  - QEPCAD (in C)



# Software Packages

- General Inputs
  - QEPCAD (in C)



# Software Packages

- General Inputs
  - QEPCAD (in C)



# Software Packages

## □ General Inputs

- QEPCAD              (in C)
- Resolve              (in Mathematica)



# Software Packages

## □ General Inputs

- **QEPCAD** (in C)
- **Resolve** (in Mathematica)
- **Regular chain** (in Maple)



# Software Packages

## □ General Inputs

- **QEPCAD** (in C)
- **Resolve** (in Mathematica)
- **Regular chain** (in Maple)
- **Discoverer** (in Maple)



# Software Packages

## □ General Inputs

- **QEPCAD** (in C)
- **Resolve** (in Mathematica)
- **Regular chain** (in Maple)
- **Discoverer** (in Maple)
- ...



# Software Packages

## General Inputs

- QEPCAD (in C)
- Resolve (in Mathematica)
- Regular chain (in Maple)
- Discoverer (in Maple)
- ...



## Special Inputs



# Software Packages

## General Inputs

- QEPCAD (in C)
- Resolve (in Mathematica)
- Regular chain (in Maple)
- Discoverer (in Maple)
- ...



## Special Inputs

- Redlog (in REDUCE)



# Software Packages

## General Inputs

- QEPCAD (in C)
- Resolve (in Mathematica)
- Regular chain (in Maple)
- Discoverer (in Maple)
- ...



## Special Inputs

- Redlog (in REDUCE)



# Software Packages

## General Inputs

- **QEPCAD** (in C)
- **Resolve** (in Mathematica)
- **Regular chain** (in Maple)
- **Discoverer** (in Maple)
- ...



## Special Inputs

- **Redlog** (in REDUCE)
- **SyNRAC** (in Maple)



# Software Packages

## General Inputs

- **QEPCAD** (in C)
- **Resolve** (in Mathematica)
- **Regular chain** (in Maple)
- **Discoverer** (in Maple)
- ...



## Special Inputs

- **Redlog** (in REDUCE)
- **SyNRAC** (in Maple)
- ...



# Software Packages

A screenshot of a web browser showing the swMATH search results for "quantifier+elimination". The search bar at the top contains the query. Below it, a search result card for "QEPCAD" is shown, followed by cards for "SACLIB", "DISCOVERER", and "SYNRAC". A red arrow points from the number "109" in the search bar to the "Results 1 to 20 of 109" text on the search results page.

Mathematical software - x

www.swmath.org/?term=quantifier+elimination&which\_search=standard&sortby=rank

About & Contact   Feedback   Contribute   Help   zbMATH

swMATH

Search   Advanced search   Browse

quantifier elimination

Results 1 to 20 of 109

Sort by: Name   Relevance

**QEPCAD** Referenced in 219 articles [sw00752]  
CADs QEPCAD is an implementation of **quantifier elimination** by partial cylindrical algebraic decomposition due originally...

**SACLIB** Referenced in 24 articles [sw00823]  
also forms the basis of the **quantifier elimination** systems QEPCAD [5] and QEPCAD ... same routines are also used in **quantifier elimination**. While runtime-tools such as Valgrind...

**DISCOVERER** Referenced in 25 articles [sw07719]  
equations and polynomial inequalities. Algorithms for **quantifier elimination** of real closed fields are the general...

**SYNRAC** Referenced in 23 articles [sw00942]  
problems. Our main tool is real **quantifier elimination** and we focus on its application...

# References

# References

- Computational Quantifier Elimination
  - *Computer Journal*
  - Edited by Hong

# References

- Computational Quantifier Elimination
  - *Computer Journal*
  - Edited by Hong
- Collins’ 65<sup>th</sup> Birthday Conference
  - *RISC monograph*
  - Edited by Johnson and Caviness

# References

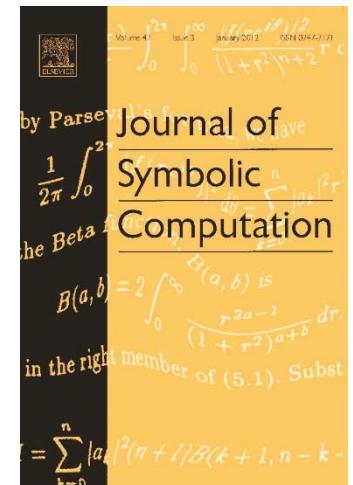
- Computational Quantifier Elimination
  - *Computer Journal*
  - Edited by Hong
- Collins’ 65<sup>th</sup> Birthday Conference
  - *RISC monograph*
  - Edited by Johnson and Caviness
- Application of Quantifier Elimination
  - *Journal of Symbolic Computation*
  - Edited by Hong

# References

- Computational Quantifier Elimination
  - *Computer Journal*
  - Edited by Hong
- Collins’ 65<sup>th</sup> Birthday Conference
  - *RISC monograph*
  - Edited by Johnson and Caviness
- Application of Quantifier Elimination
  - *Journal of Symbolic Computation*
  - Edited by Hong
- Algorithms in Real Algebraic Geometry
  - *Springer*
  - Authored by Basu, Pollack, Roy

# References

- Computational Quantifier Elimination
  - Computer Journal
  - Edited by Hong
- Collins’ 65<sup>th</sup> Birthday Conference
  - RISC monograph
  - Edited by Johnson and Caviness
- Application of Quantifier Elimination
  - Journal of Symbolic Computation
  - Edited by Hong
- Algorithms in Real Algebraic Geometry
  - Springer
  - Authored by Basu, Pollack, Roy
- Numerous individual articles
  - Journal of Symbolic Computation



# Quantifier Elimination

“Scientific Overview”

# Mathematical Theories

- Algebraic geometry
- Real algebraic geometry
- Commutative algebra
- Complex analysis
- Real analysis
- Topology

# Plan

# Plan

*Tarski's*

# Plan

- *Tarski's*
  - *Original*

# Plan

- *Tarski's*

- *Original*
- *Rereading (Polynomial remaindering - Sign table)*

# Plan

- *Tarski's*

- *Original*
- *Rereading (Polynomial remaindering - Sign table)*

- *Collins*

# Plan

- *Tarski's*

- *Original*
- *Rereading (Polynomial remaindering - Sign table)*

- *Collins*

- *Original*

# Plan

## Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original
- Repackaging (*Critical points - Morse complex*)

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original
- Repackaging (*Critical points - Morse complex*)

## □ New methods

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original
- Repackaging (*Critical points - Morse complex*)

## □ New methods

- *Error function, Numeric projection*

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original
- Repackaging (*Critical points - Morse complex*)

## □ New methods

- Error function, Numeric projection
- Signature, Group analysis

# Plan

## □ Tarski's

- Original
- Repackaging (*Polynomial remaindering - Sign table*)

## □ Collins

- Original
- Repackaging (*Prolongation - Relaxation*)

## □ Canny, Grigor'ev, Basu, Roy, et al

- Original
- Repackaging (*Critical points - Morse complex*)

## □ New methods

- Error function, Numeric projection
- Signature, Group analysis
- .....

Tarski

Key Idea

## Key Idea

1. Reduce to

"Generalized Root Counting"

## Key Idea

1. Reduce to

"Generalized Root Counting"

2. Solve it using

"Sturm - Sylvester Theorem"

# Generalized Root Counting

# Generalized Root Counting

In:  $f(x), g(x)$

# Generalized Root Counting

In:  $f(x), g(x)$

Out:  $N = N_+ - N_-$

where

$$N_+ = \#\{x : f(x)=0 \wedge g(x)>0\}$$

$$N_- = \#\{x : f(x)=0 \wedge g(x)<0\}$$

# Generalized Root Counting

In:  $f(x), g(x)$

Out:  $N = N_+ - N_-$

where

$$N_+ = \#\{x : f(x)=0 \wedge g(x)>0\}$$

$$N_- = \#\{x : f(x)=0 \wedge g(x)<0\}$$

## Example

In:  $f(x) = x^3 - 4x, g(x) = x - 1$

Out:

# Generalized Root Counting

In:  $f(x), g(x)$

Out:  $N = N_+ - N_-$

where

$$N_+ = \#\{x : f(x) = 0 \wedge g(x) > 0\}$$

$$N_- = \#\{x : f(x) = 0 \wedge g(x) < 0\}$$

## Example

In:  $f(x) = x^3 - 4x, g(x) = x - 1$

Out:  $N =$

# Generalized Root Counting

In:  $f(x), g(x)$

Out:  $N = N_+ - N_-$

where

$$N_+ = \#\{x : f(x) = 0 \wedge g(x) > 0\}$$

$$N_- = \#\{x : f(x) = 0 \wedge g(x) < 0\}$$

## Example

In:  $f(x) = x^3 - 4x, g(x) = x - 1$

Out:  $N = -1$

# Reduction

# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) > 0$$

# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) > 0$$



# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) > 0$$



$$\frac{N(f, g) + N(f, g^2)}{2} > 0$$

# Reduction

# Reduction

$$\exists x \ f(x) = 0 \ \wedge \ g(x) < 0$$

# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) < 0$$



# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) < 0$$



$$\frac{-N(f, g) + N(f, g^2)}{2} > 0$$

# Reduction

# Reduction

$$\exists x \ f(x) = 0 \ \wedge \ g(x) = 0$$

# Reduction

$$\exists x \ f(x) = 0 \ \wedge \ g(x) = 0$$



# Reduction

$$\exists x \ f(x) = 0 \wedge g(x) = 0$$



$$N(f, 1) - N(f, g^2) > 0$$

# *Sturm - Sylvester*



# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

⋮

# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

$\vdots$



# Sturm - Sylvester

Let

$$\begin{array}{ll} P_1 = f & | \quad P_1(-\infty) \\ P_2 = f' \cdot g & | \quad P_2(-\infty) \\ P_3 = - \operatorname{rem}(P_1, P_2) & | \quad P_3(-\infty) \\ P_4 = - \operatorname{rem}(P_2, P_3) & | \quad P_4(-\infty) \\ \vdots & | \quad \vdots \end{array}$$

# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

⋮

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
⋮	⋮

# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

⋮

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
⋮	⋮
$\nabla_-$	

# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

⋮

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
⋮	⋮
$V_-$	$V_+$

# Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\text{rem}(P_1, P_2)$$

$$P_4 = -\text{rem}(P_2, P_3)$$

⋮

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
⋮	⋮
$V_-$	$V_+$

Then

$$N = V_- - V_+$$

# Challenges

- Remove redundancy
- Remove inconsistency
- Make incremental

# Rereading

*Polynomial remaindering - Sign table*

# Observe

# Observe

Tarski

# Observe

Tarski

Sturm-Sylvester

# Observe

Tarski

Sturm-Sylvester

Polynomial  
remainder

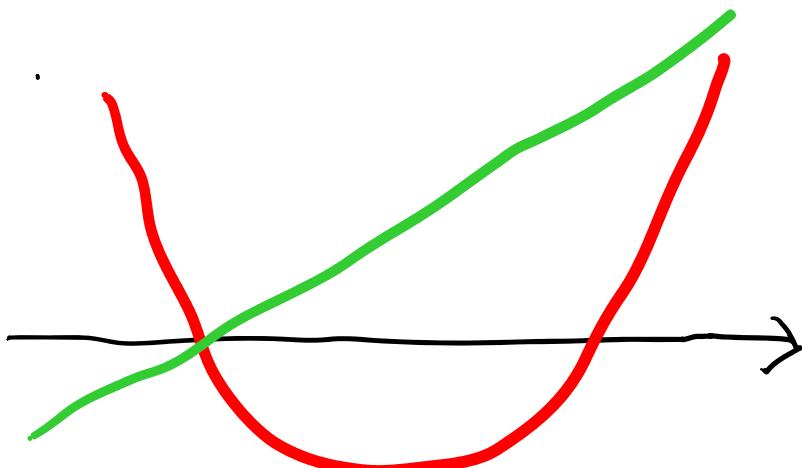
# *Remaindering - Sign table*

## Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

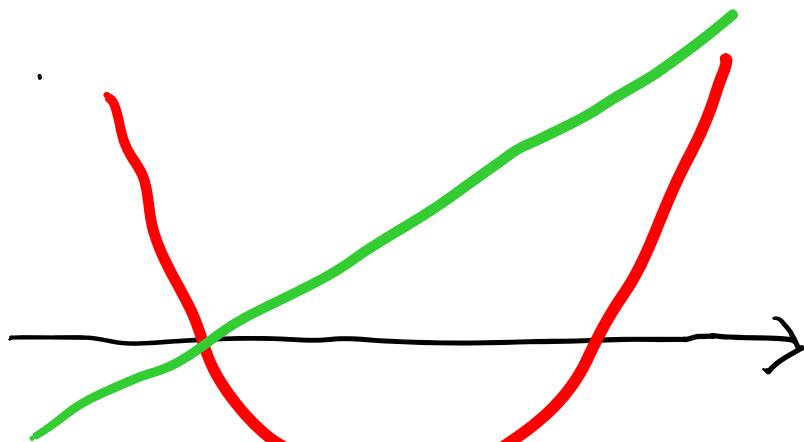
# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



# Remaindering - Sign table

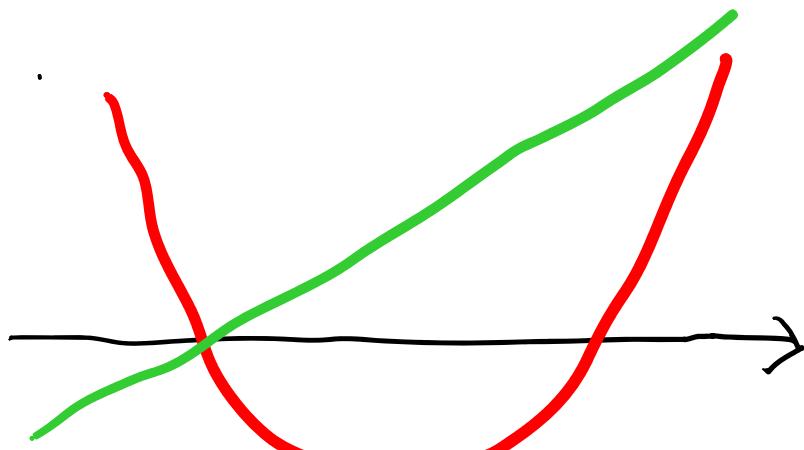
$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



$f$	+
$g$	-

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



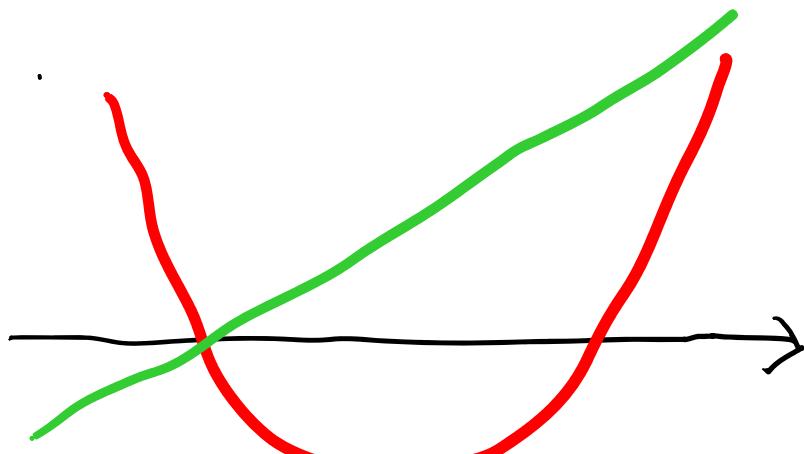
---

$f$	$+ 0$
$g$	$- 0$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



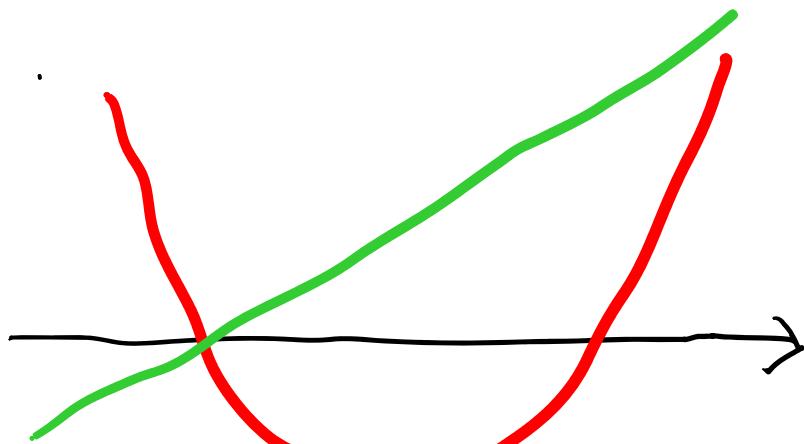
---

$f$	$+$	$0$	$-$
$g$	$-$	$0$	$+$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



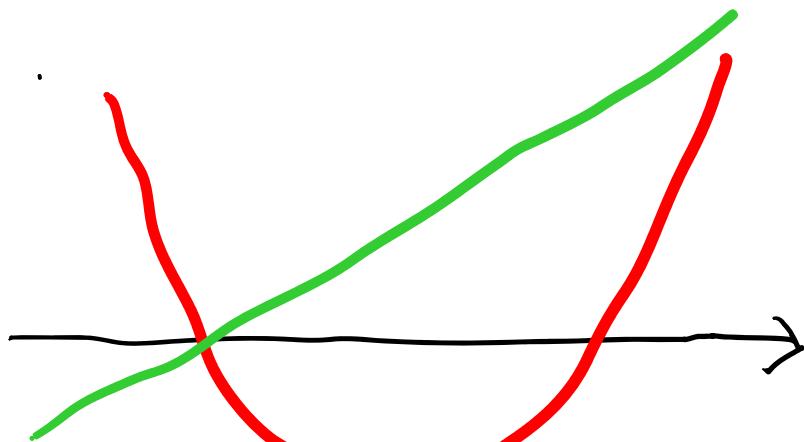
---

$f$	$+$	$0$	$-$	$0$
$g$	$-$	$0$	$+$	$+$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



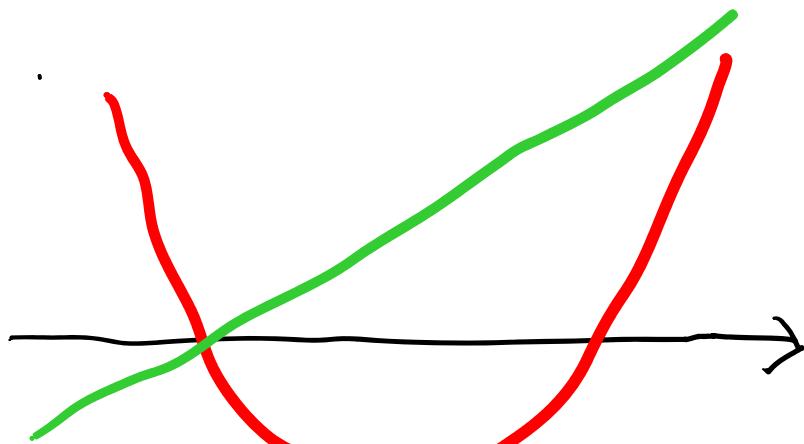
---

f	+ 0	-	0 +
g	- 0	+	++

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

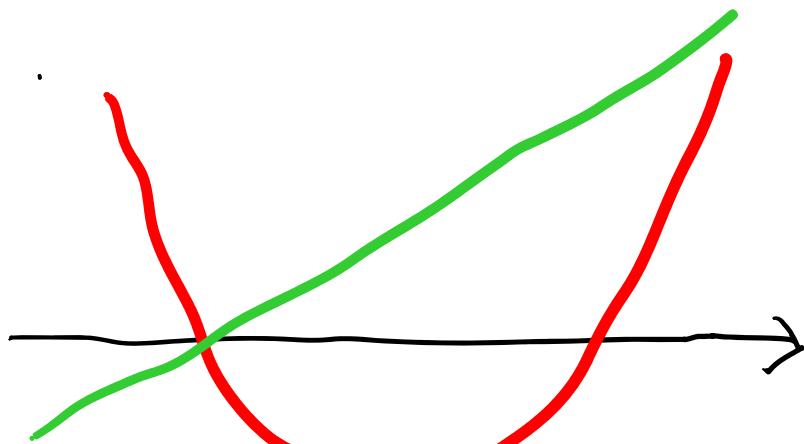


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$				
$r(f, g)$	$0$	$0$	$0$		
$f$					
$g$					

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

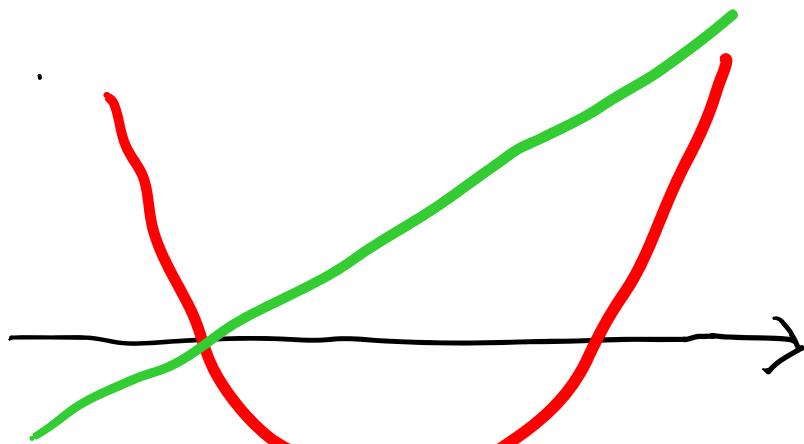


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

		$\circ$			
$f'$	$-$	$-$	$-$	$\circ$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$				
$r(f, g)$	$0$	$0$	$0$		
$f$					
$g$	$-$	$0$	$+$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

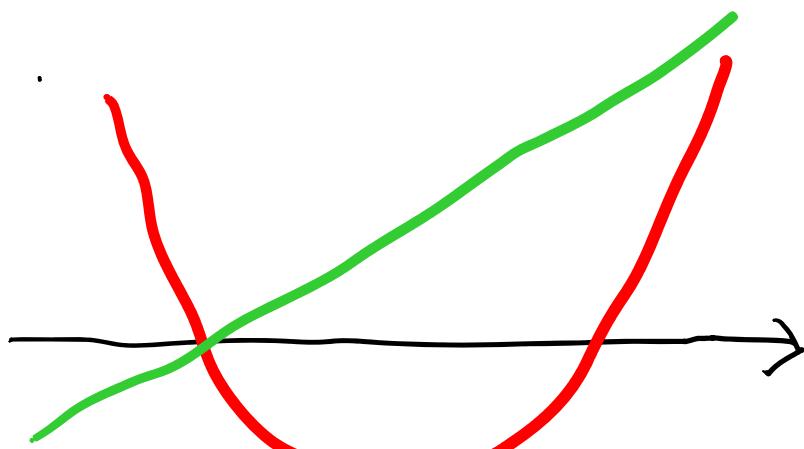


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

		$\circ$			
$f'$	$-$	$-$	$-$	$\circ$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$-$	$-$
$r(f, g)$	$0$	$0$	$0$		
$f$		$+$			$+$
$g$		$-$	$0$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

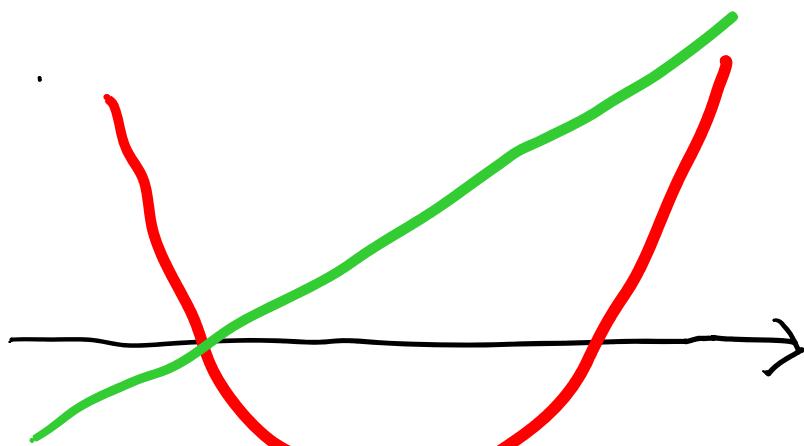


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

	$\circ$					
$f'$	$-$	$-$	$-$	$\circ$	$+$	
$g$	$-$	$0$	$+$	$+$	$+$	
$r(f, f')$	$-$					
$r(f, g)$	$0$	$0$	$0$			
$f$		$+$				$+$
$g$		$-$	$0$	$+$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

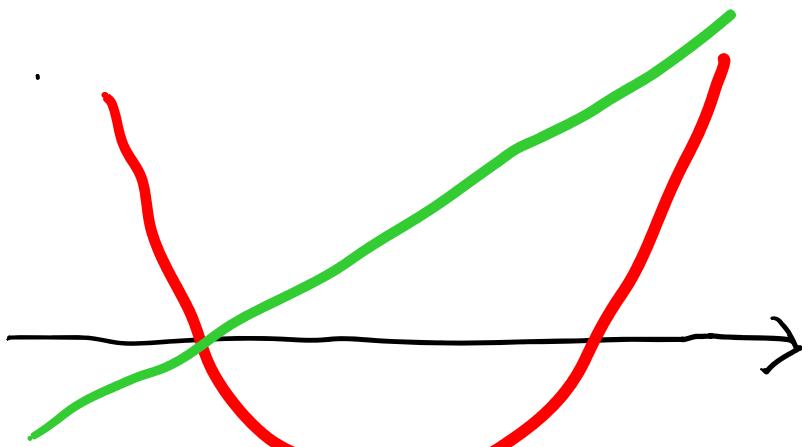


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

	$\circ$					
$f'$	$-$	$-$	$-$	$\circ$	$+$	
$g$	$-$	$0$	$+$	$+$	$+$	
$r(f, f')$	$-$					
$r(f, g)$	$0$	$0$	$0$			
$f$	$+$	$0$			$+$	
$g$	$-$	$0$	$+$	$+$	$+$	

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

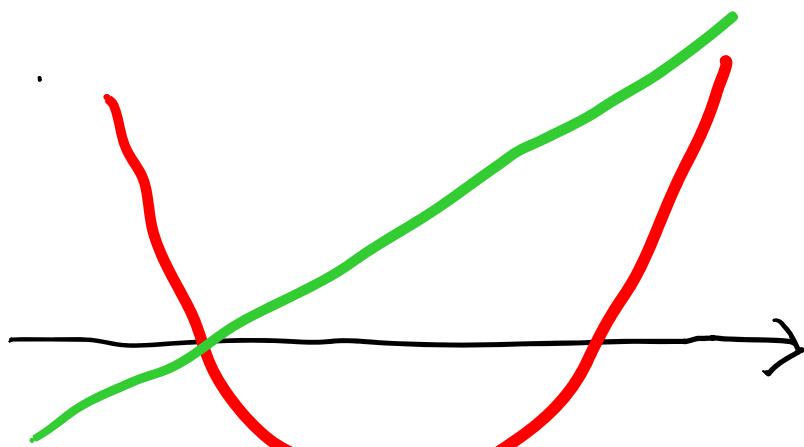


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$0$	$-$	$-$	$-$	$0$	$+$
$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$-$	$-$
$r(f, g)$	$0$	$0$	$0$		
$f$	$+$	$0$			$+$
$g$	$-$	$0$	$+$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

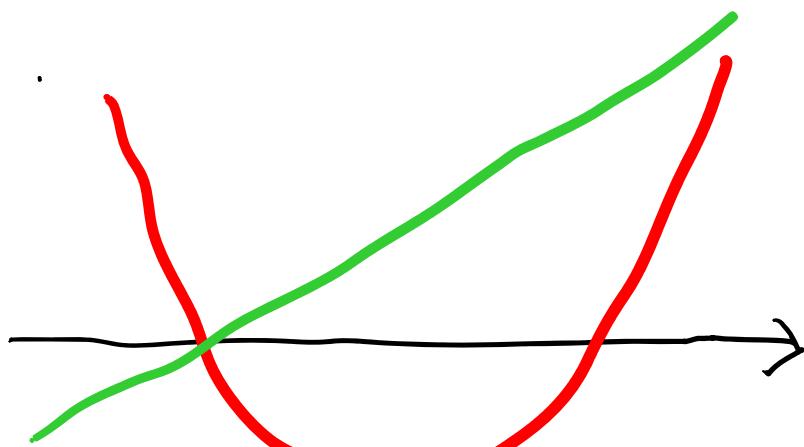


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$0$					
$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$				
$r(f, g)$	$0$	$0$	$0$		
$f$	$+$	$0$			$+$
$g$	$-$	$0$	$+$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

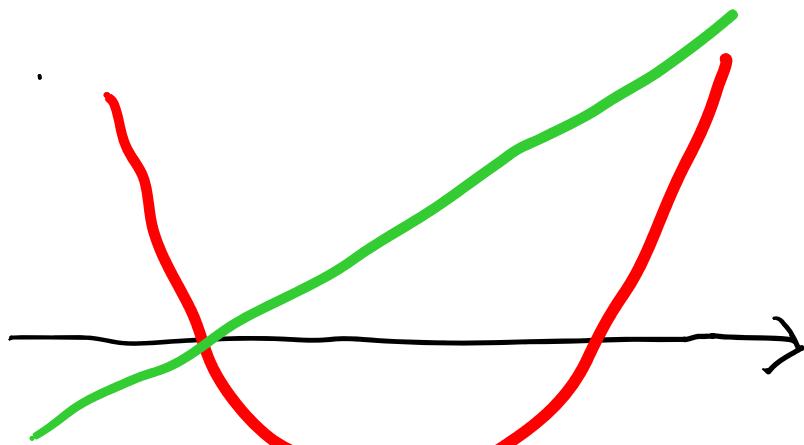


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$0$					
$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$				
$r(f, g)$	$0$	$0$	$0$		
$f$	$+$	$0$	$-$		$+$
$g$	$-$	$0$	$+$	$+$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$0$					
$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$				
$r(f, g)$	$0$	$0$	$0$		
$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

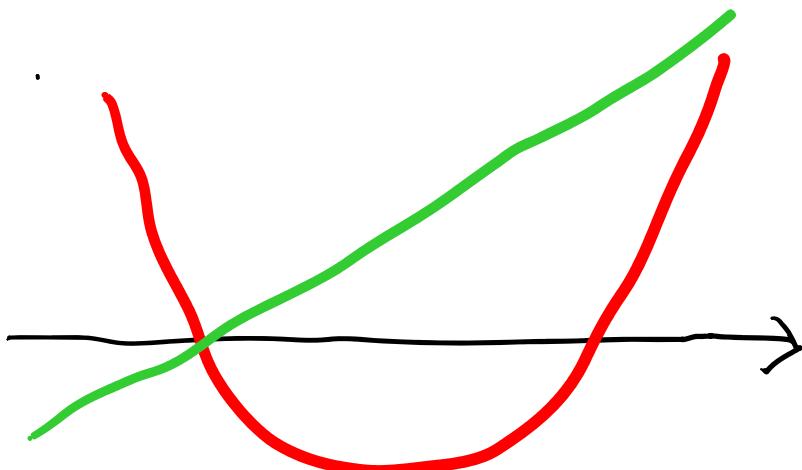
# *Remaindering - Sign table*

## Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

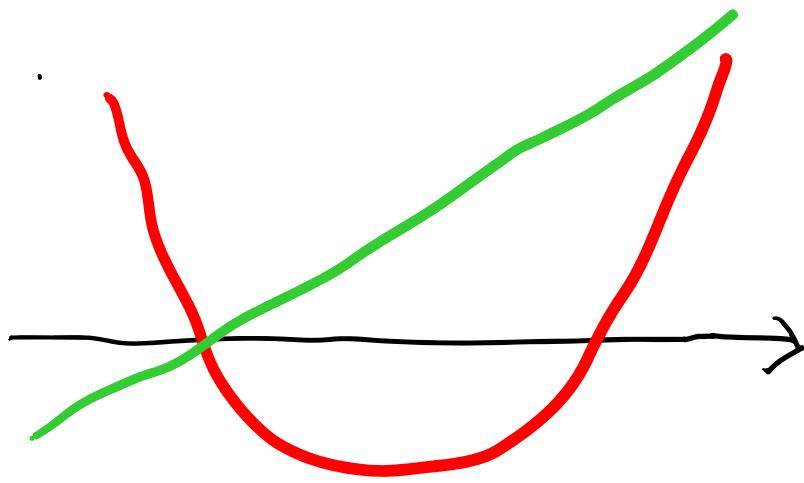
# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



# Remaindering - Sign table

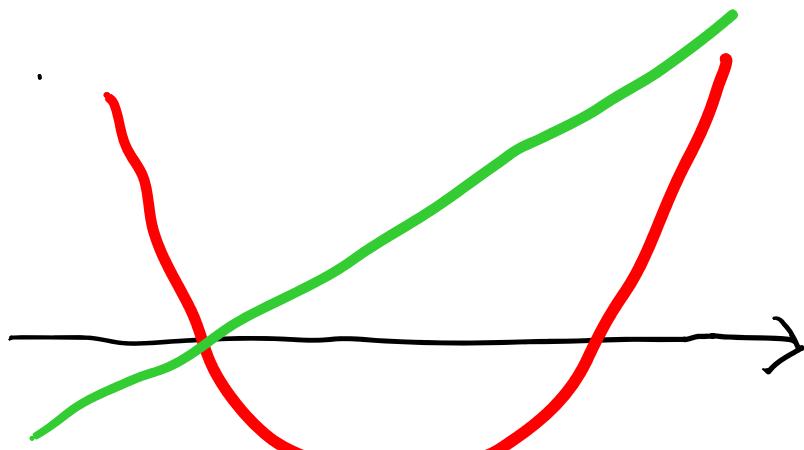
$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \quad \wedge \quad \underbrace{x}_{g} > 0$$



$f$	+
$g$	-

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



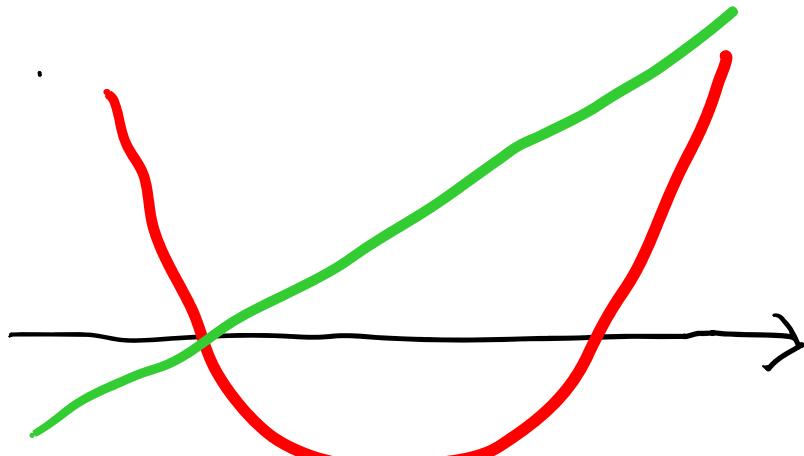
---

$f$	$+ 0$
$g$	$- 0$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



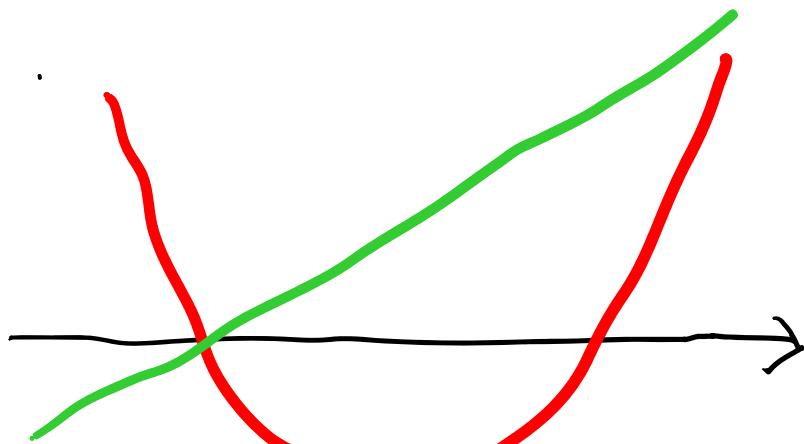
---

$f$	$+$	$0$	$-$
$g$	$-$	$0$	$+$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



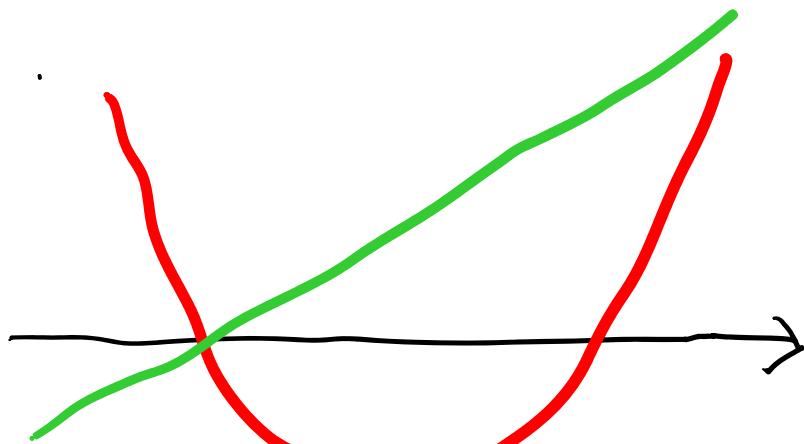
---

$f$	$+$	$0$	$-$	$0$
$g$	$-$	$0$	$+$	$+$

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



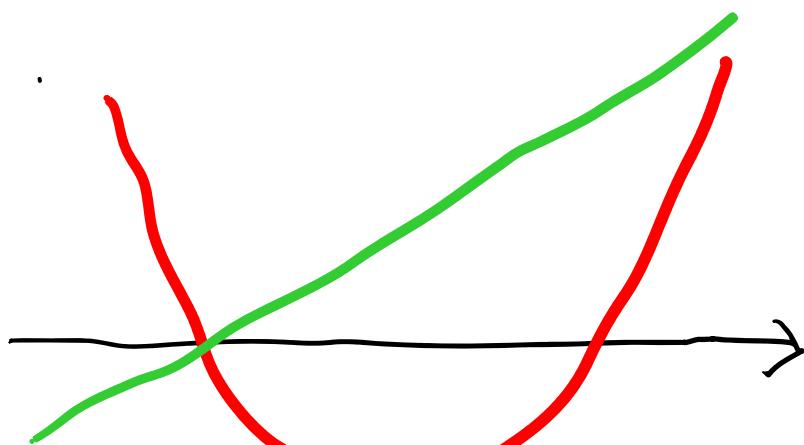
---

f	+ 0	-	0 +
g	- 0	+	++

---

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

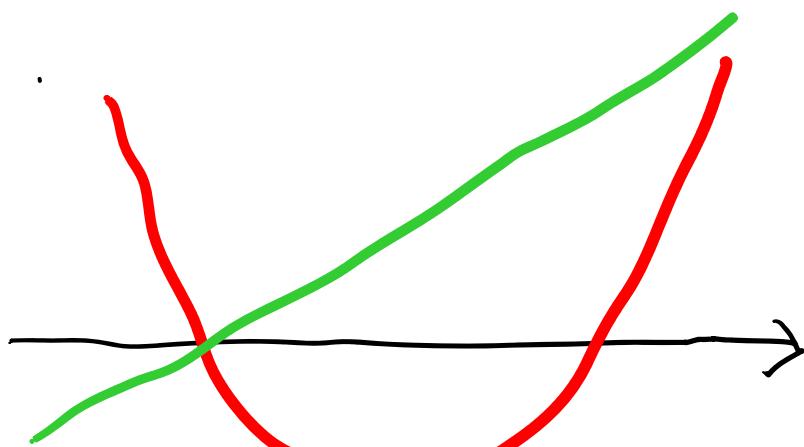


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$0$	$+$
$r(f, g)$	$-$	$0$	$+$		
$f$					
$g$					

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

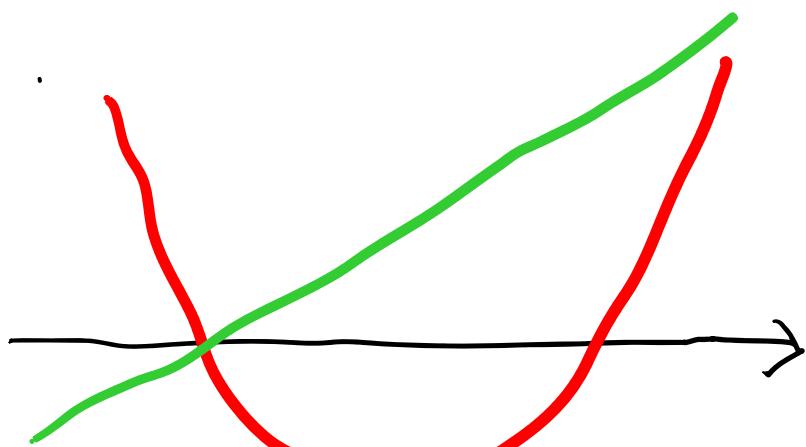


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$0$	$+$
$r(f, g)$	$-$	$0$	$0$	$0$	
$f$	$+$				$+$
$g$	$-$				$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

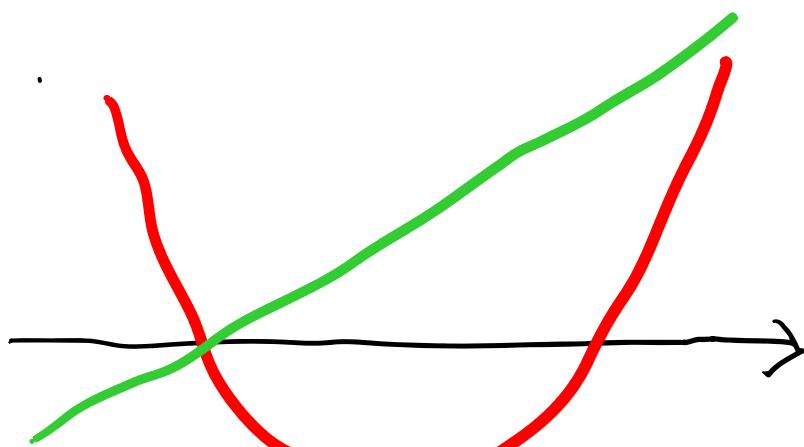


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$f'$	$-$			$0$	$+$
$r(f, f')$	$-$			$-$	$-$
$g$	$-$	$0$	$+$		
$r(f, g)$	$0$	$0$	$0$		
$f$	$+$				$+$
$g$	$-$				$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

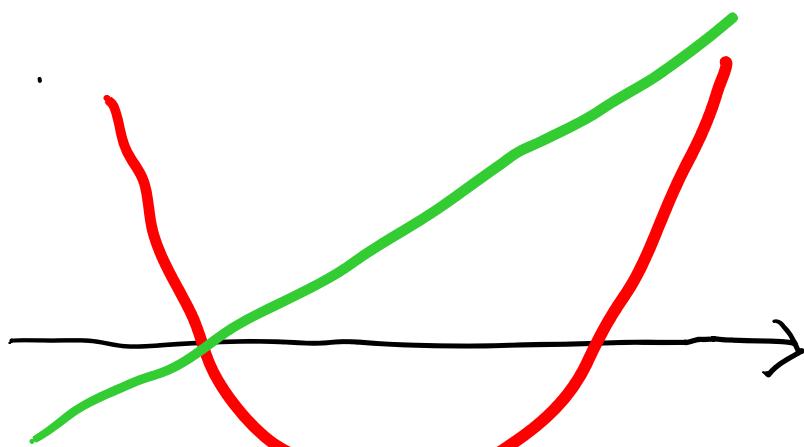


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$		$0$	$+$	
$r(f, g)$	$-$	$0$	$0$	$0$	
$f$	$+$				$+$
$g$	$-$				$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

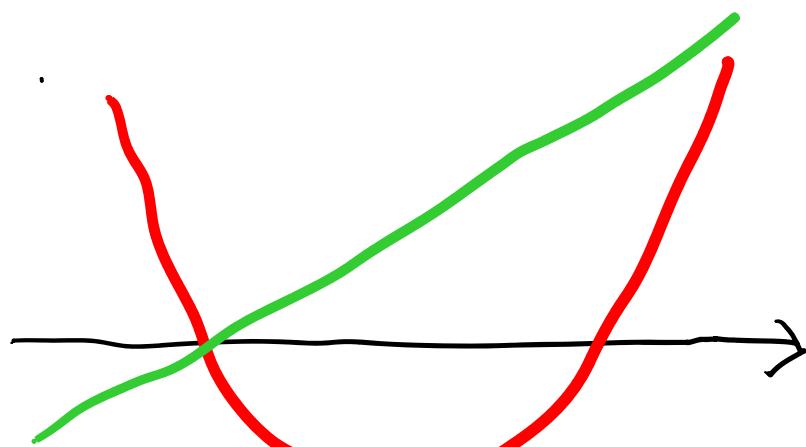


$f$	$+ 0$	$-$	$0 +$
$g$	$- 0$	$+$	$++$

$f'$	$- - -$	$0$	$+$
$g$	$- 0$	$+$	$++$
$f'$	$-$	$0 +$	
$r(f, f')$	$-$	$- -$	
$g$	$-$	$0 +$	
$r(f, g)$	$0$	$0 0$	$0$
$f$	$+$	$0$	$+$
$g$	$-$	$0$	$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

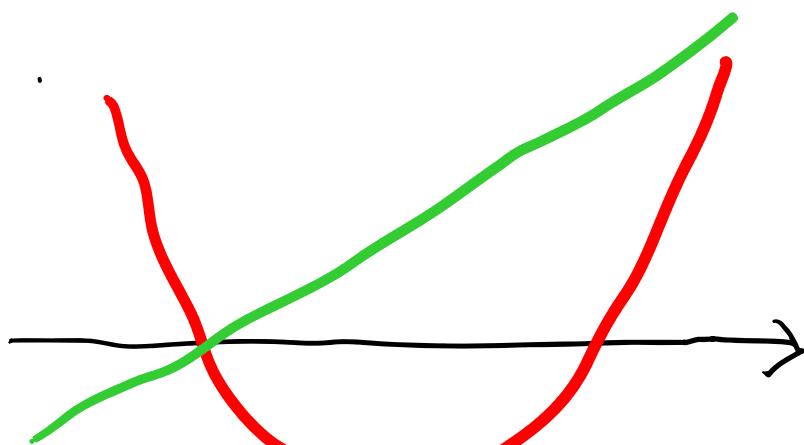


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$		$0$	$+$	
$r(f, g)$	$-$	$0$	$0$	$0$	
$f$	$+$	$0$			$+$
$g$	$-$	$0$			$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

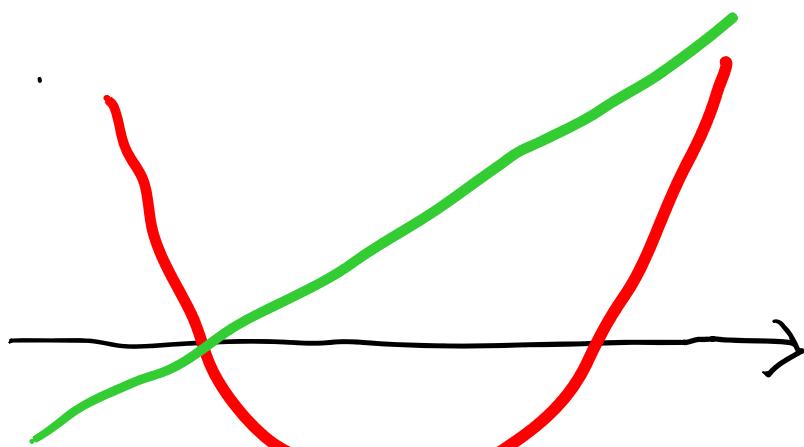


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$0$	$+$
$r(f, g)$	$-$	$0$	$0$	$0$	$-$
$f$	$+$	$0$			$+$
$g$	$-$	$0$			$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$

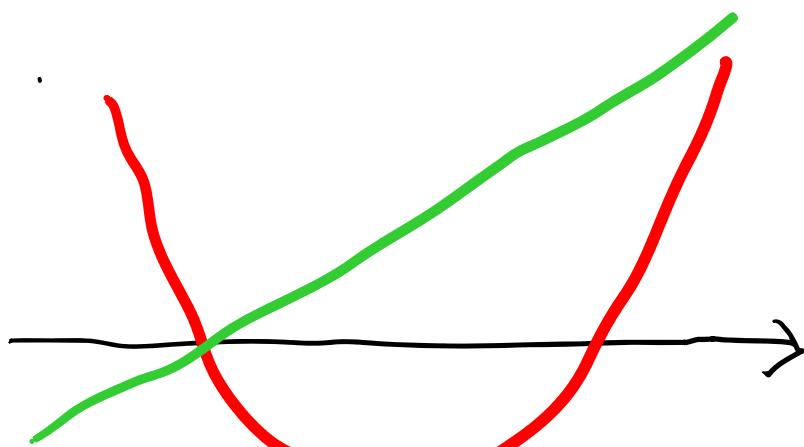


$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$			$0$	$+$
$r(f, g)$	$0$	$0$	$0$	$0$	$-$
$f$	$+$	$0$	$-$		$+$
$g$	$-$	$0$	$+$		$+$

# Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_f > 0 \quad \wedge \quad \underbrace{x}_g > 0$$



$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

$f'$	$-$	$-$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$
$r(f, f')$	$-$		$0$	$+$	
$r(f, g)$	$0$	$0$	$0$	$0$	
$f$	$+$	$0$	$-$	$0$	$+$
$g$	$-$	$0$	$+$	$+$	$+$

# Challenges

- Remove redundancy
- Remove inconsistency
- Incremental

Collins

Key Idea

Key Idea

$$\exists x \in R \quad F(x)$$

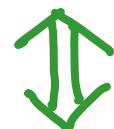
Key Idea

infinite set

$$\exists x \in \mathbb{R} \quad f(x)$$

## Key Idea

$$\exists x \in \mathbb{R} \quad F(x)$$



$$\exists x \in S \quad F(x)$$

infinite set

finite set, "samples"

## Key Idea

$$\exists x \in \mathbb{R} \quad F(x)$$

$\Updownarrow$

$$\exists x \in S \quad F(x)$$

$\Updownarrow$

$$\bigvee_{x \in S} F(x)$$

infinite set



finite set, "samples"

Key Idea

$$\exists x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad F(x, y)$$

Key Idea

$$\exists x \in \mathbb{R} \quad \forall y \in \mathbb{R} \quad F(x, y)$$

Infinite sets

Key Idea

$$\exists x \in \mathbb{R} \forall y \in \mathbb{R} F(x, y)$$

Infinite sets

$$\Leftrightarrow \exists x \in S \forall y \in T_x F(x, y)$$

finite sets

## Key Idea

$$\exists x \in \mathbb{R} \forall y \in \mathbb{R} F(x, y)$$

Infinite sets

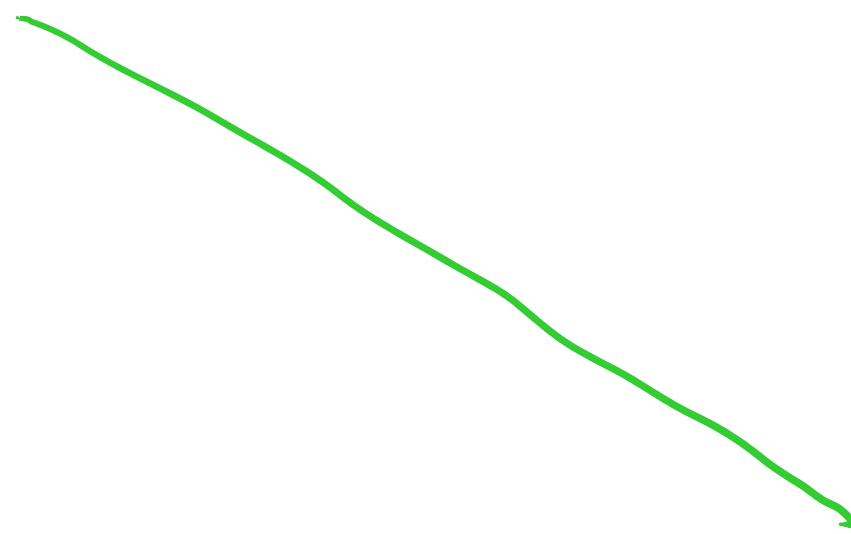
$$\Leftrightarrow \exists x \in S \forall y \in T_x F(x, y)$$

finite sets

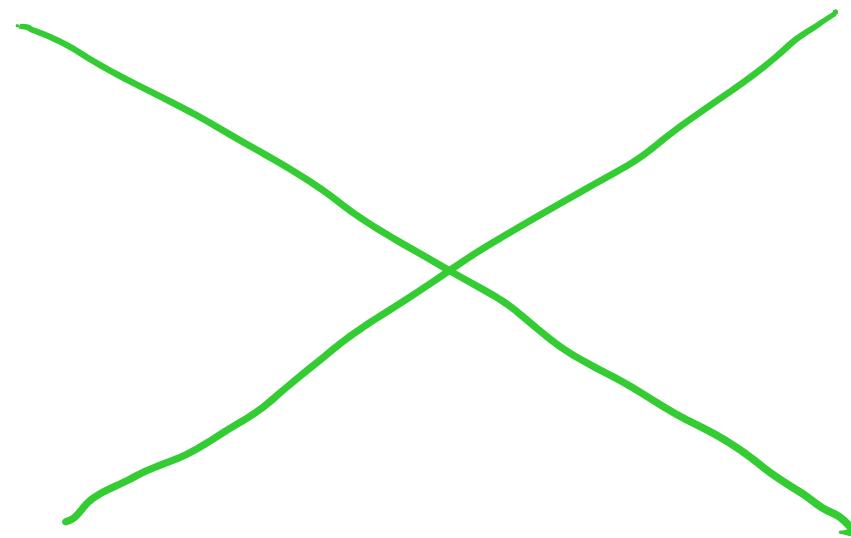
$$\bigvee_{x \in S} \bigwedge_{y \in T_x} F(x, y)$$

$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$

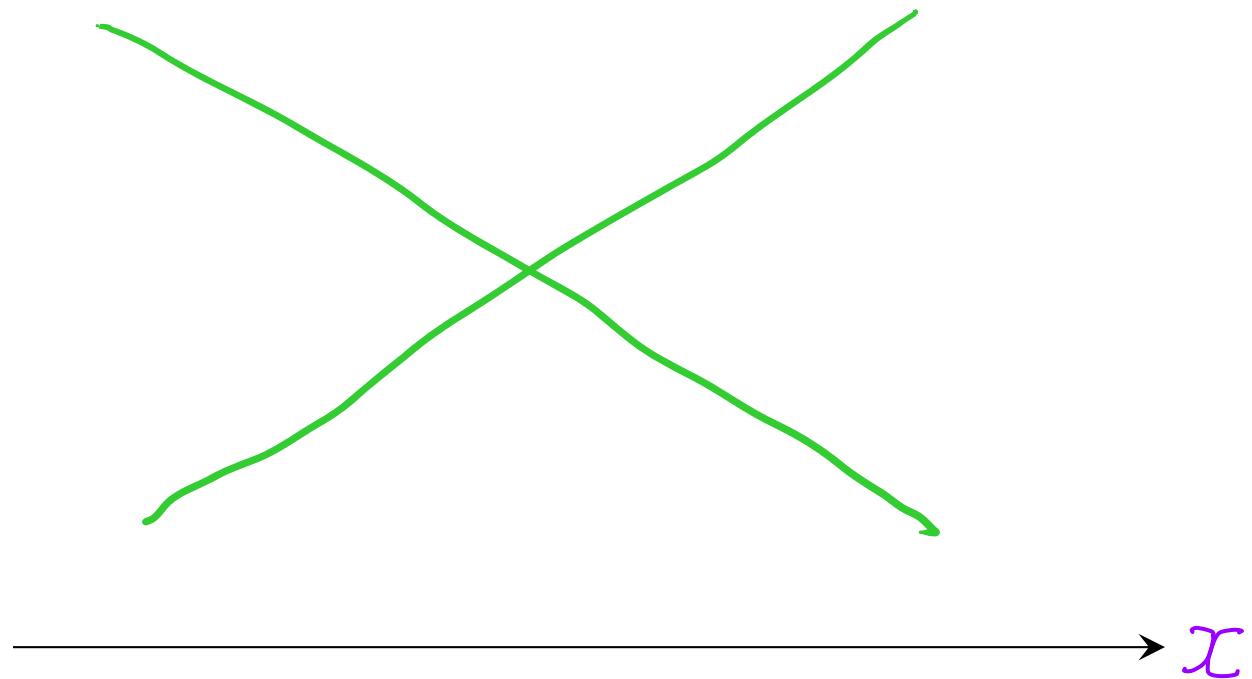
$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$



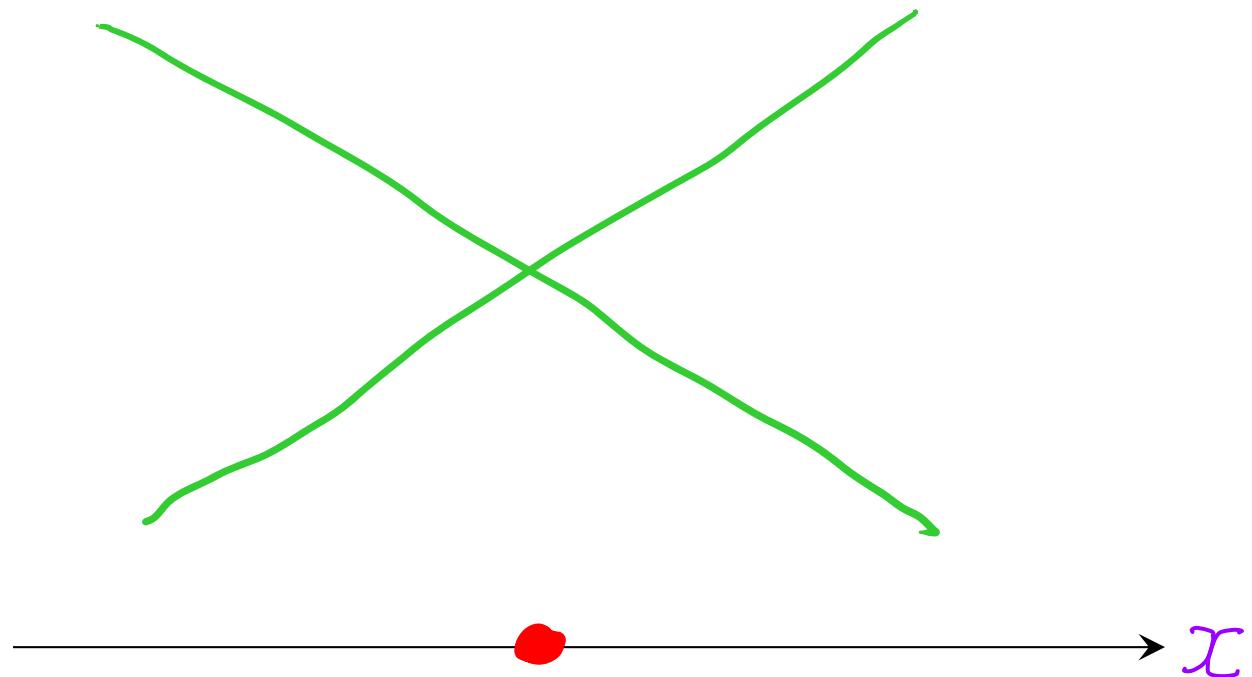
$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$



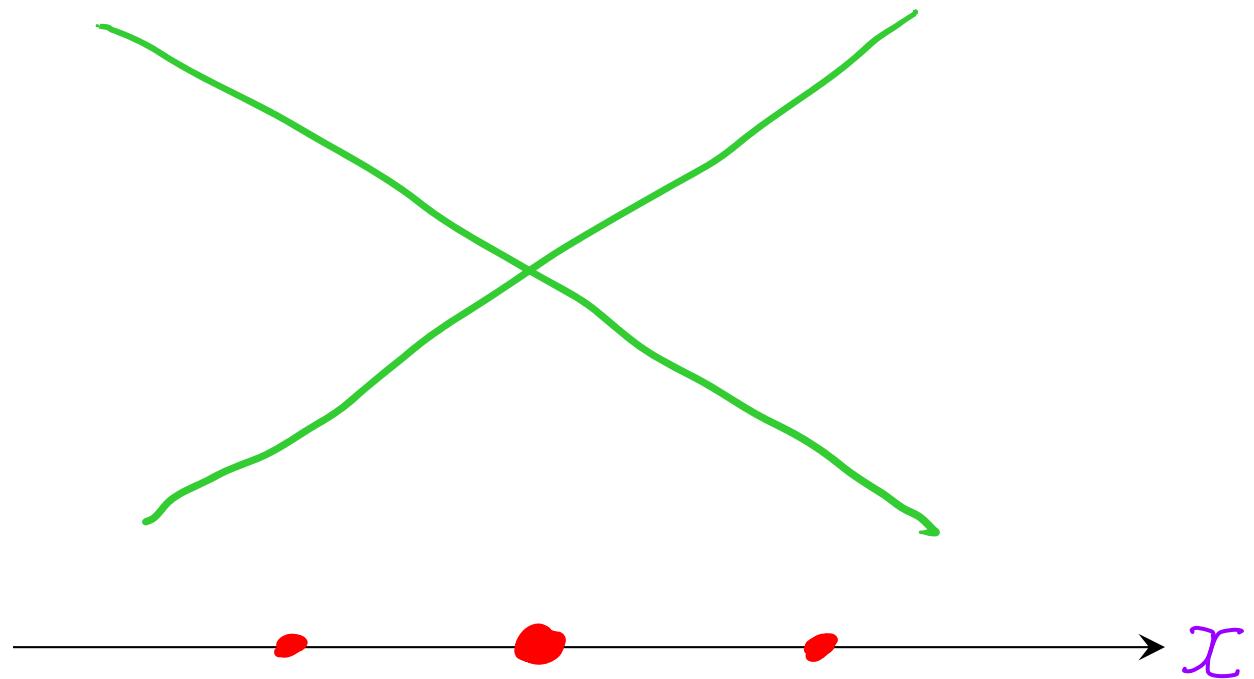
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



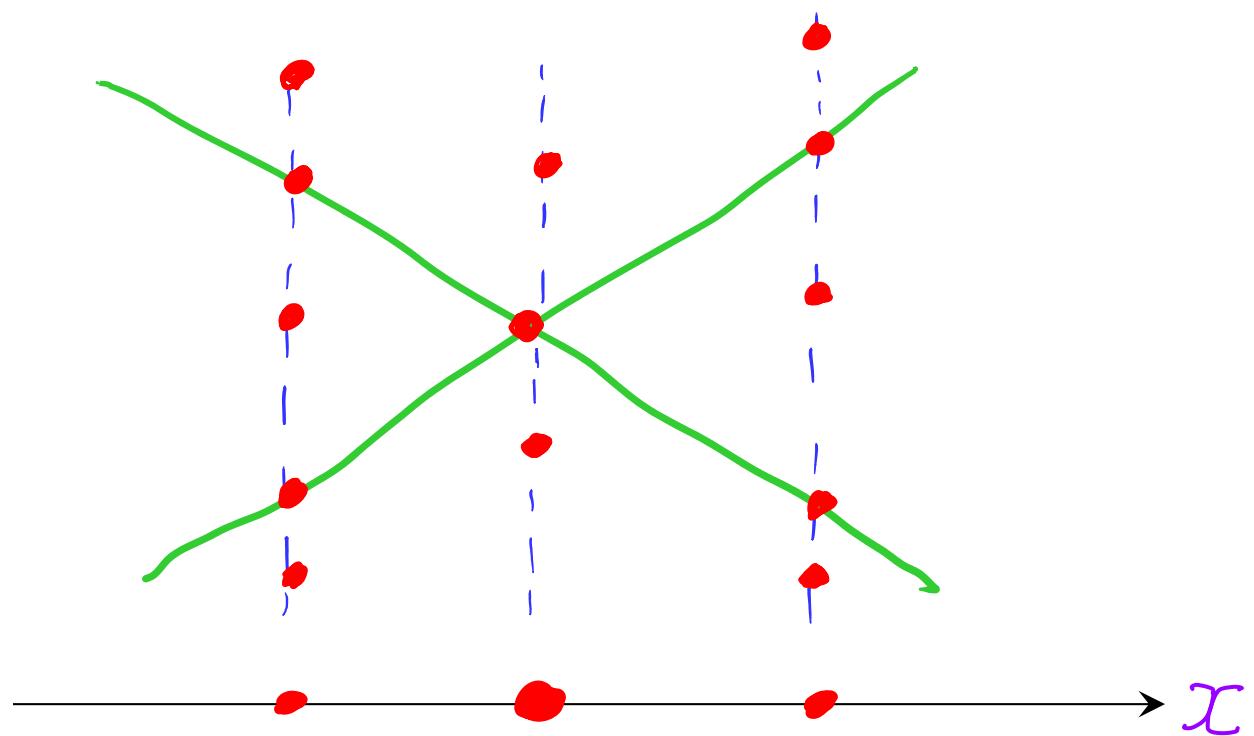
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



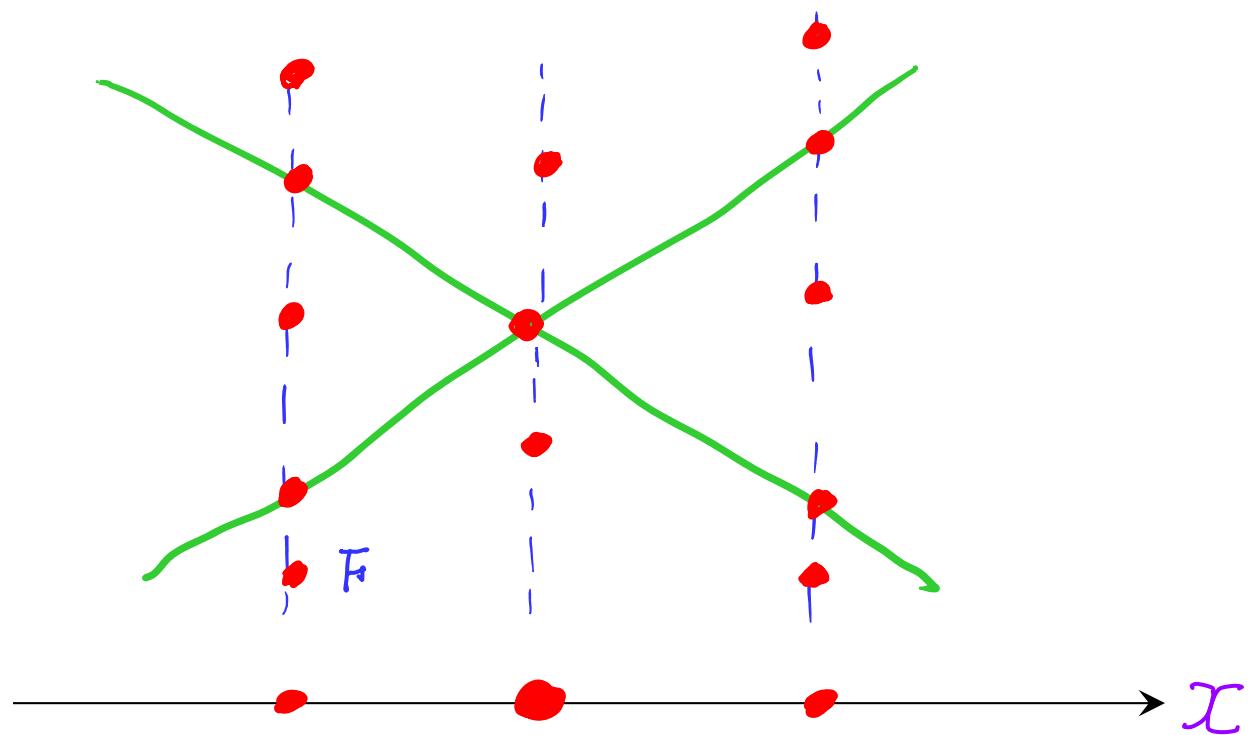
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



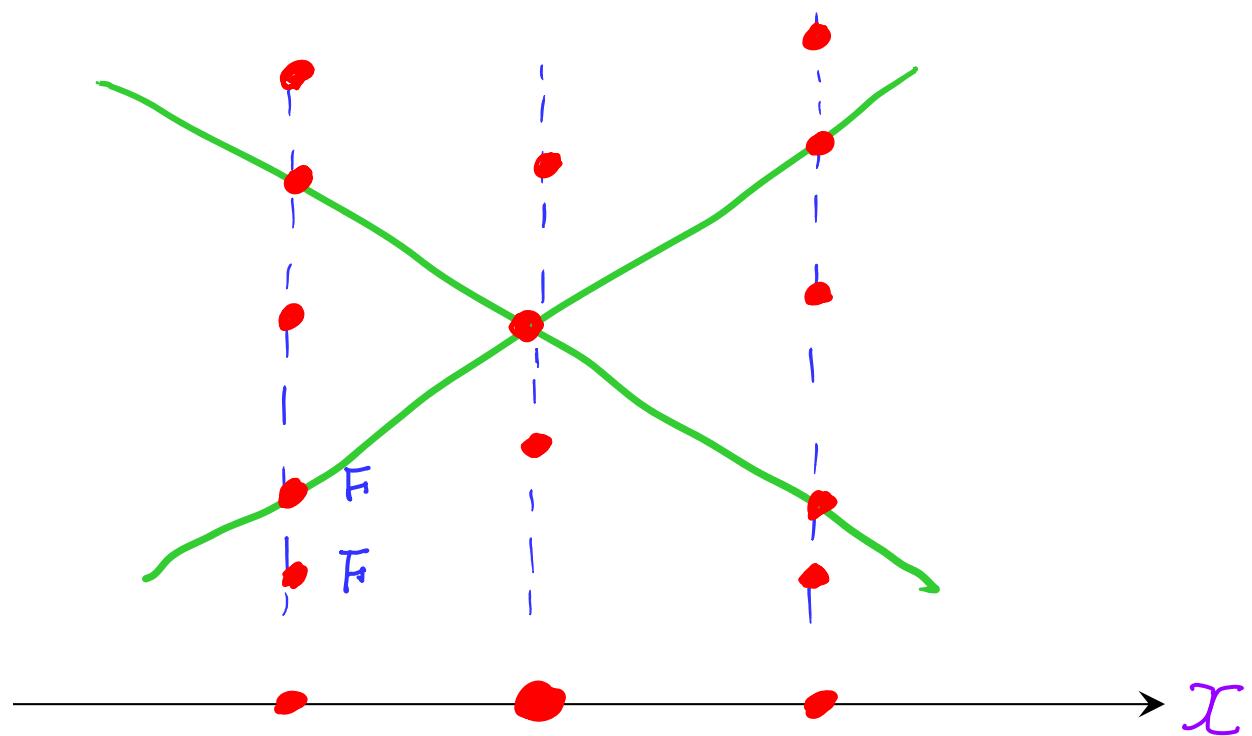
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



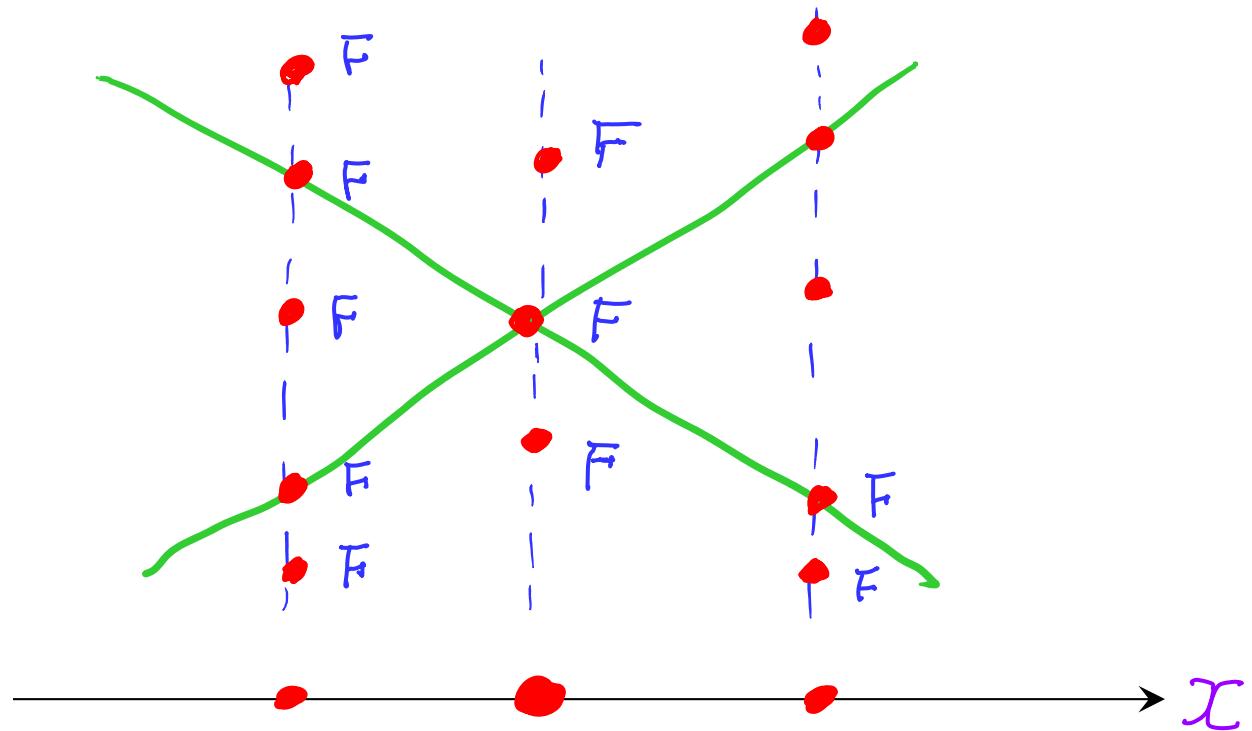
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



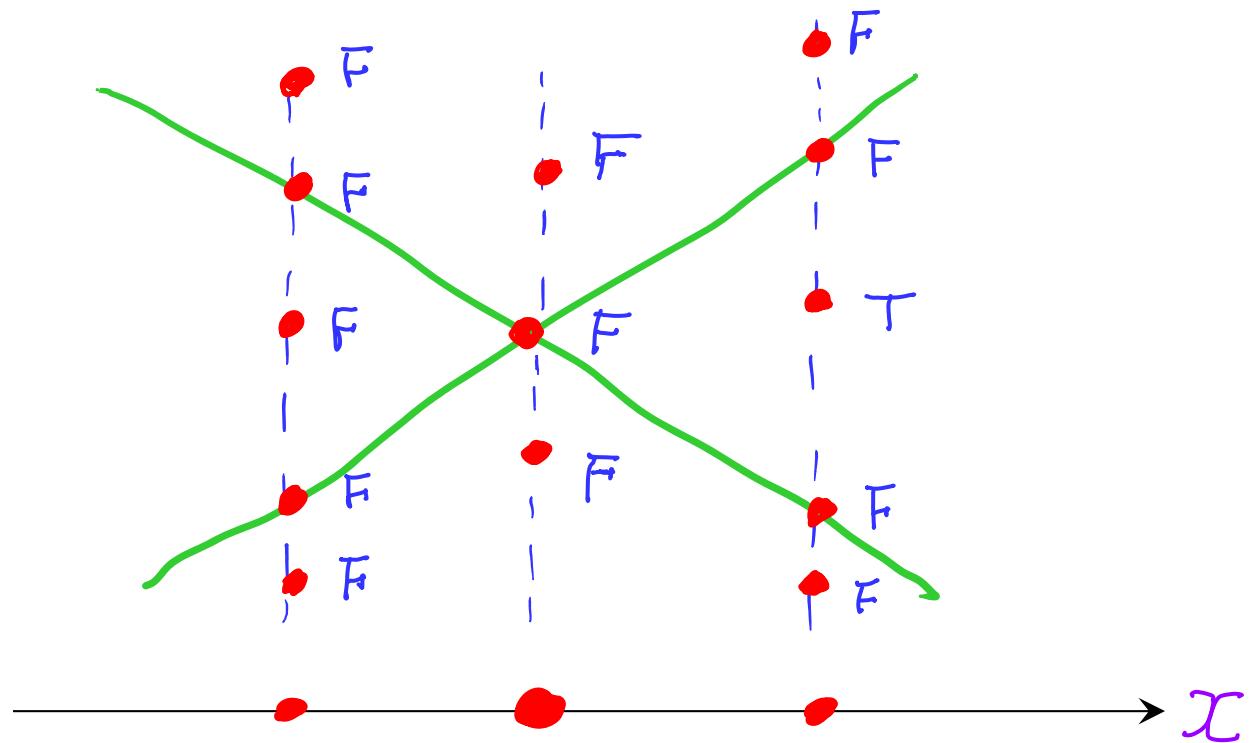
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



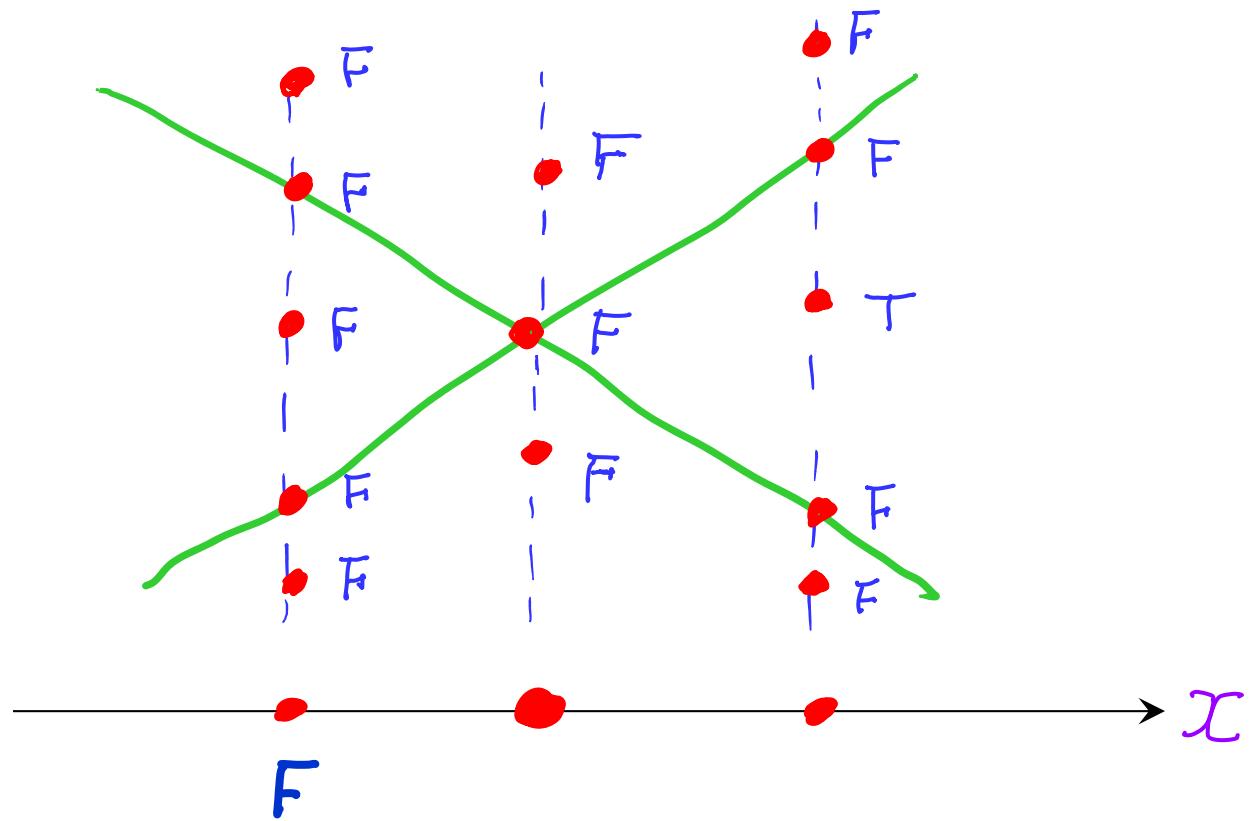
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



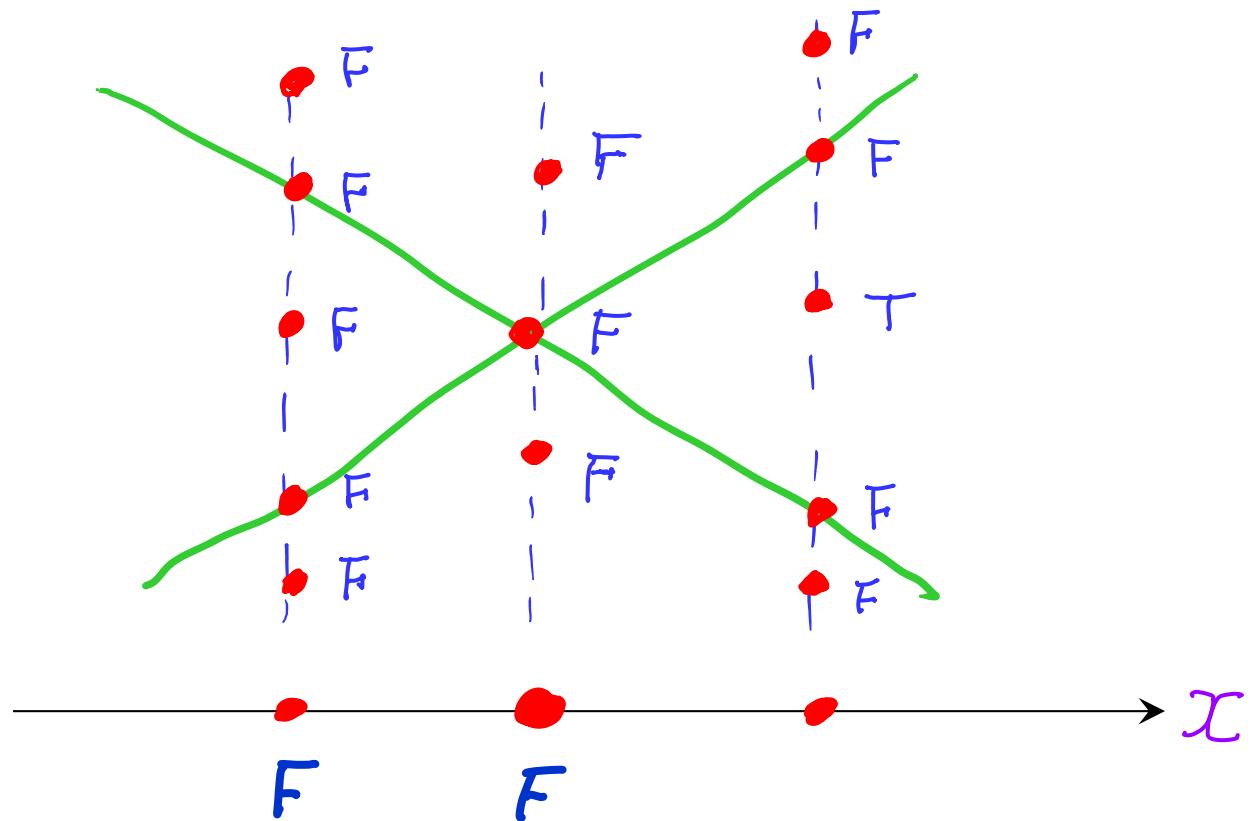
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



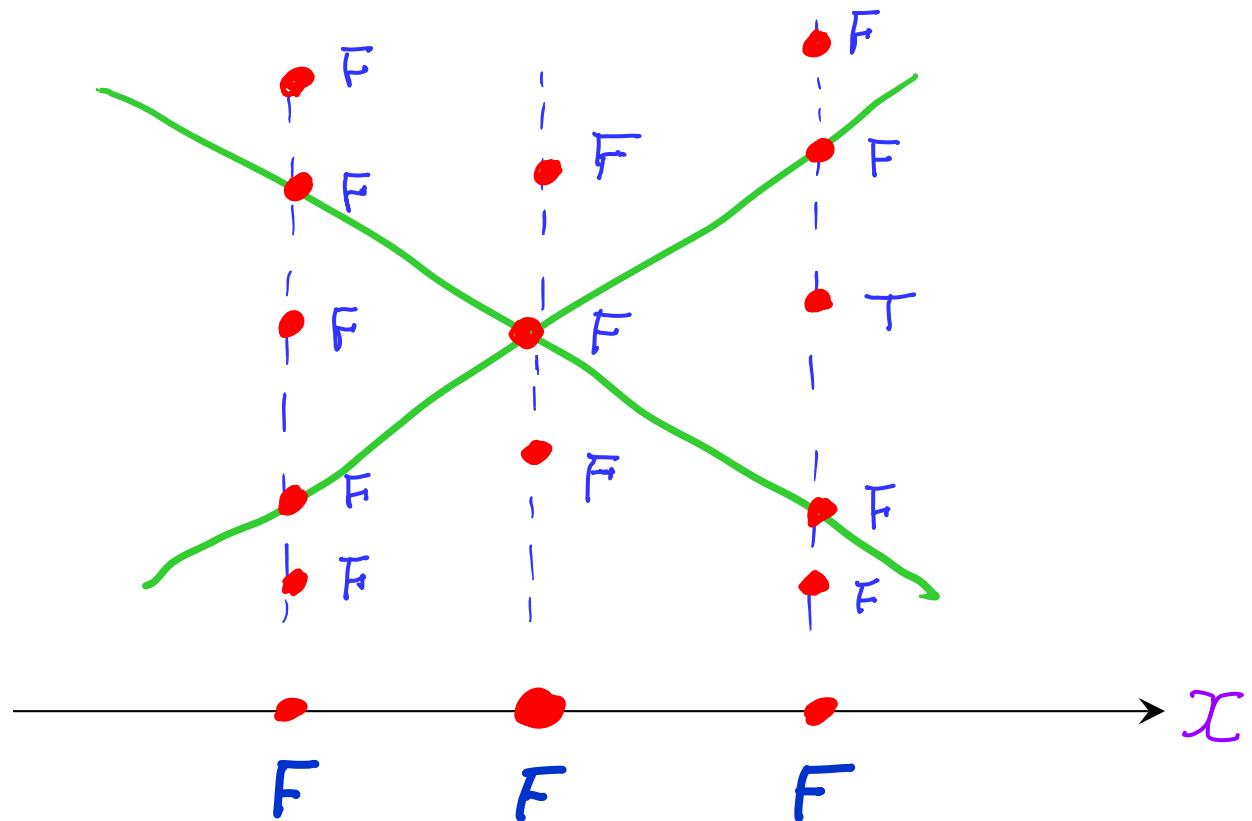
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



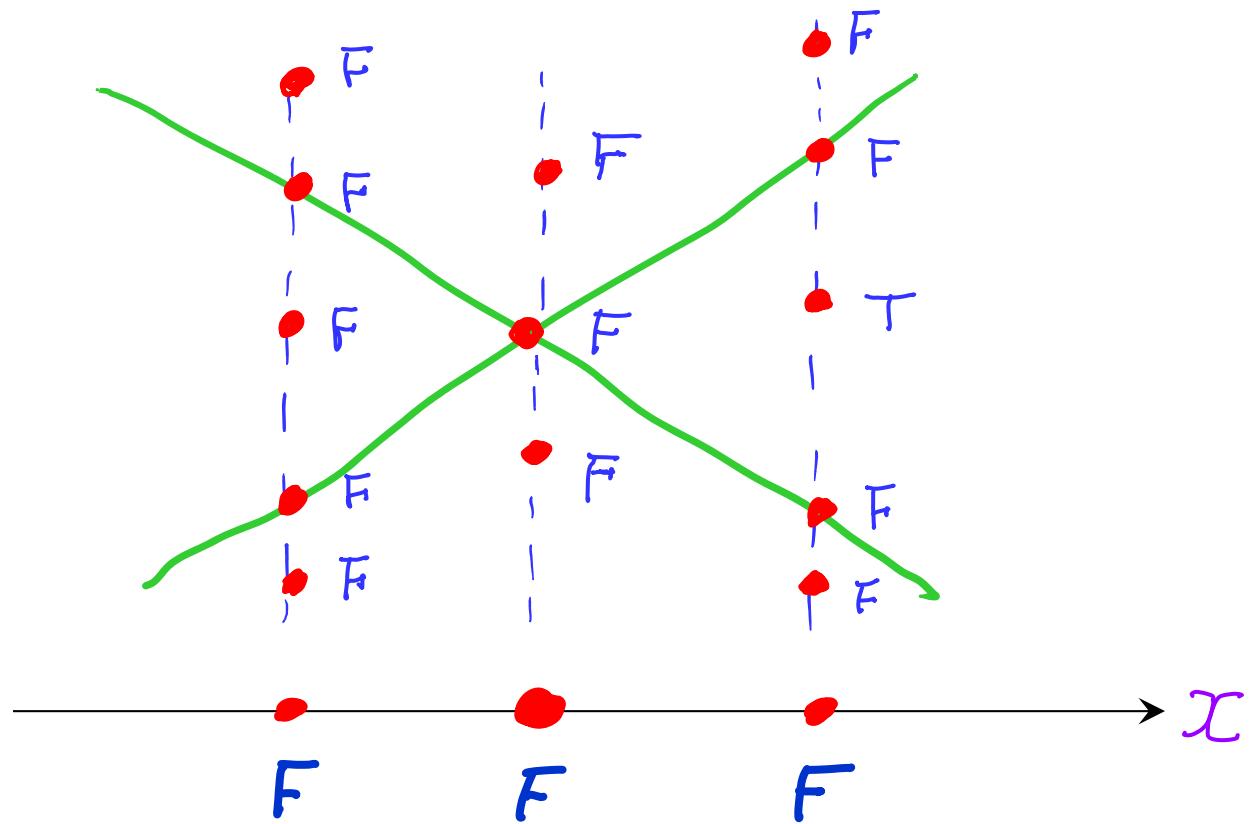
$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$

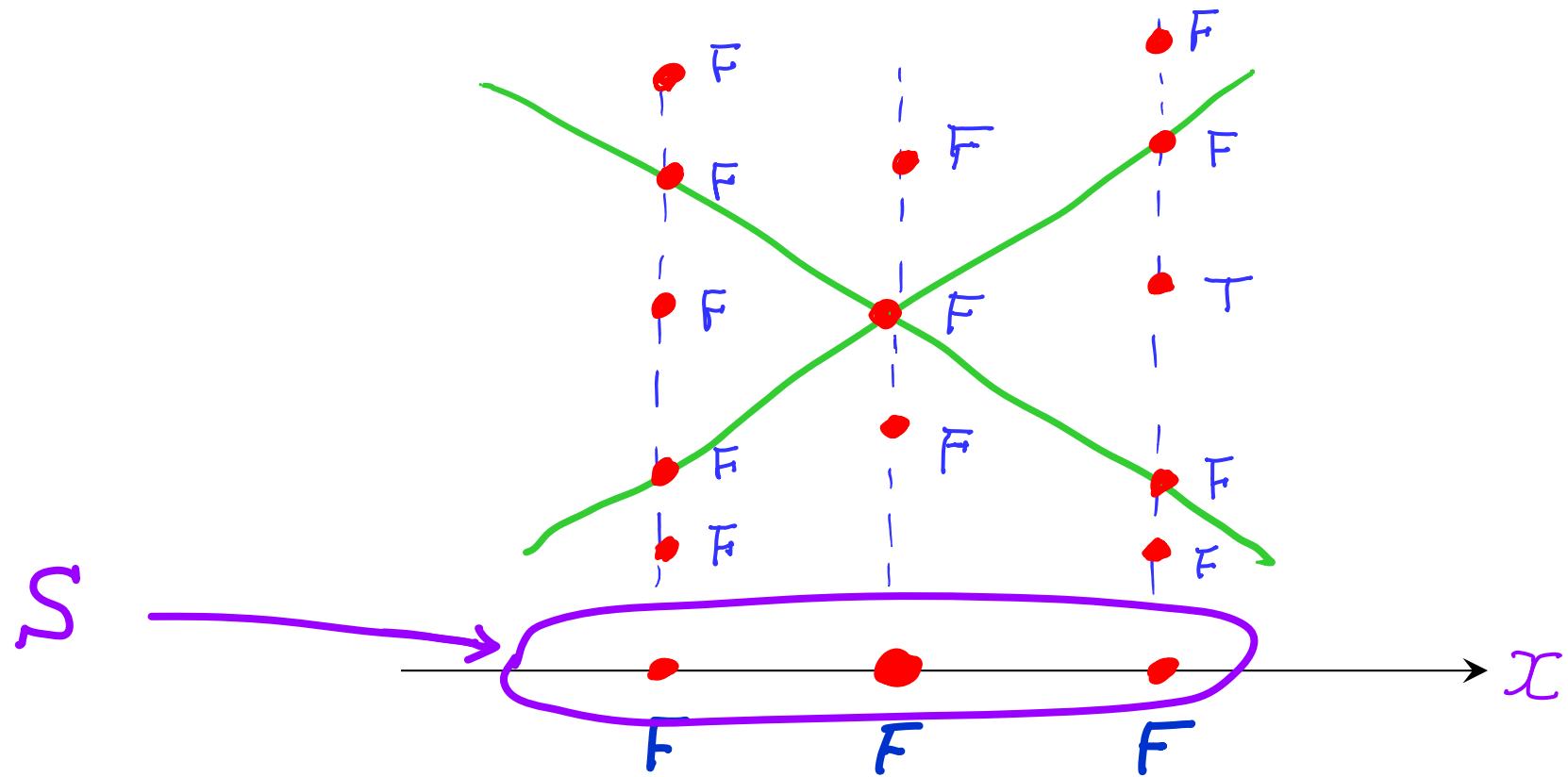


$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



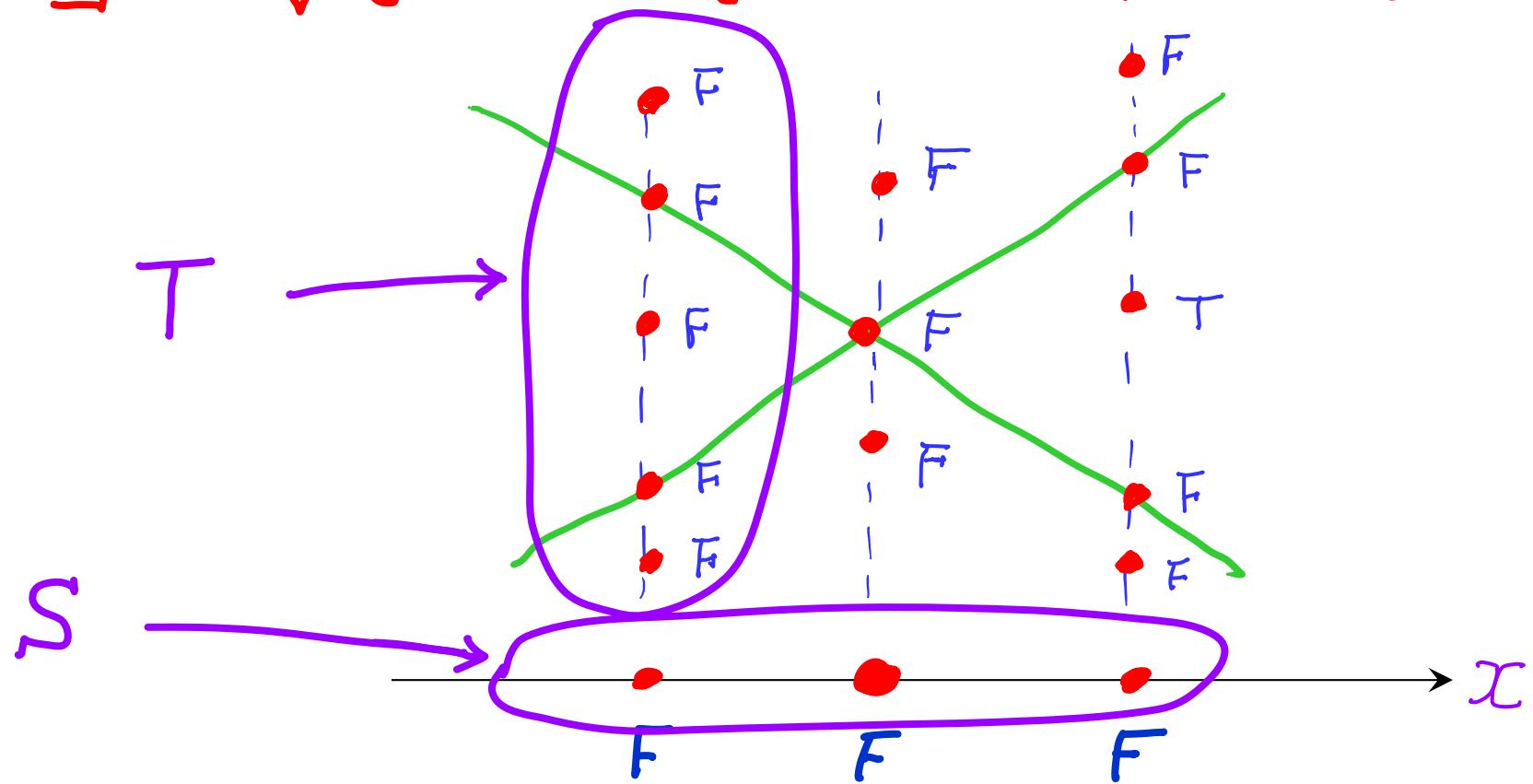
Out: False

$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



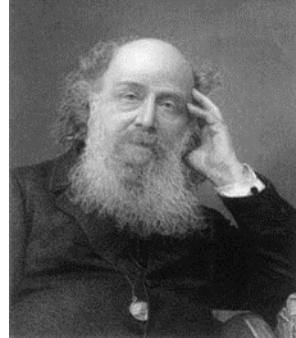
Out: False

$$I_n: \exists x \forall y \quad x+y > 0 \quad \wedge \quad x-y > 0$$



Out: False

# Resultant



# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_{\cancel{x}}(f, g) =$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) =$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) = \left| \begin{matrix} a_3 & a_2 & a_1 & a_0 \\ \end{matrix} \right|$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \end{vmatrix}$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & \end{vmatrix}$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$\text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$h = \text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$h = \text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

Theorem

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$h = \text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & \\ b_2 & b_1 & b_0 & \\ b_2 & b_1 & b_0 & \end{vmatrix}$$

Theorem

$$\exists x \in \mathbb{C} \ f(x) = 0 \wedge g(x) = 0$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$h = \text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

Theorem

$$\exists x \in \mathbb{C} \ f(x) = 0 \wedge g(x) = 0 \iff$$

# Resultant

$$f = a_3 \cancel{x^0} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = b_2 \cancel{x^0} x^2 + b_1 x + b_0$$

$$h = \text{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \\ b_2 & b_1 & b_0 \end{vmatrix}$$

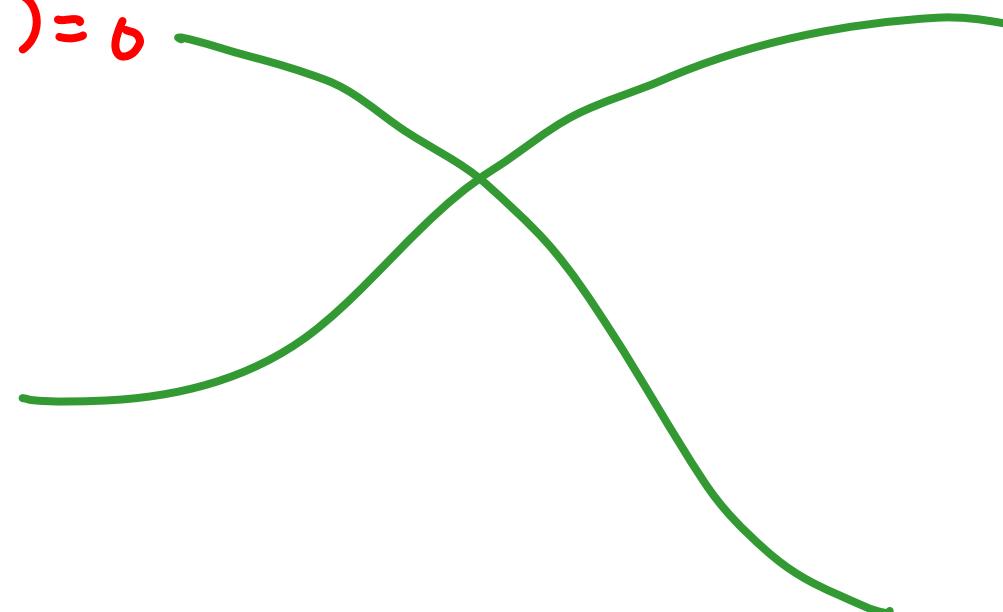
Theorem

$$\exists x \in \mathbb{C} \ f(x) = 0 \wedge g(x) = 0 \iff h = 0$$

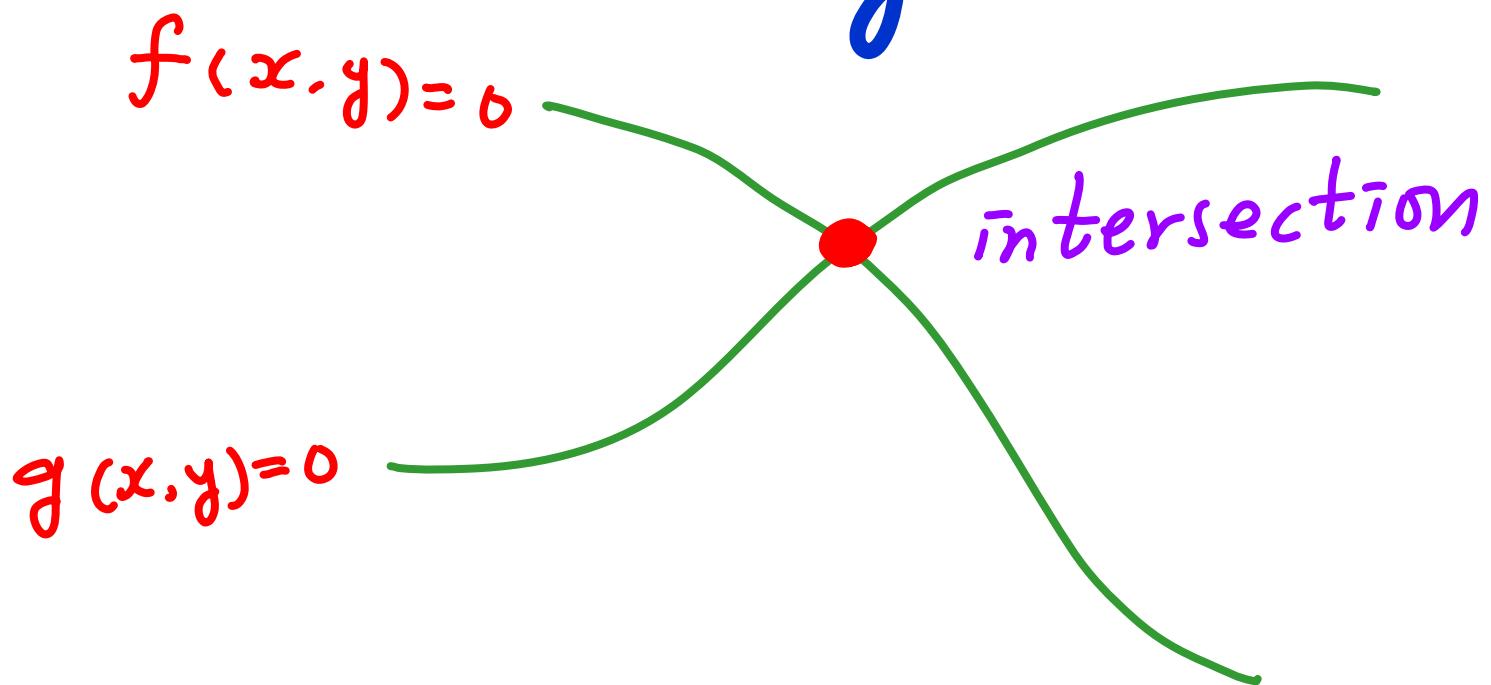
# Interesting Points

$$f(x,y) = 0$$

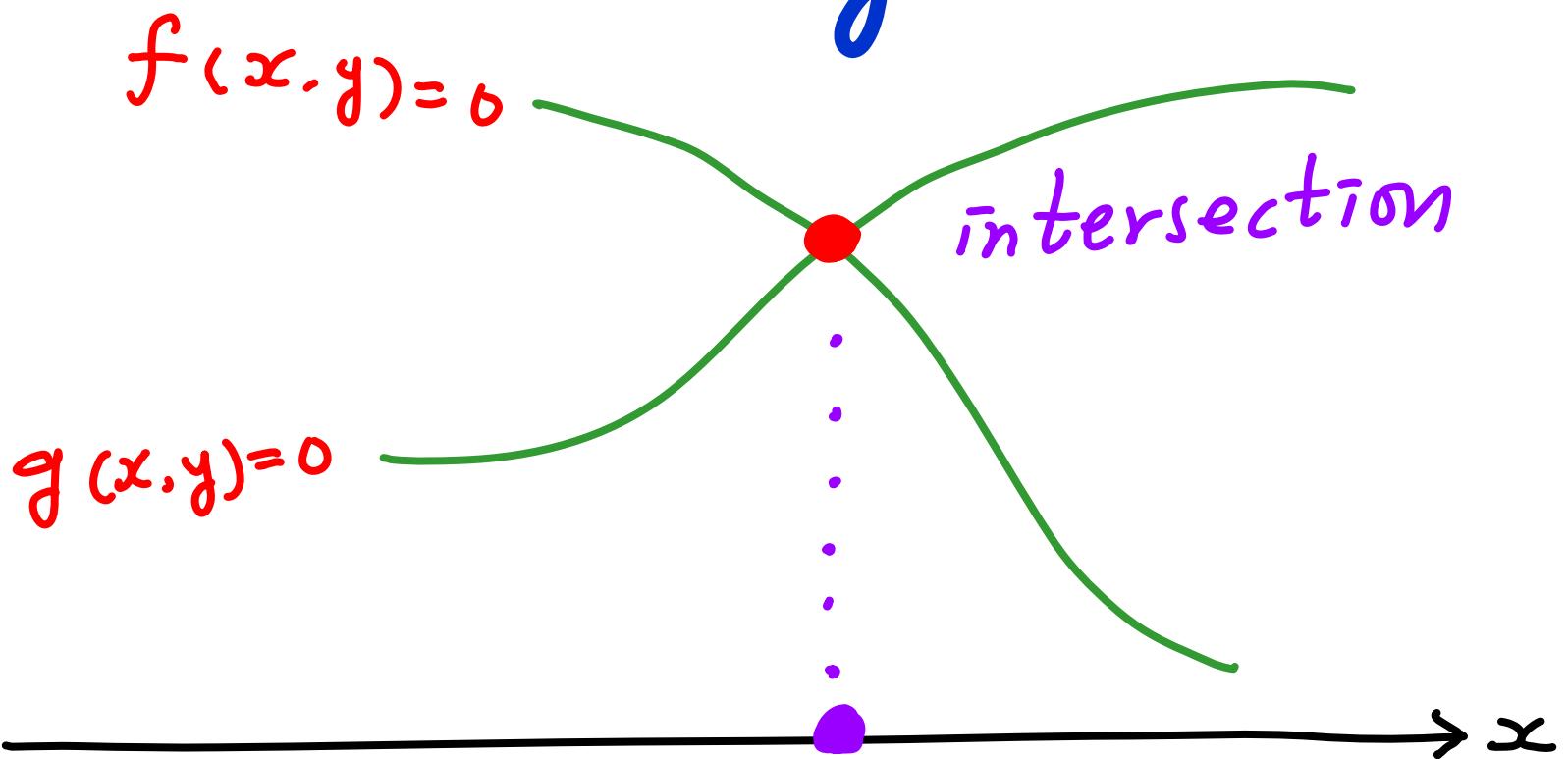
$$g(x,y) = 0$$



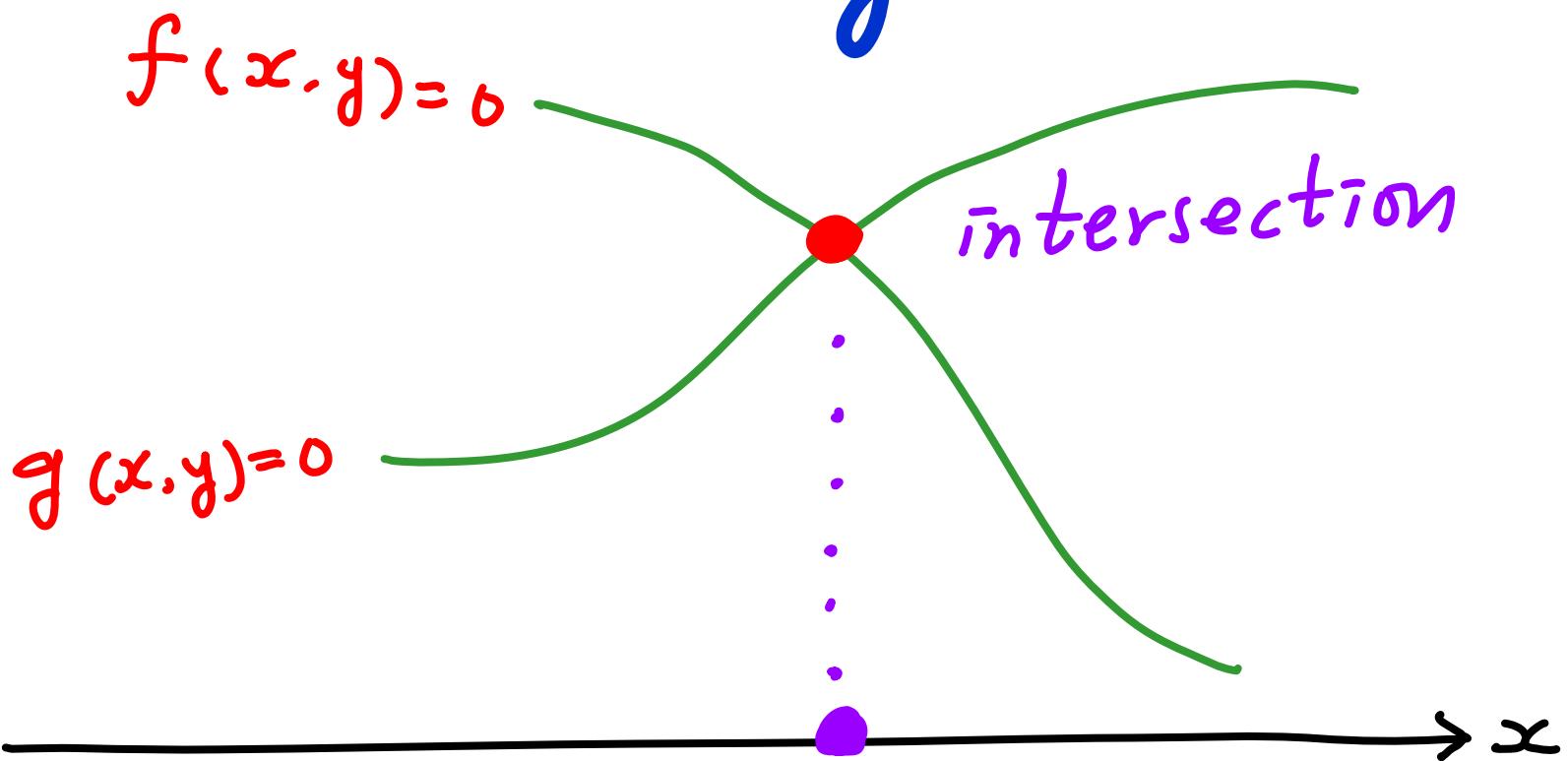
# Interesting Points



# Interesting Points



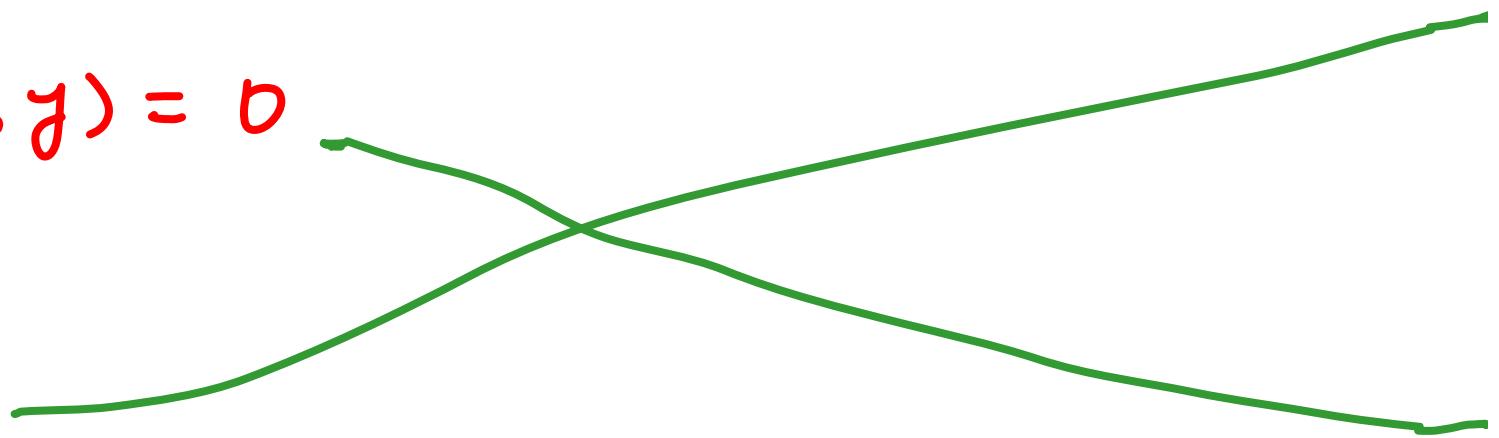
# Interesting Points



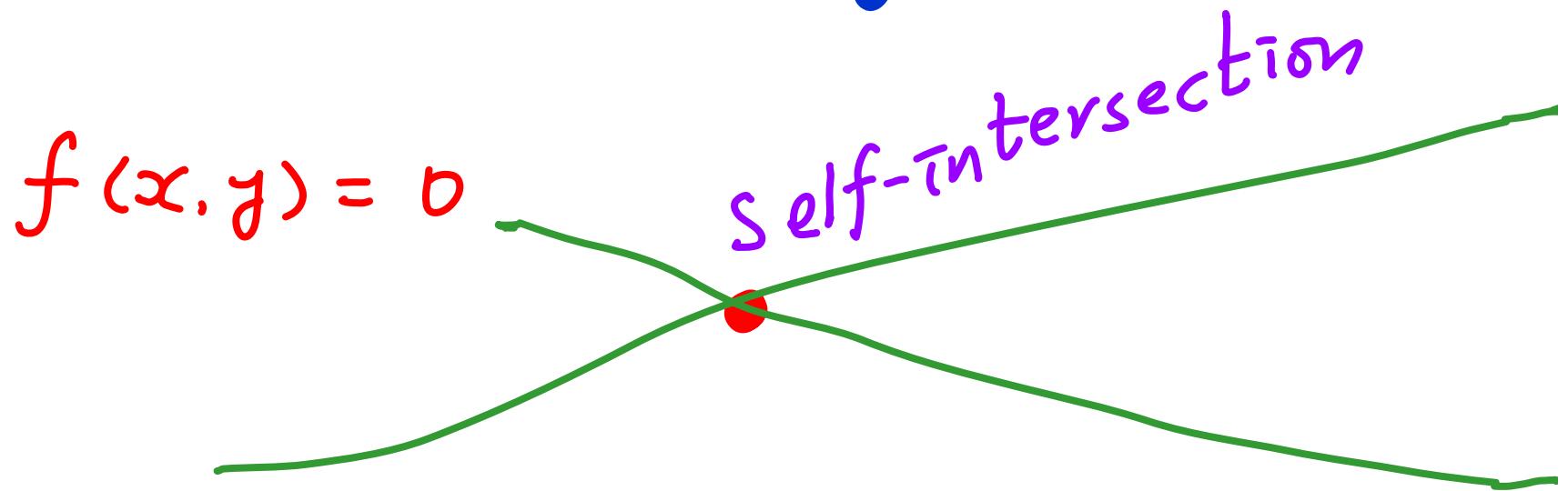
$$r(x) = \text{res}_y(f, g)$$

# Interesting Points

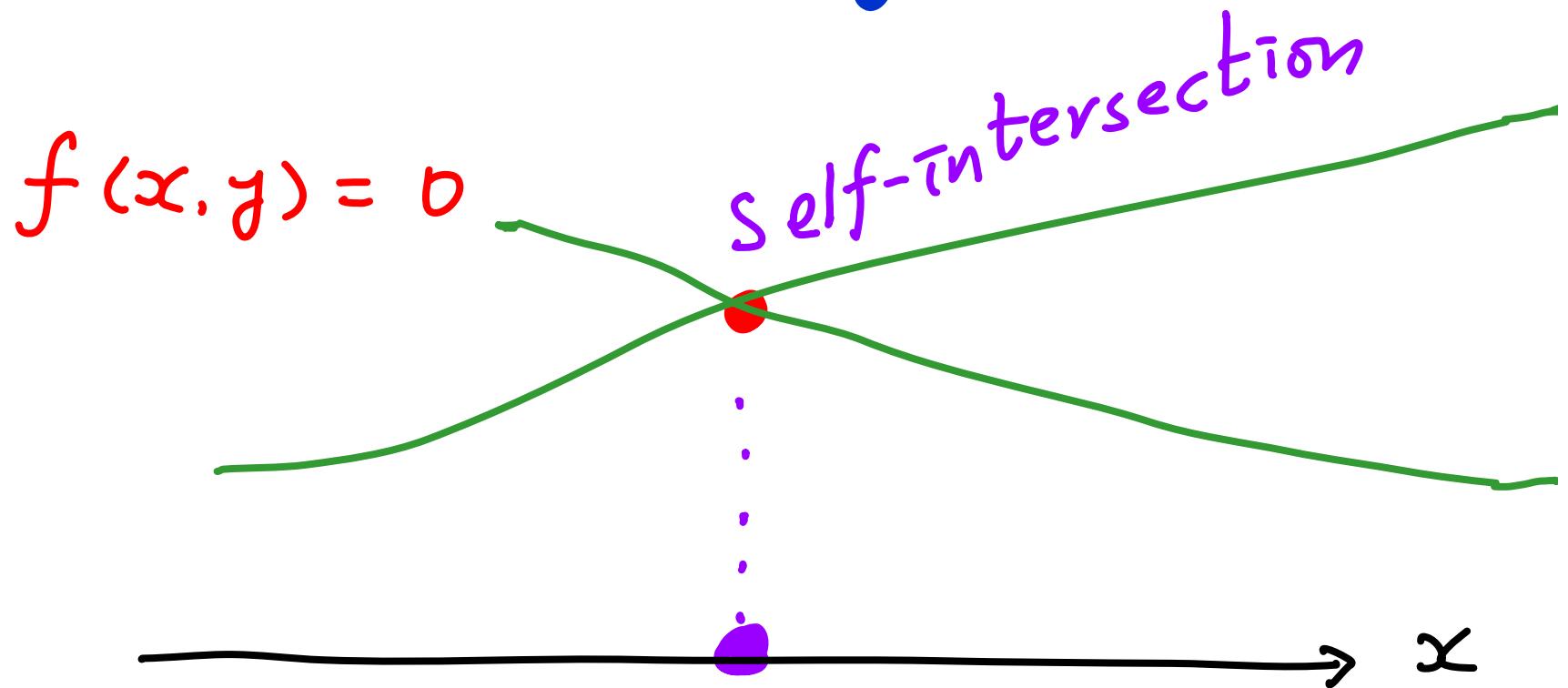
$$f(x, y) = 0$$



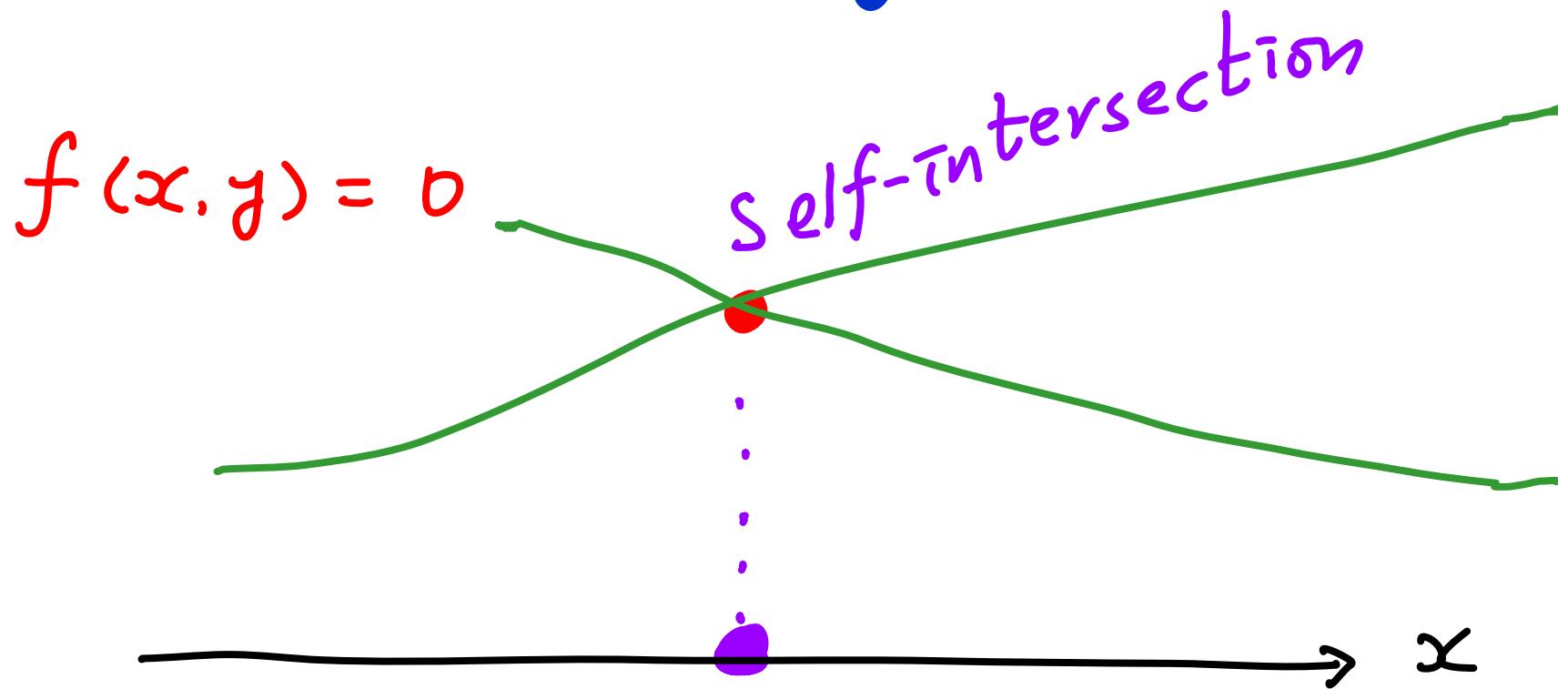
# Interesting Points



# Interesting Points

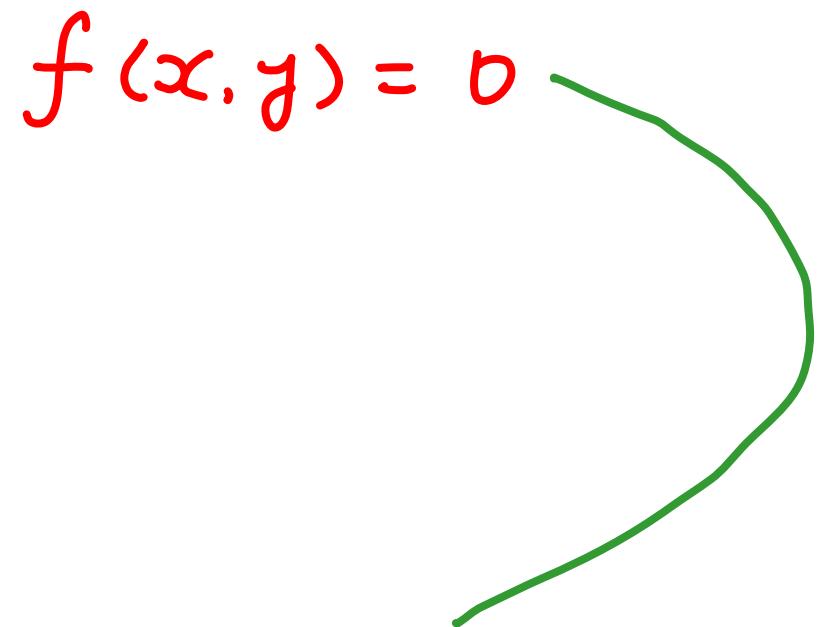


# Interesting Points

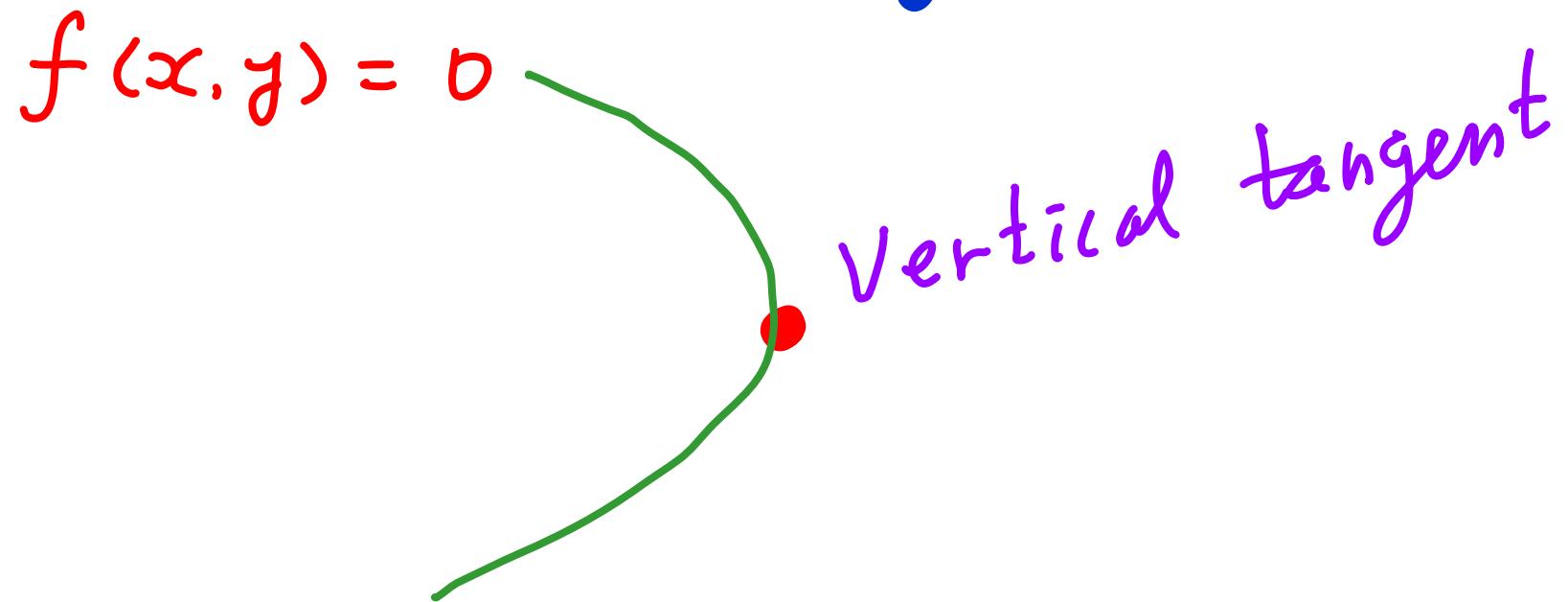


$$d(x) = \text{res}_y \left( f, \frac{\partial f}{\partial y} \right)$$

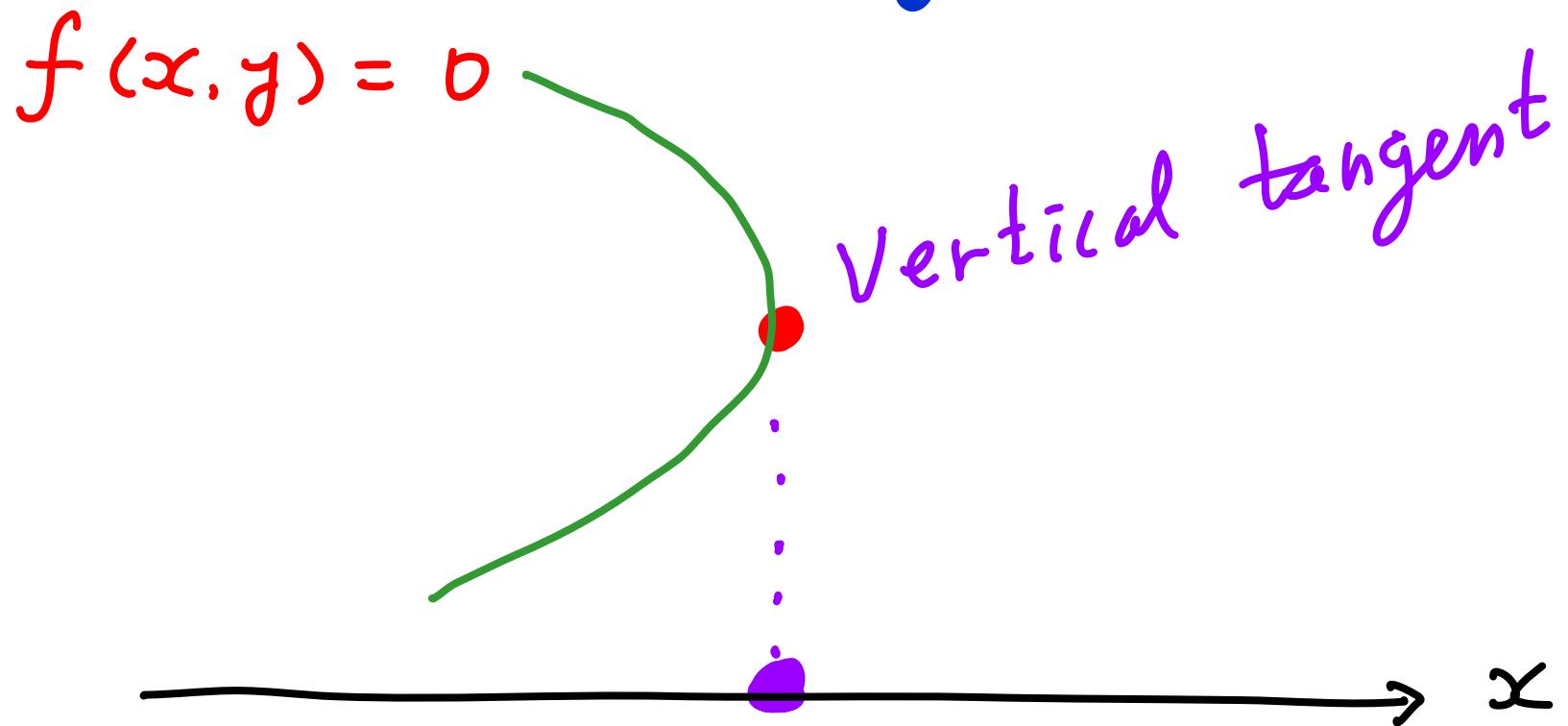
# Interesting Points



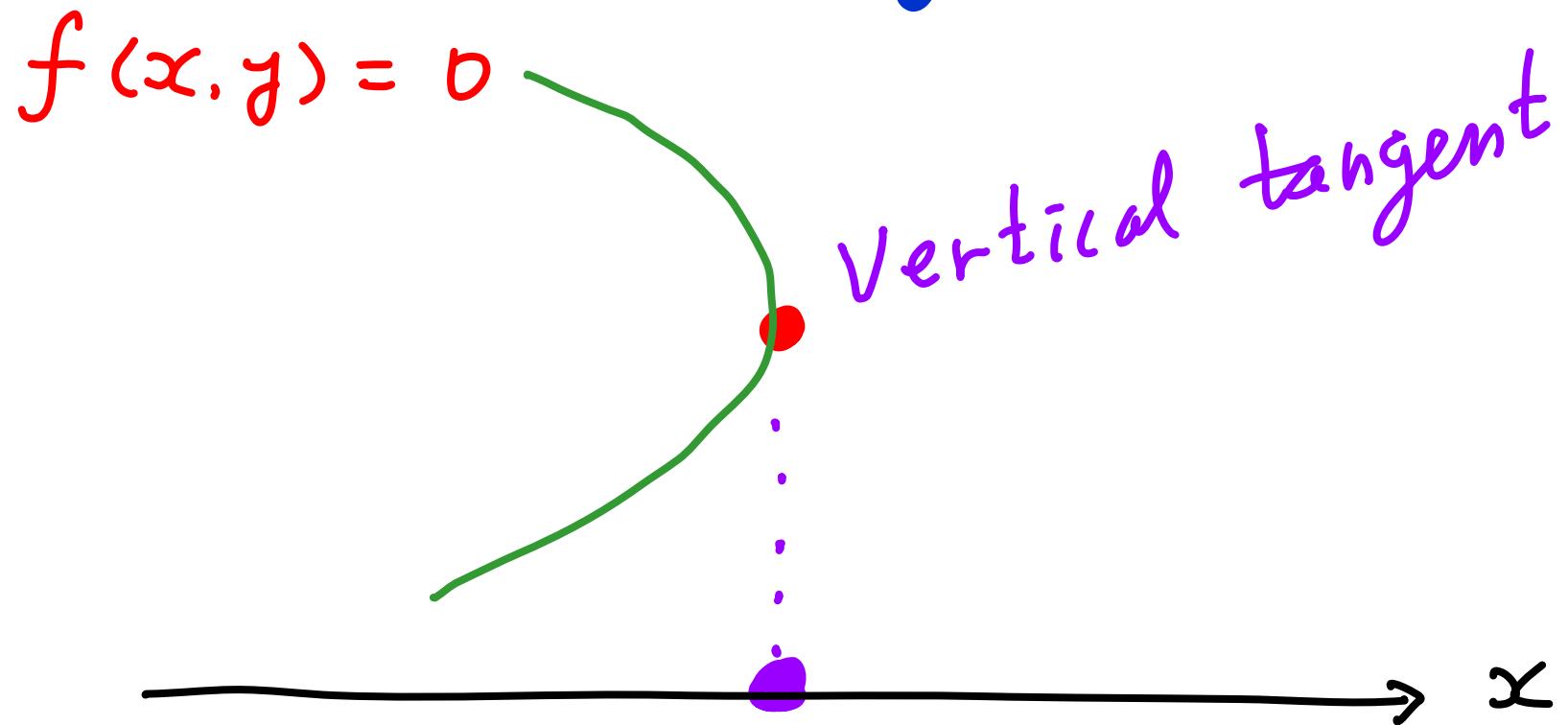
# Interesting Points



# Interesting Points



# Interesting Points



$$d(x) = \text{res}_y \left( f, \frac{\partial f}{\partial y} \right)$$

# Interesting Points

projection  
polys

$$\text{res}_y(f, g)$$

$$\text{res}_y(f, \frac{\partial f}{\partial g})$$

$$\text{res}_y(g, \frac{\partial g}{\partial f})$$

# Challenges

- Reduce the # of projection polys
- Projection "on demand"
- Simplify the output formula
- make incremental

# Rereading

*Prolongation - Relaxation*

# Observe

# Observe

Collins

# Observe

Collins

Resultant

# Observe

Collins

Resultant

Prolongation

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases} \quad \text{prolongation}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$



$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases} \quad \text{prolongation}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$



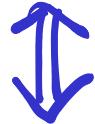
$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases} \quad \text{prolongation}$$

$$a_2 \ a_1 \ a_0$$

$$b_1 \ b_0$$

$$b_1 \ b_0$$

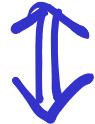
$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{array} \right.$$



$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{array} \right. \quad \text{prolongation}$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{array} \right.$$



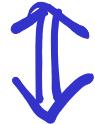
$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{array} \right. \quad \text{prolongation}$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$



$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix}$$

$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{array} \right.$$



$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{array} \right. \quad \text{prolongation}$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$\Downarrow$   $\uparrow C$

$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix}$$

$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{array} \right.$$



$$\exists x \left\{ \begin{array}{l} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{array} \right. \quad \text{prolongation}$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} \quad \begin{bmatrix} x_2 \\ x_1 \\ 1 \end{bmatrix}$$



$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 \\ b_1 & b_0 \end{bmatrix}$$

relaxation

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$$\exists x \in \mathbb{R}^+ \quad \begin{array}{l} x^2 - x - 1 = 0 \\ \wedge x^3 - x^2 + x = 0 \end{array}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$$\exists x \in \mathbb{R}^+ \quad \begin{matrix} x^2 - x - 1 = 0 \\ \wedge x^3 - x^2 + x = 0 \end{matrix} \quad \leftarrow \text{prolongation}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$\Updownarrow$

$$\exists x \in \mathbb{R}^+ \quad \begin{matrix} x^2 - x - 1 = 0 \\ \wedge x^3 - x^2 + x = 0 \end{matrix} \quad \leftarrow \text{prolongation}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$\Updownarrow$

$$\exists x \in \mathbb{R}^+ \quad \begin{array}{l} x^2 - x - 1 = 0 \\ \wedge x^3 - x^2 + x = 0 \end{array} \quad \leftarrow \text{prolongation}$$

$$\exists \vec{x} \in \mathbb{R}^{+^3} \quad \begin{array}{l} x_2 - x_1 + 1 = 0 \\ \wedge x_3 - x_2 + x_1 = 0 \end{array}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$\Updownarrow$

$$\exists x \in \mathbb{R}^+ \quad \begin{array}{l} x^2 - x - 1 = 0 \\ \wedge x^3 - x^2 + x = 0 \end{array} \quad \leftarrow \text{prolongation}$$

$$\exists \vec{x} \in \mathbb{R}^{+^3} \quad \begin{array}{l} x_2 - x_1 + 1 = 0 \\ \wedge x_3 - x_2 + x_1 = 0 \end{array} \quad \leftarrow \text{Relaxaction}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

↔

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \quad \wedge \quad x^3 - x^2 + x = 0 \quad \leftarrow \text{prolongation}$$

↓

$$\exists \vec{x} \in \mathbb{R}^{+3} \quad x_2 - x_1 + 1 = 0 \quad \wedge \quad x_3 - x_2 + x_1 = 0 \quad \leftarrow \text{Relaxation}$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$\Updownarrow$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \quad \wedge \quad x^3 - x^2 + x = 0 \quad \leftarrow \text{prolongation}$$

$\Downarrow$

$$\exists \vec{x} \in \mathbb{R}^{+^3} \quad x_2 - x_1 + 1 = 0 \quad \wedge \quad x_3 - x_2 + x_1 = 0 \quad \leftarrow \text{Relaxation}$$

$\Updownarrow$

$\leftarrow$  Linear method

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

↑  
↓

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \quad \wedge \quad x^3 - x^2 + x = 0 \quad \leftarrow \text{prolongation}$$

↓

$$\exists \vec{x} \in \mathbb{R}^{+3} \quad x_2 - x_1 + 1 = 0 \quad \wedge \quad x_3 - x_2 + x_1 = 0 \quad \leftarrow \text{Relaxaction}$$

↑  
← Linear method

false

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0 \quad \text{false}$$

$\Updownarrow$

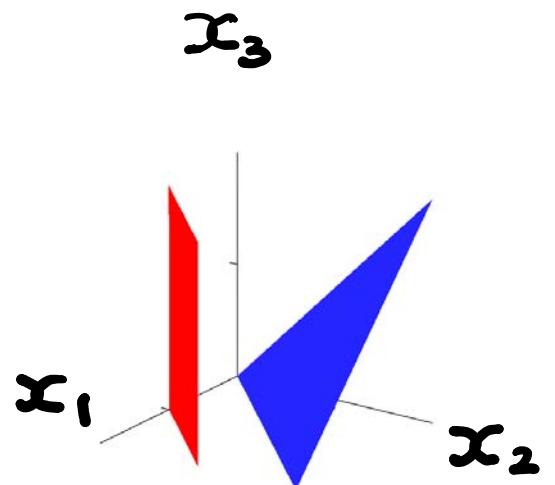
$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \quad \wedge \quad x^3 - x^2 + x = 0 \quad \leftarrow \text{prolongation}$$

$\Downarrow$

$$\exists \vec{x} \in \mathbb{R}^{+3} \quad x_2 - x_1 + 1 = 0 \quad \wedge \quad x_3 - x_2 + x_1 = 0 \quad \leftarrow \text{Relaxaction}$$

$\Updownarrow$   $\leftarrow$  Linear method

false



Cany, Grigor'ev, Roy, Renegar

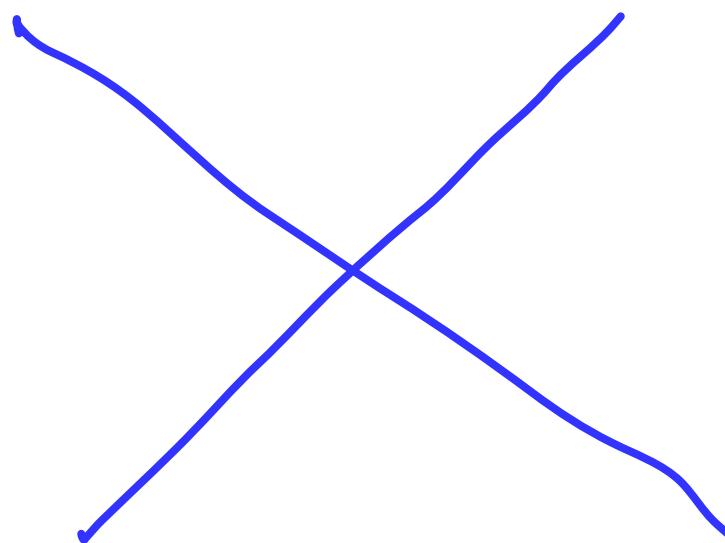
$$\exists x \exists y \underbrace{x+y}_{f} \geq 0 \wedge \underbrace{x-y}_{g} \geq 0$$

$$\exists x \exists y \underbrace{x+y}_{f} \geq 0 \wedge \underbrace{x-y}_{g} \geq 0$$

$$f \cdot g = 0$$

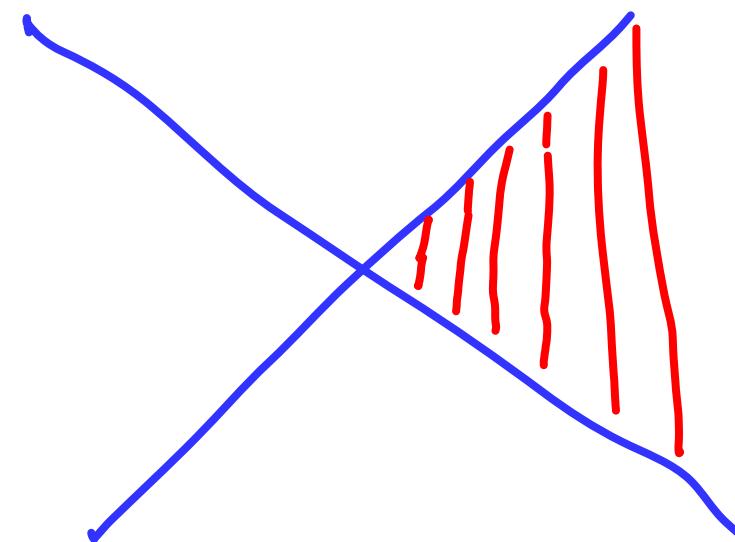
$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

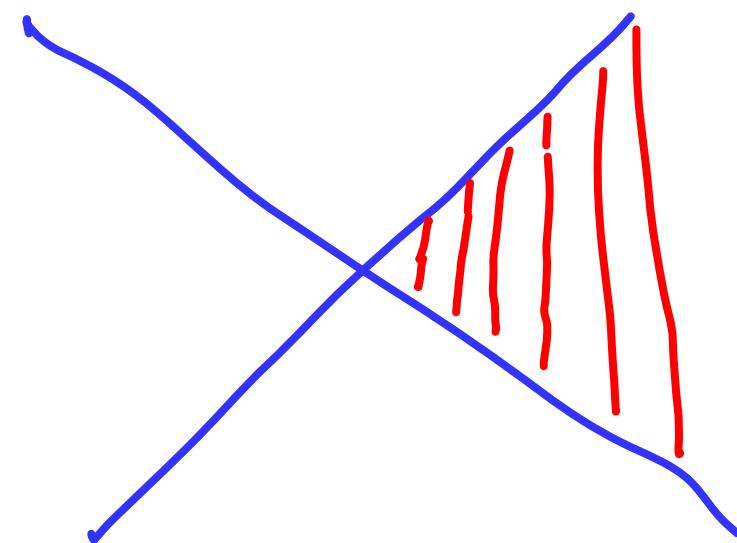
$$f \cdot g = 0$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

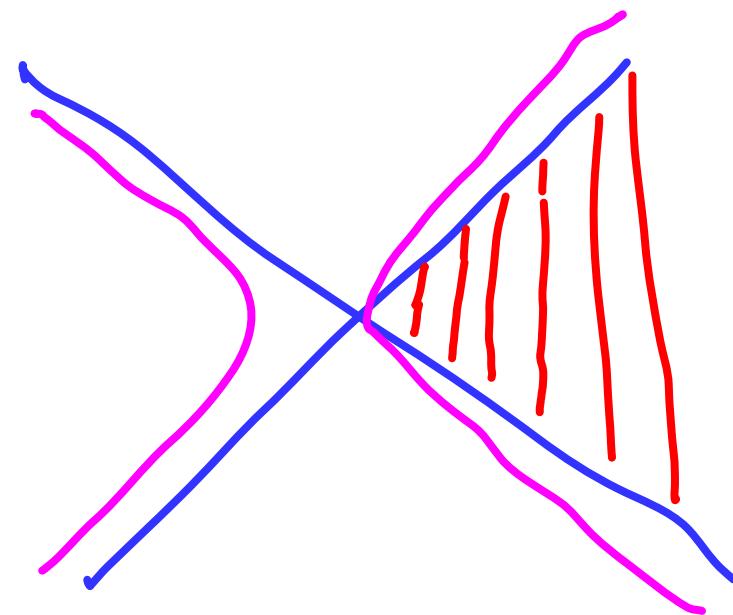
$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

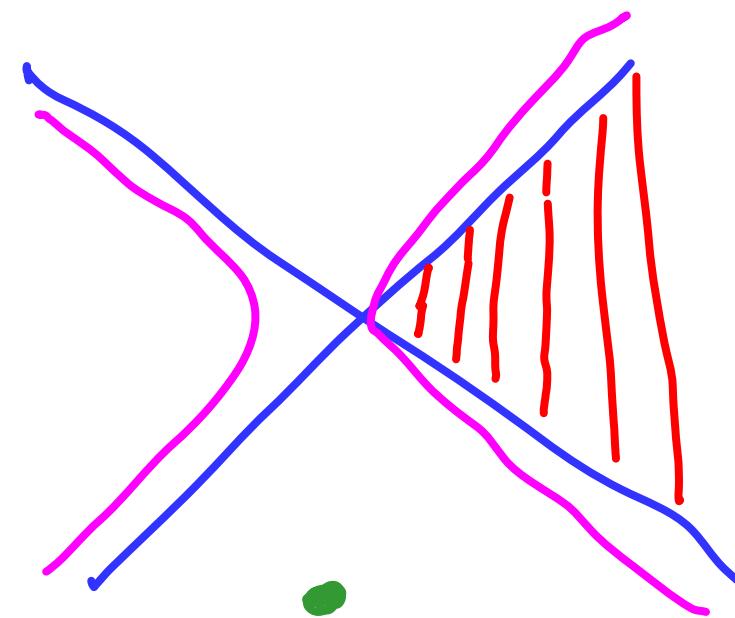
$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

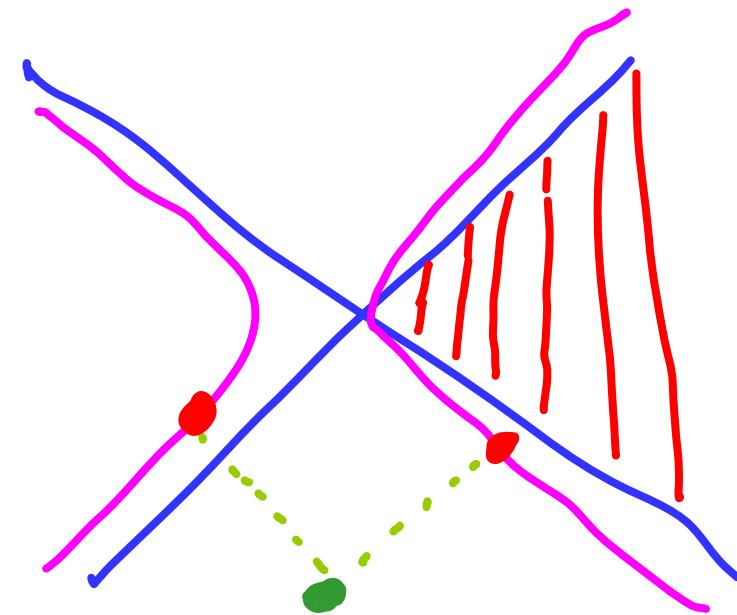
$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

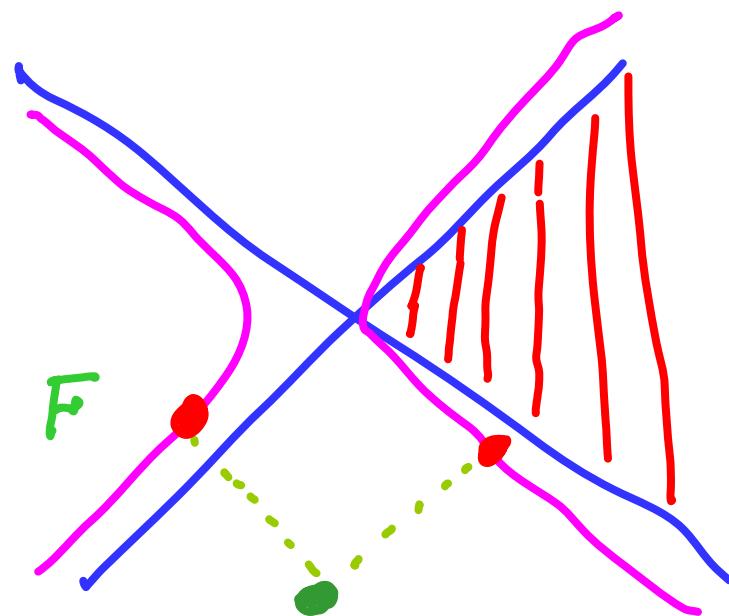
$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

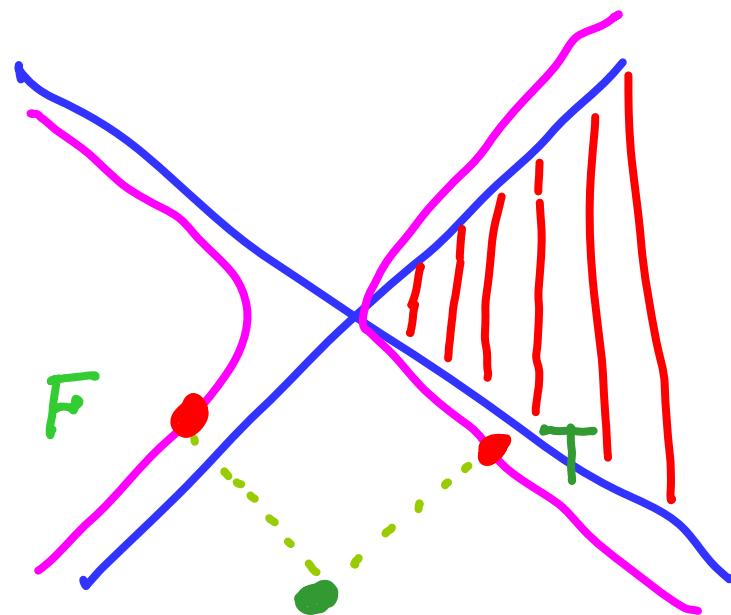
$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$



$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

$$f \cdot g = 0$$

$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$

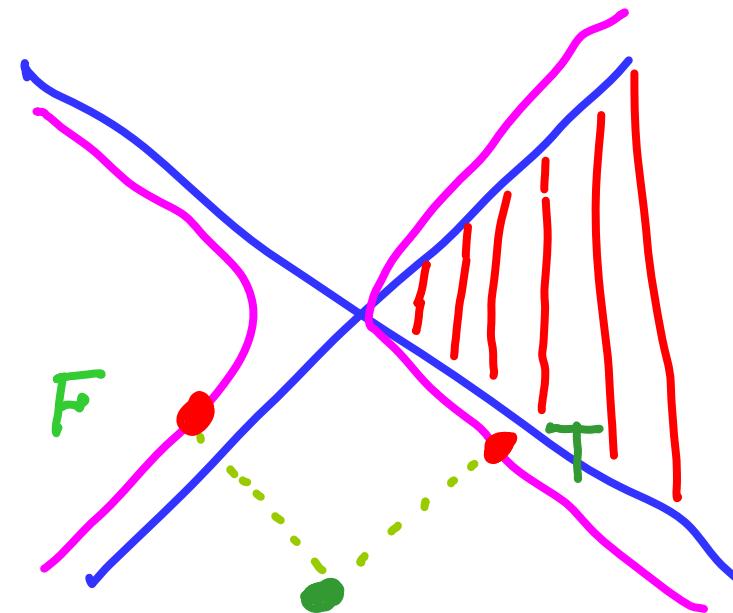


$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

True!

$$f \cdot g = 0$$

$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2$$

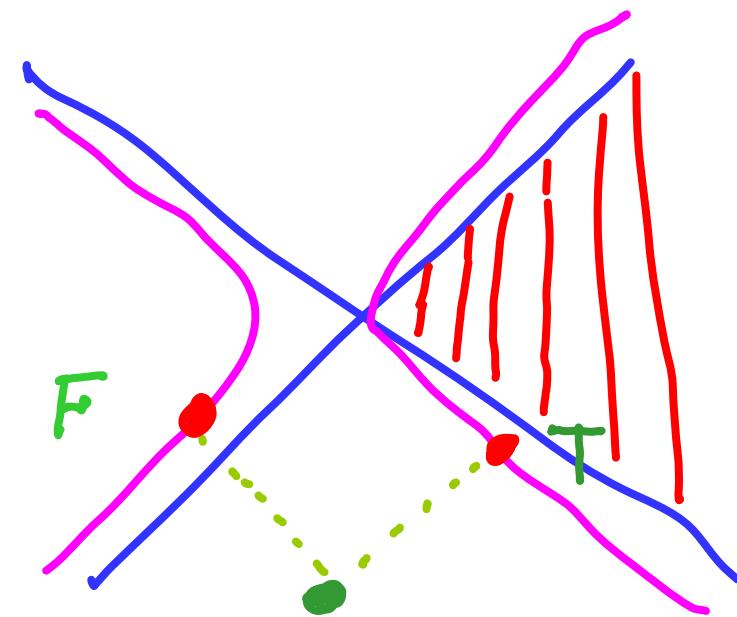


$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

True!

$$f \cdot g = 0$$

$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2 + \delta$$



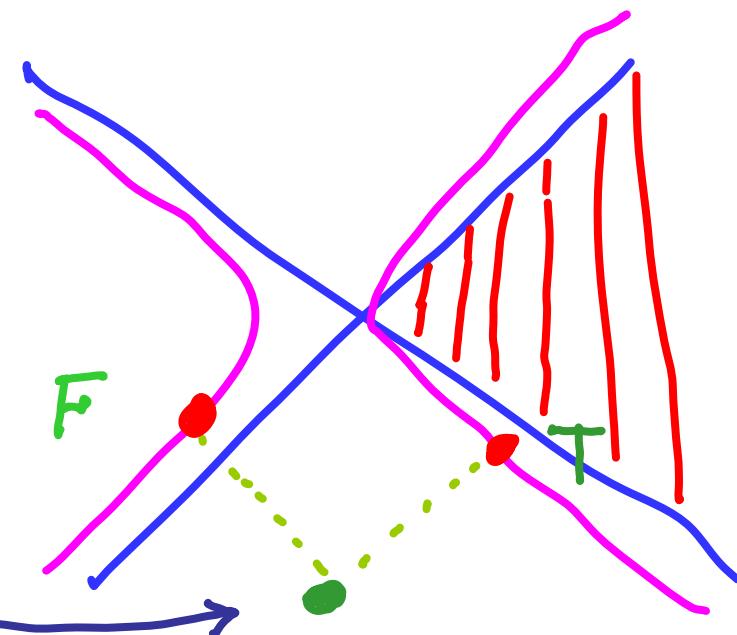
$$\exists x \exists y \underbrace{x+y \geq 0}_{f} \wedge \underbrace{x-y \geq 0}_{g}$$

True!

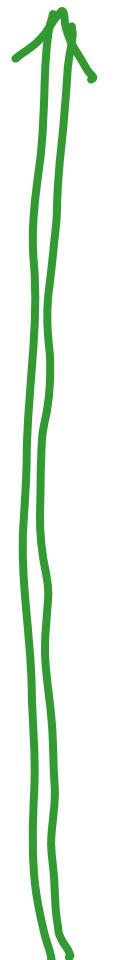
$$f \cdot g = 0$$

$$(f + \varepsilon)(g + \varepsilon) = \varepsilon^2 + \delta$$

"generically"



$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$



$$\forall x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\epsilon)} \quad \text{real algebraic closure}$$

$$\boxed{\exists} x \in S \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{R(\varepsilon)} \quad K = \overline{L(\delta)}$$

real algebraic  
closure

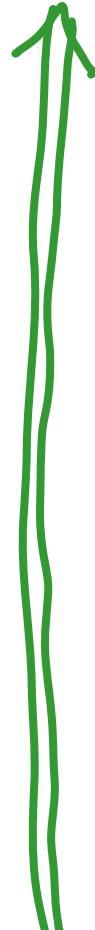
$$\boxed{\exists} x \in S \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\varepsilon)} \quad \text{real algebraic closure}$$
$$K = \overline{L(\delta)}$$
$$g = \prod_{i=1}^r (f_i + \varepsilon) - \varepsilon^r - \delta$$

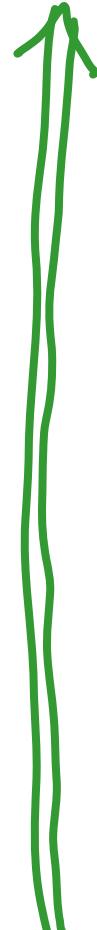
$$\forall x \in S \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$


 $L = \overline{\mathbb{R}(\varepsilon)}$  ← real algebraic closure  
 $K = \overline{L(\delta)}$   
 $g = \prod_{i=1}^r (f_i + \varepsilon) - \varepsilon^r - \delta$   
 $T = \text{Solutions } x \in K \text{ of Lagrange mult. eq.}$   
 for "distance" from a point  $p$ .


 $\boxed{\exists x \in S \ f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0}$

$$\exists x \in \mathbb{R}^n \ f_1(x) \geq 0 \wedge \dots \wedge f_r(x) \geq 0$$

- 
 $L = \overline{\mathbb{R}(\varepsilon)}$  ← real algebraic closure
- $K = \overline{L(\delta)}$
- $g = \prod_{i=1}^r (f_i + \varepsilon) - \varepsilon^r - \delta$
- $T =$  Solutions  $\lambda \in K^n$  of Lagrange mult. eq.  
 for "distance" from a point  $p$ .
- $S =$  standard part of  $T$
- $\boxed{\exists x \in S \ f_1(x) \geq 0 \wedge \dots \wedge f_r(x) \geq 0}$

# Challenges

- Efficient computation with infinitimals
- Reducing the size of sample set.

# Rereading

*Critical points - Morse complex*

# Problem

# Problem

Input

# Problem

Input

$$\textcolor{red}{f} \in R[x_1, \dots, x_n]$$

# Problem

Input

$$\textcolor{red}{f} \in R[x_1, \dots, x_n]$$

Toy Example

# Problem

Input

$$\textcolor{red}{f} \in R[x_1, \dots, x_n]$$

Toy Example

$$\textcolor{red}{f} = x^3 - x^2 + y^2$$

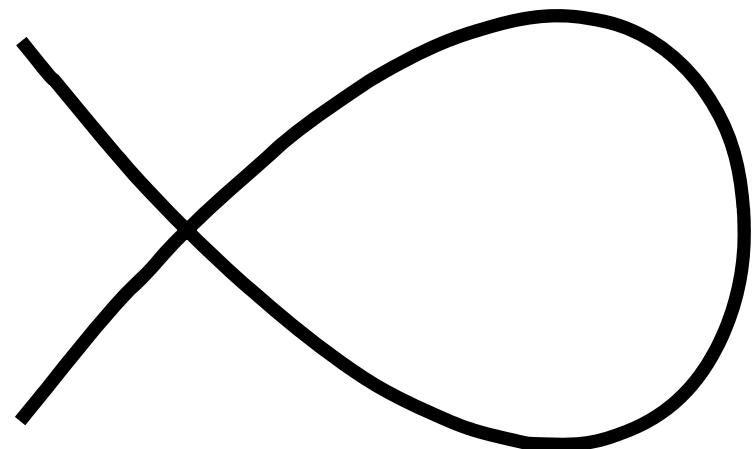
# Problem

Input

$$f \in R[x_1, \dots, x_n]$$

Toy Example

$$f = x^3 - x^2 + y^2$$



# Problem

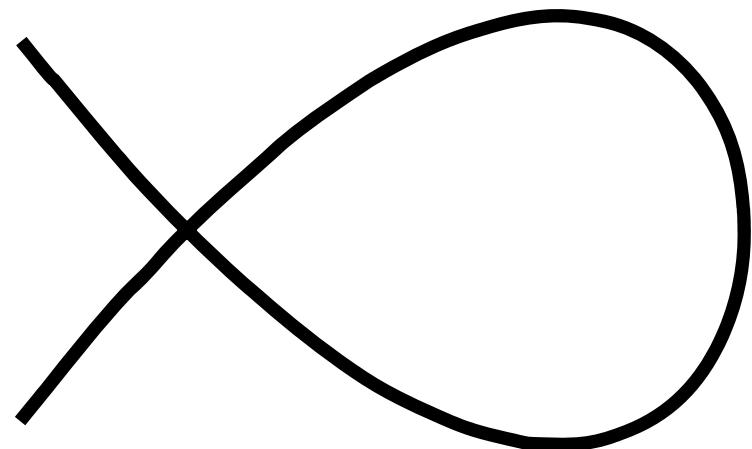
Input

$$f \in R[x_1, \dots, x_n]$$

Output

Toy Example

$$f = x^3 - x^2 + y^2$$



# Problem

Input

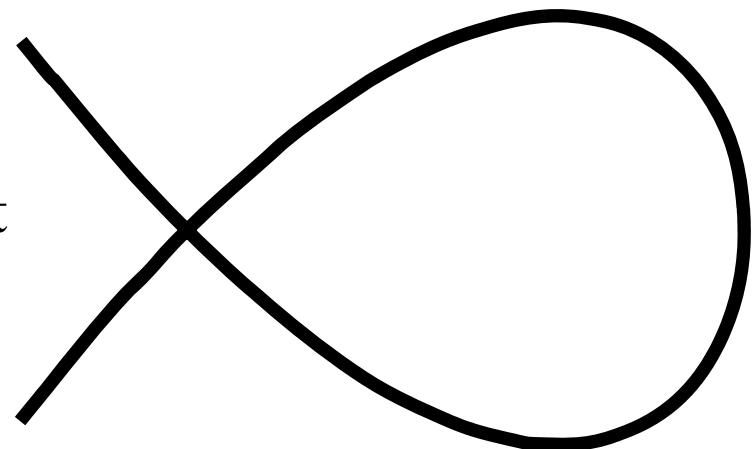
$$f \in R[x_1, \dots, x_n]$$

Toy Example

$$f = x^3 - x^2 + y^2$$

Output

A point in each connected component



# Problem

Input

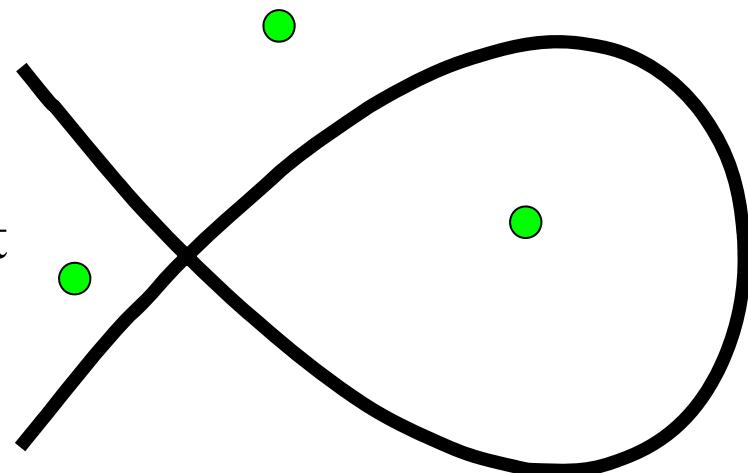
$$f \in R[x_1, \dots, x_n]$$

Toy Example

$$f = x^3 - x^2 + y^2$$

Output

A point in each connected component



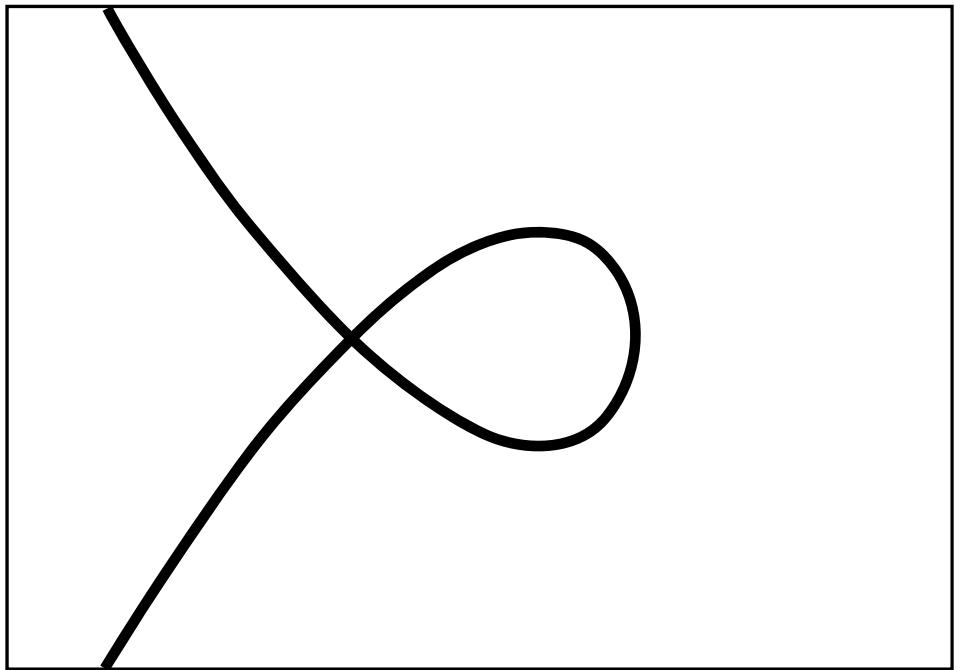
# Idea

# Idea

*f*

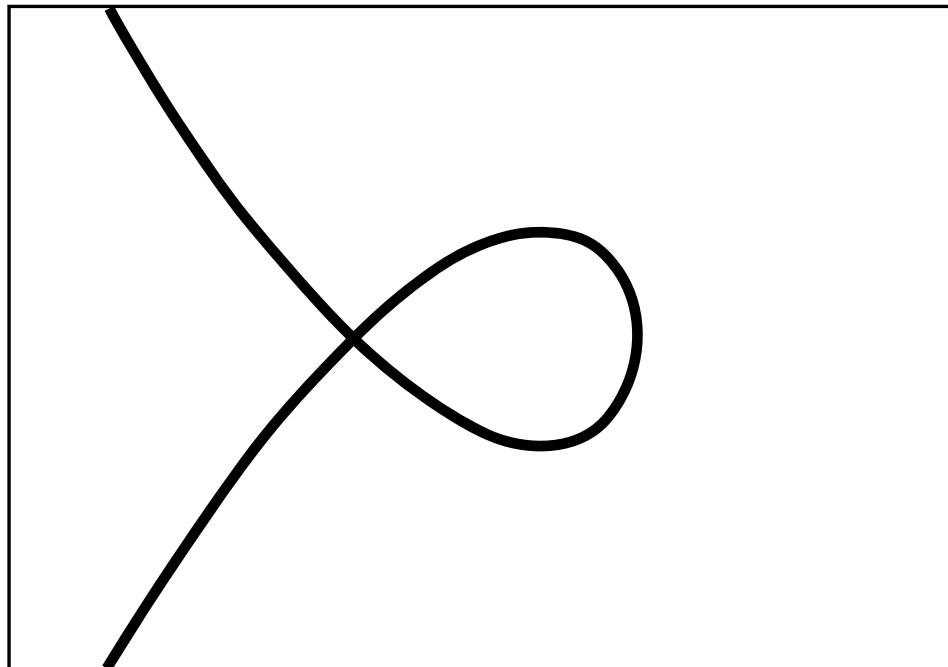
# Idea

$f$



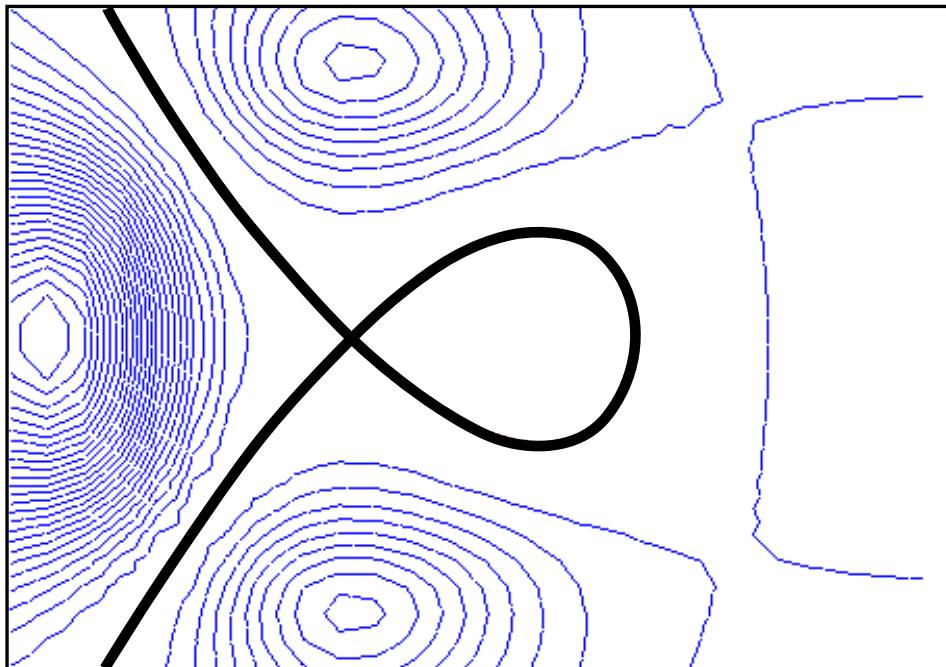
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



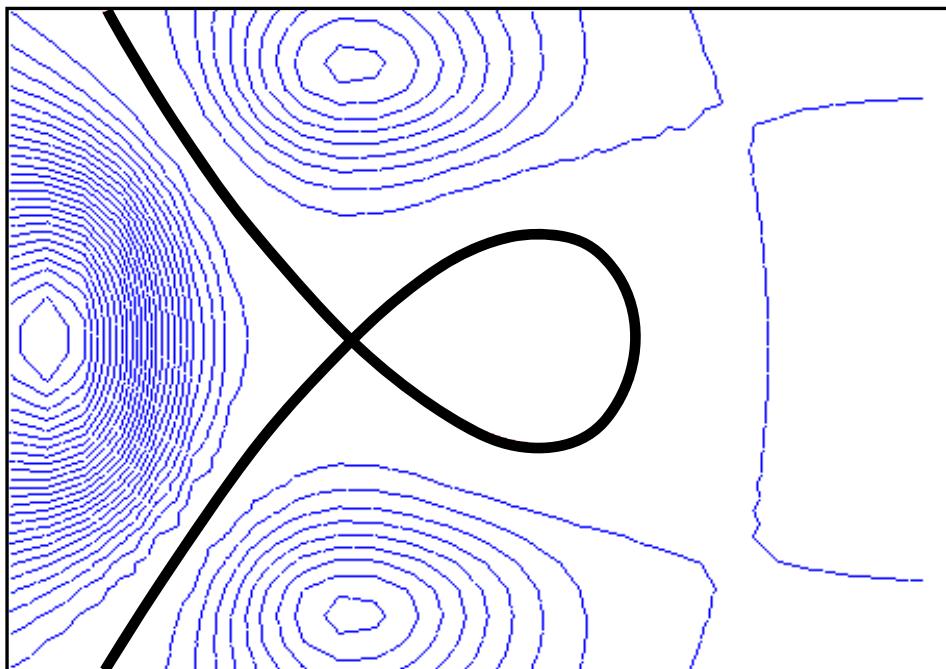
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$



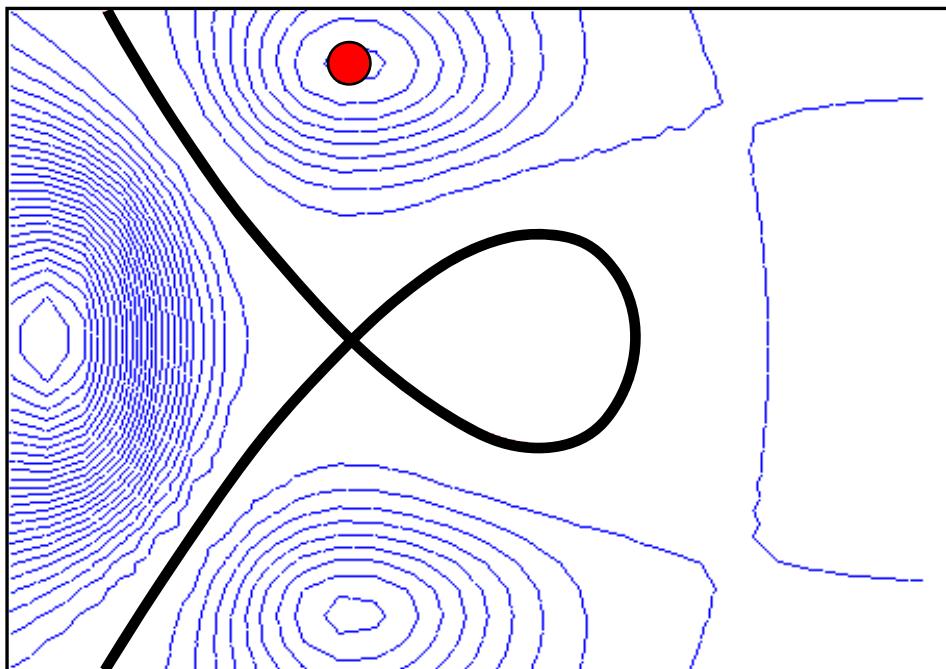
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$



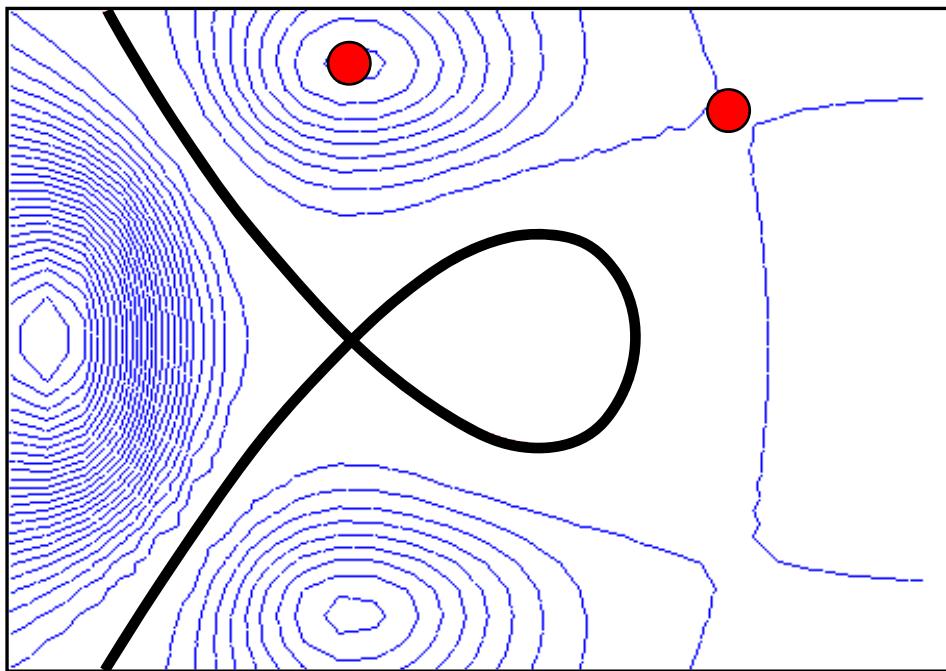
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$



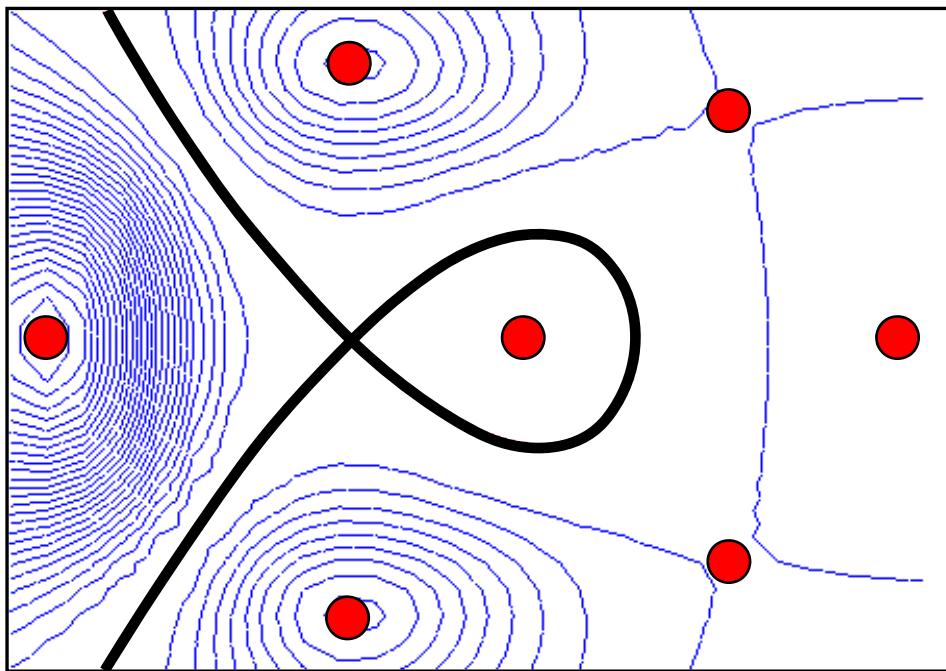
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$



# Idea

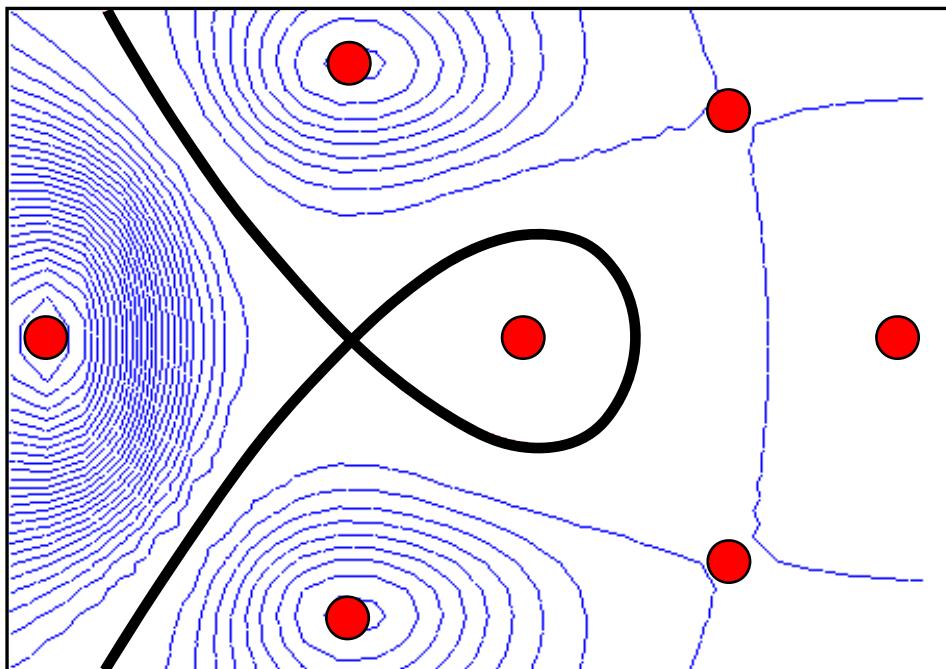
$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$

Hybrid method



# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

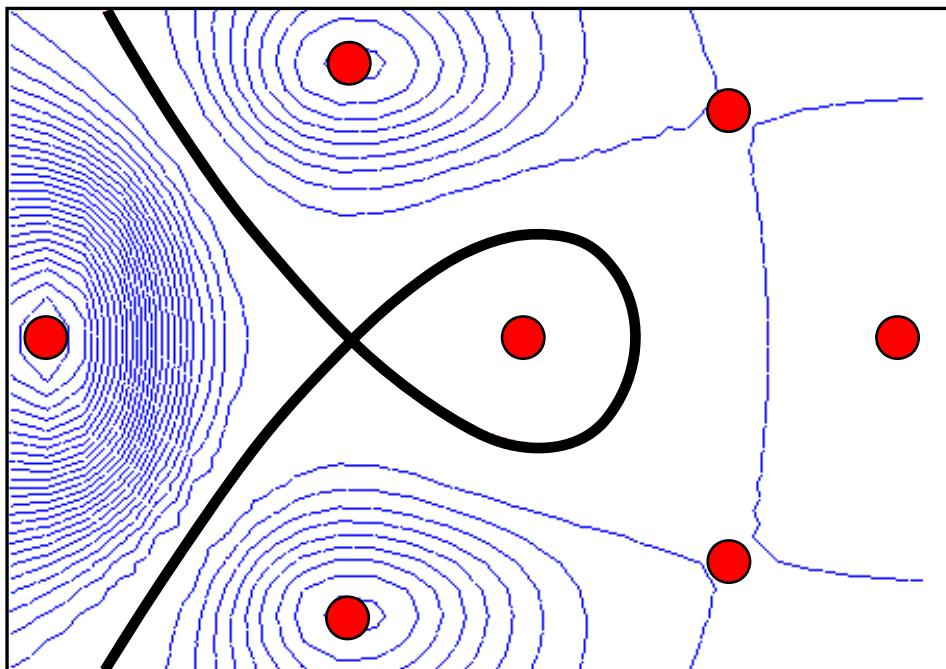
$$g_x = g_y = 0$$

and

$$f \neq 0$$

Hybrid method

- Symbolic: Resultant



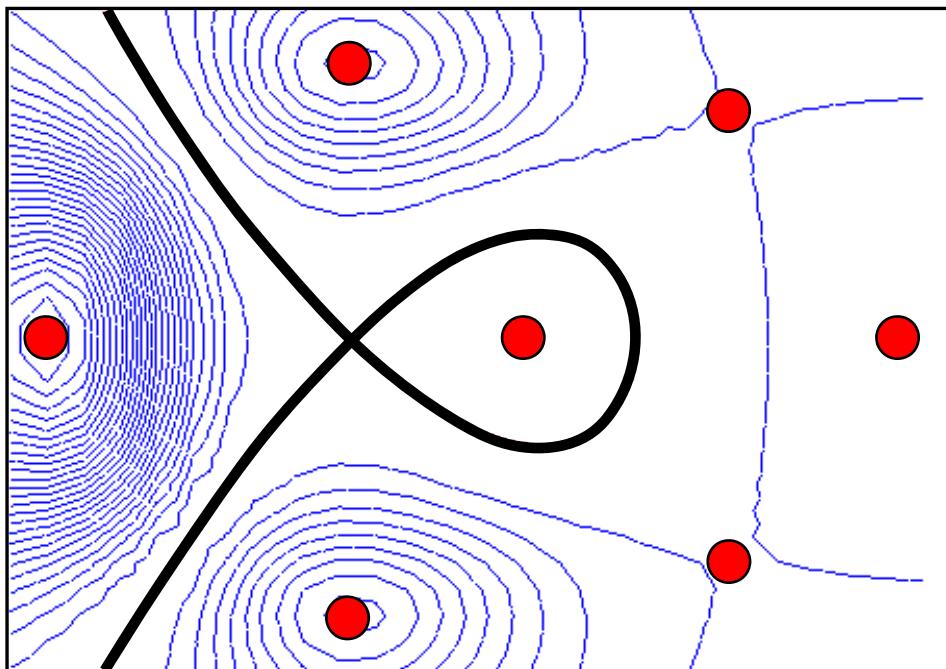
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$



Hybrid method

- **Symbolic:** Resultant
- **Numeric :** Interval (HSO)

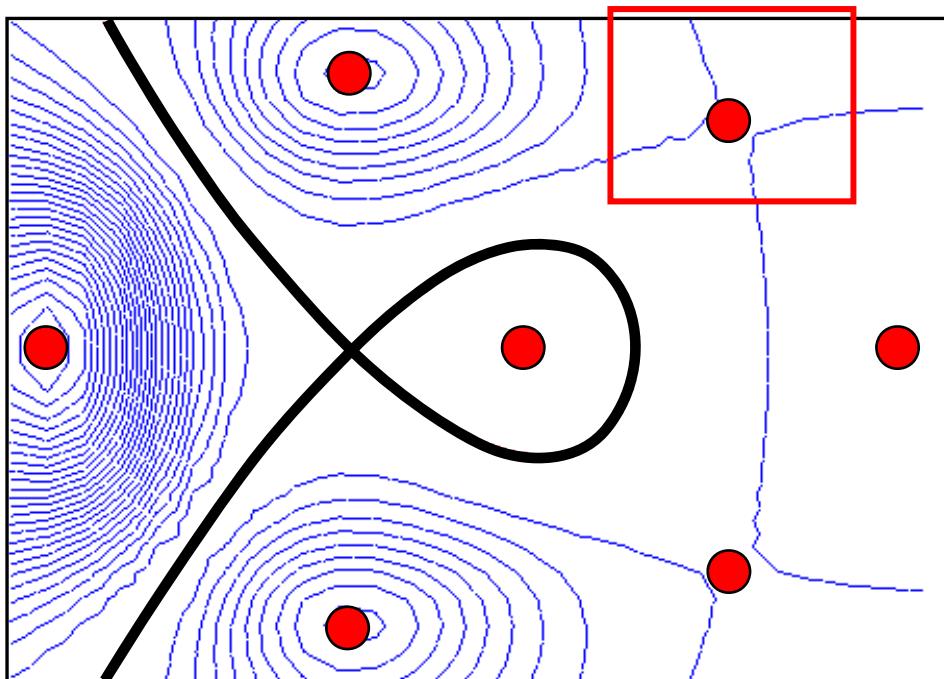
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

$$f \neq 0$$

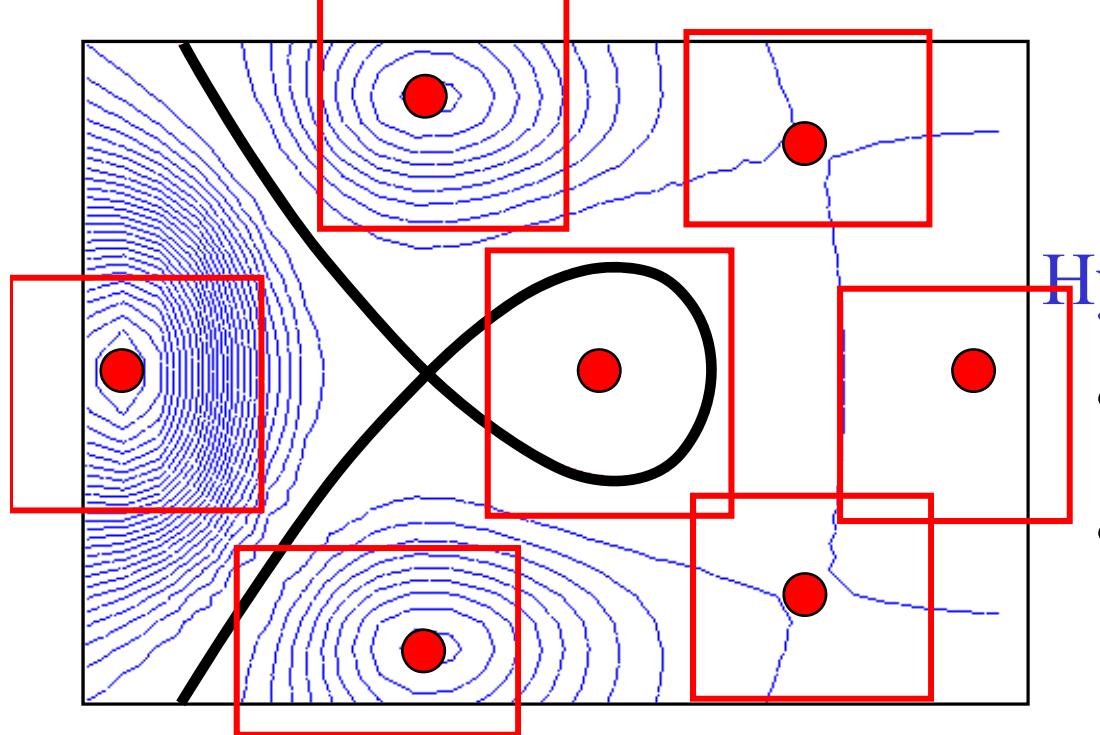


Hybrid method

- **Symbolic:** Resultant
- **Numeric :** Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

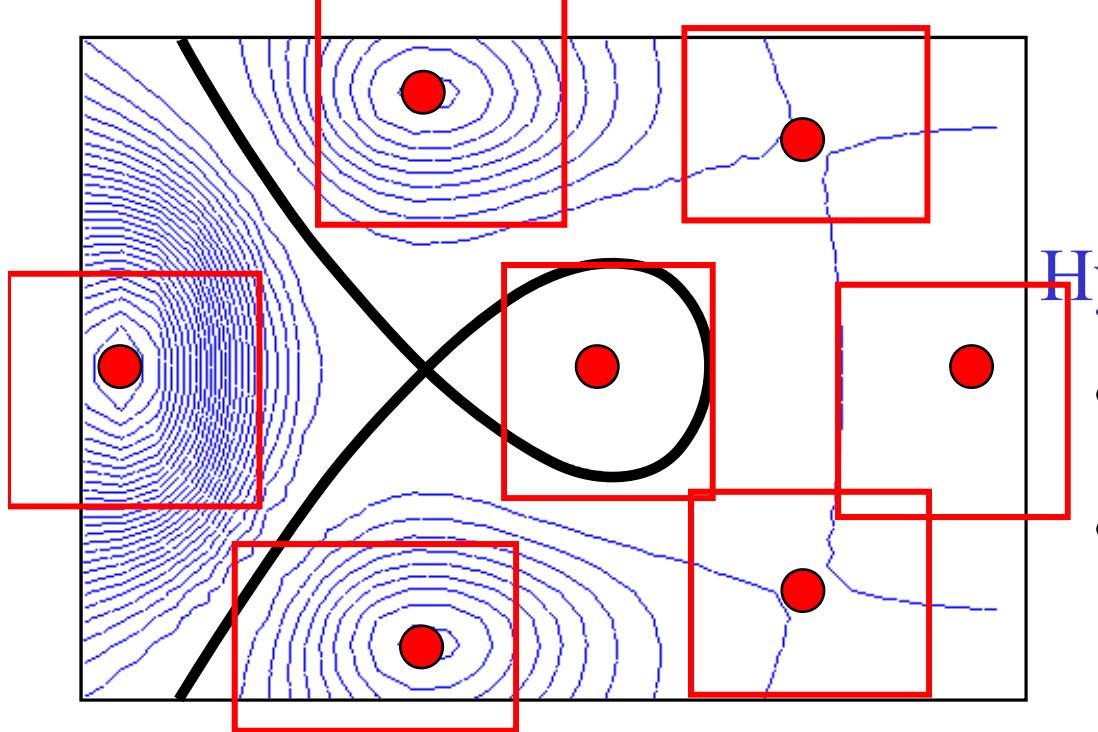
$$f \neq 0$$

Hybrid method

- Symbolic: Resultant
- Numeric : Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

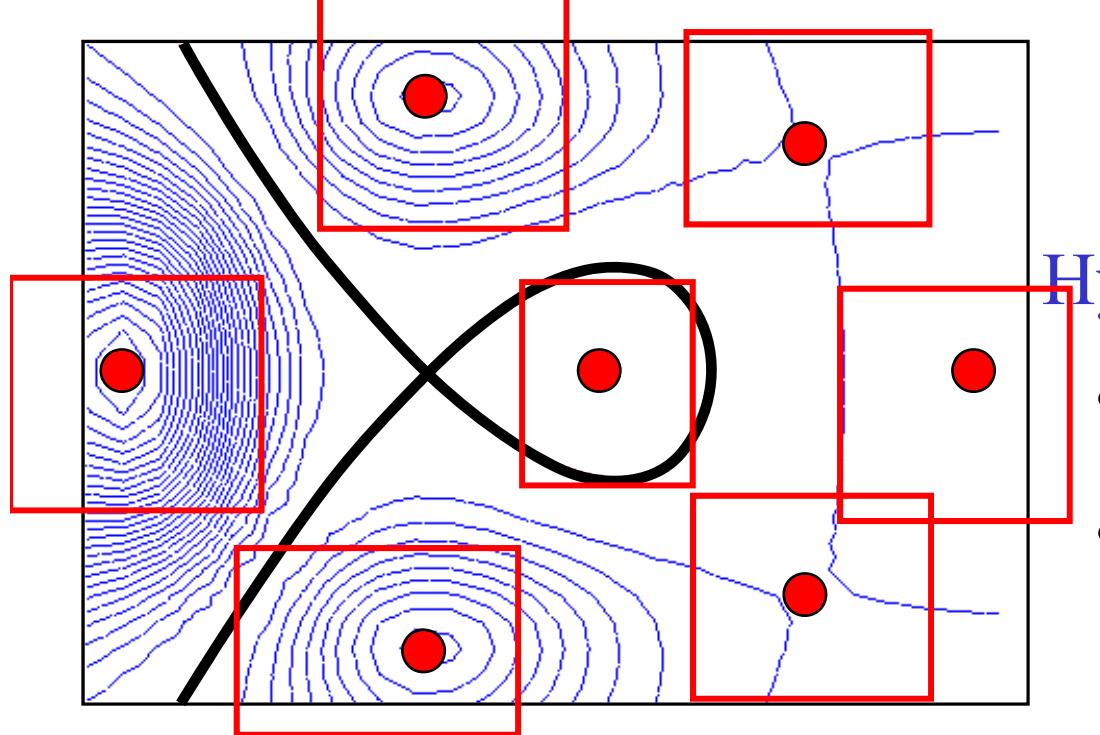
$$f \neq 0$$

Hybrid method

- Symbolic: Resultant
- Numeric : Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

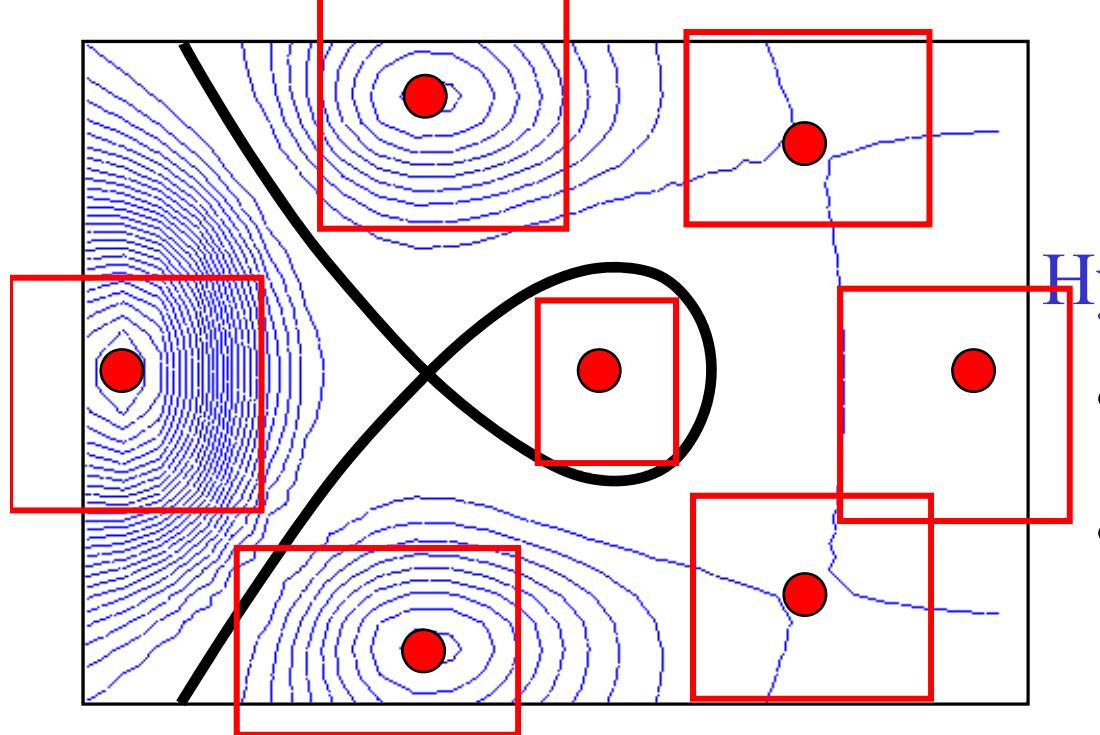
$$f \neq 0$$

Hybrid method

- Symbolic: Resultant
- Numeric : Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

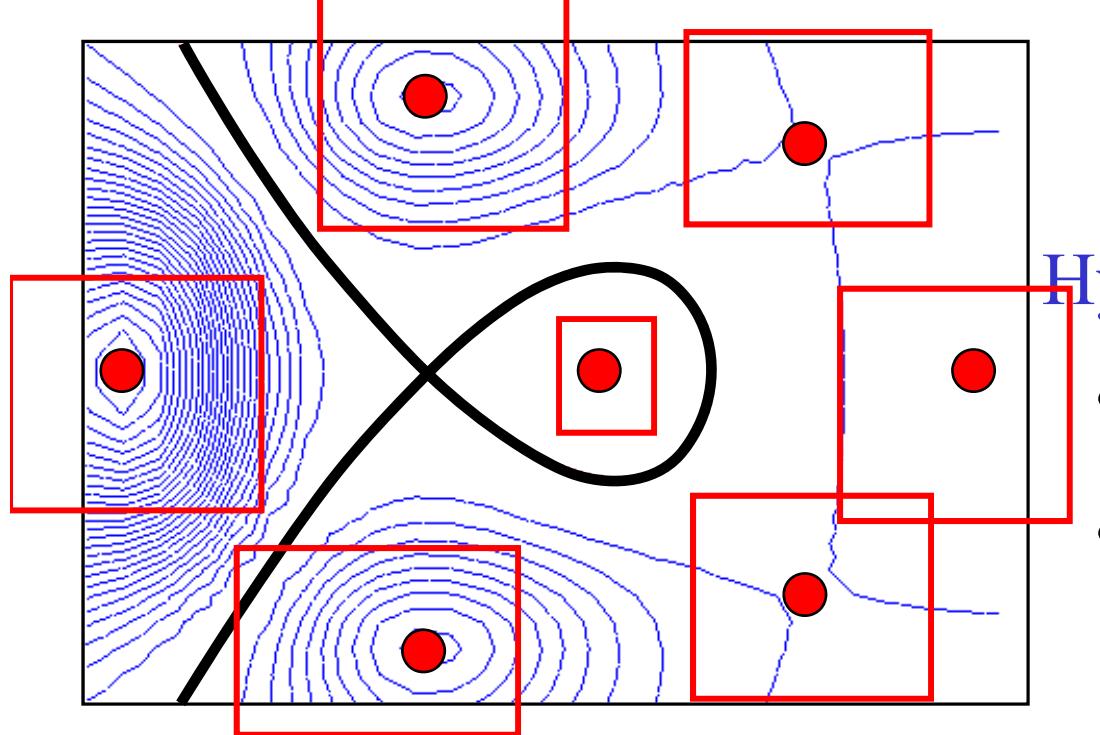
$$f \neq 0$$

Hybrid method

- Symbolic: Resultant
- Numeric : Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

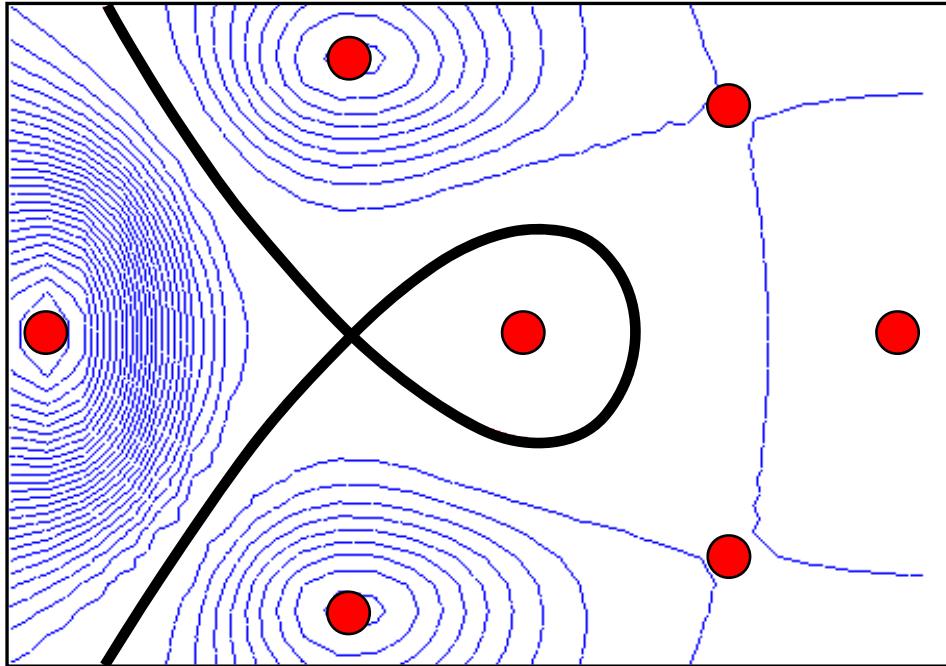
$$f \neq 0$$

Hybrid method

- Symbolic: Resultant
- Numeric : Interval (HSO)

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

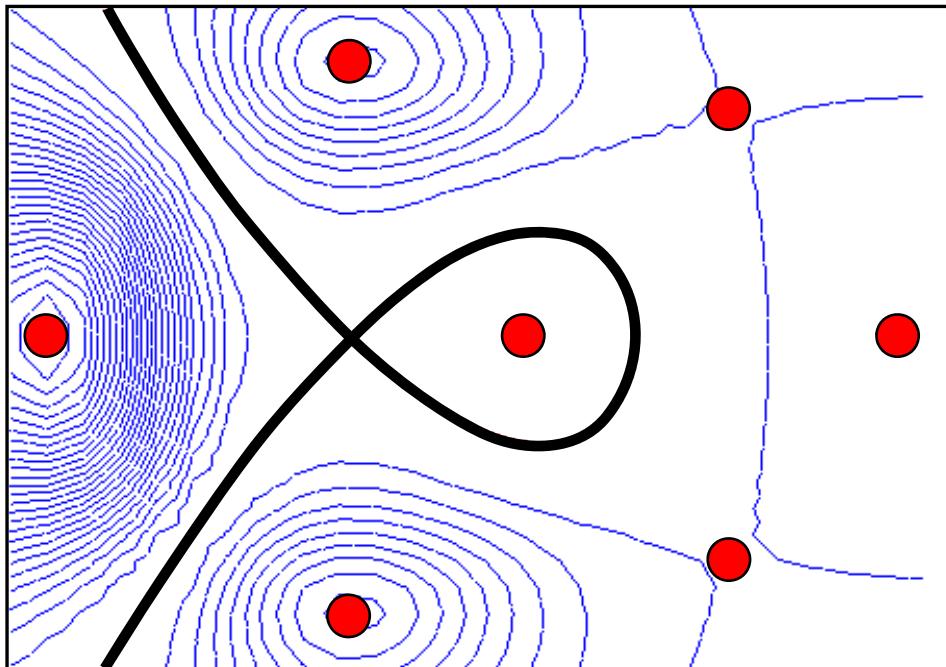
and

$$f \neq 0$$

Hessian of  $g$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

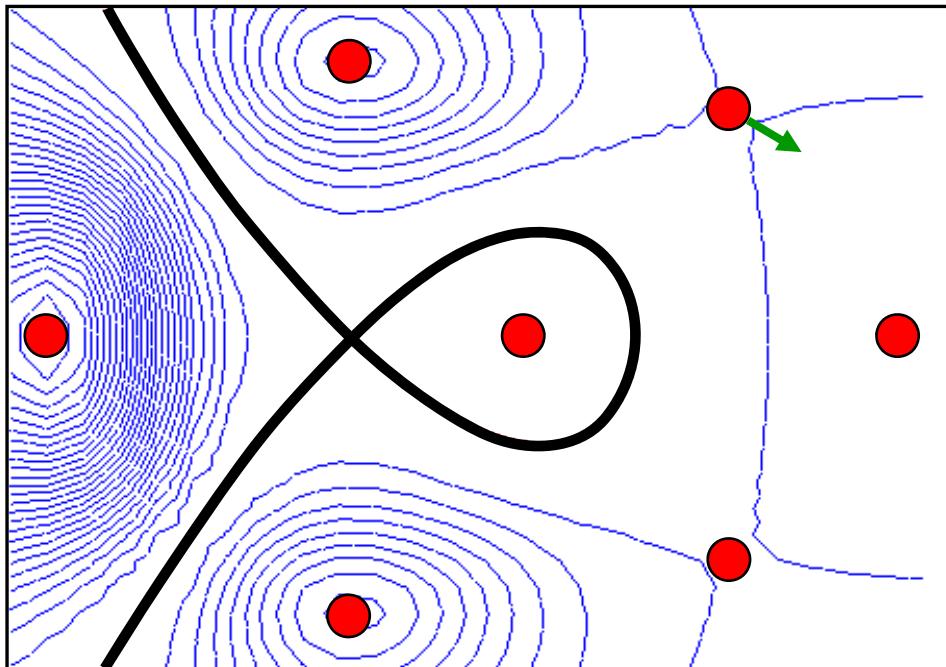
$$f \neq 0$$

Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

$$f \neq 0$$

Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

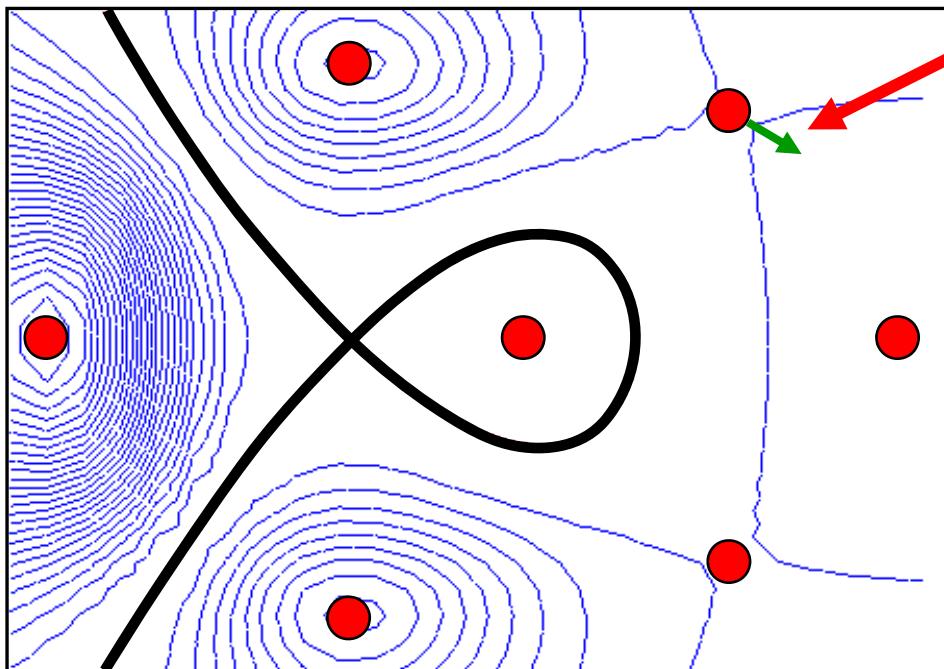
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

eigenvector of  $H$   
with positive eigenvalue



Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

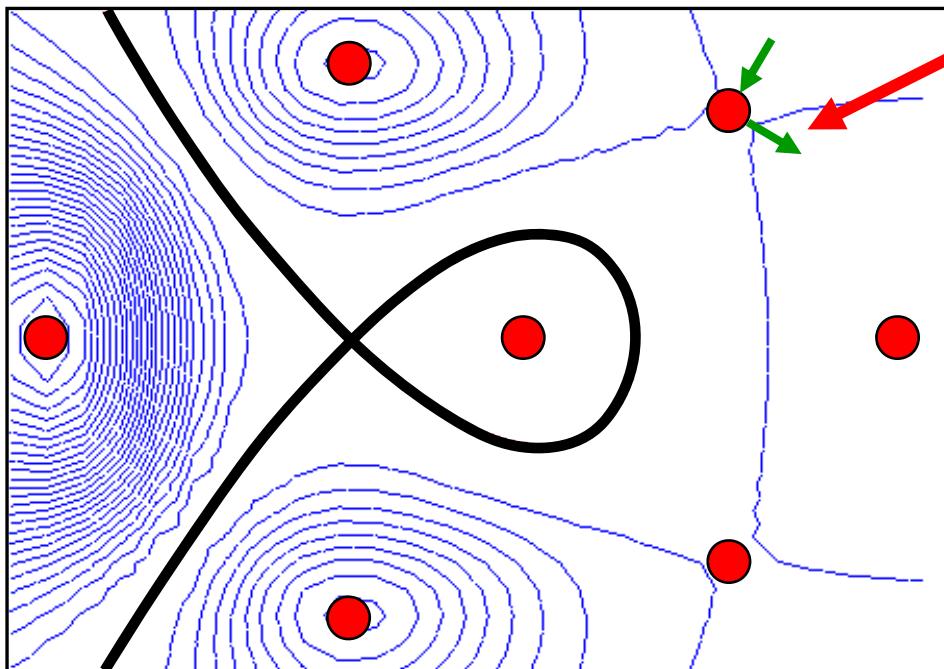
# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$

$$g_x = g_y = 0$$

and

eigenvector of  $H$   
with positive eigenvalue

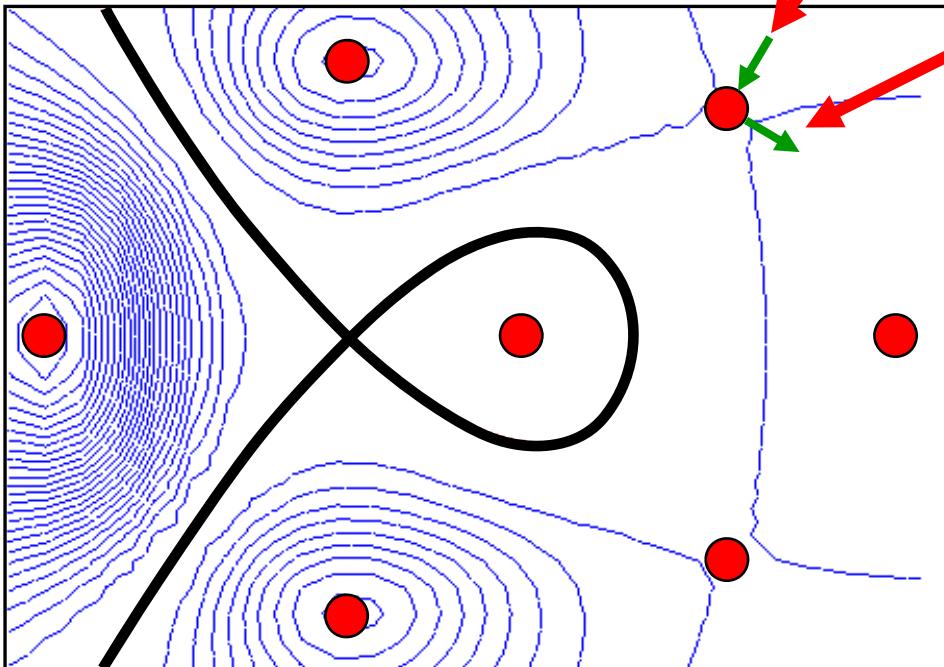


Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



eigenvector of  $H$   
with negative eigenvalue  
and

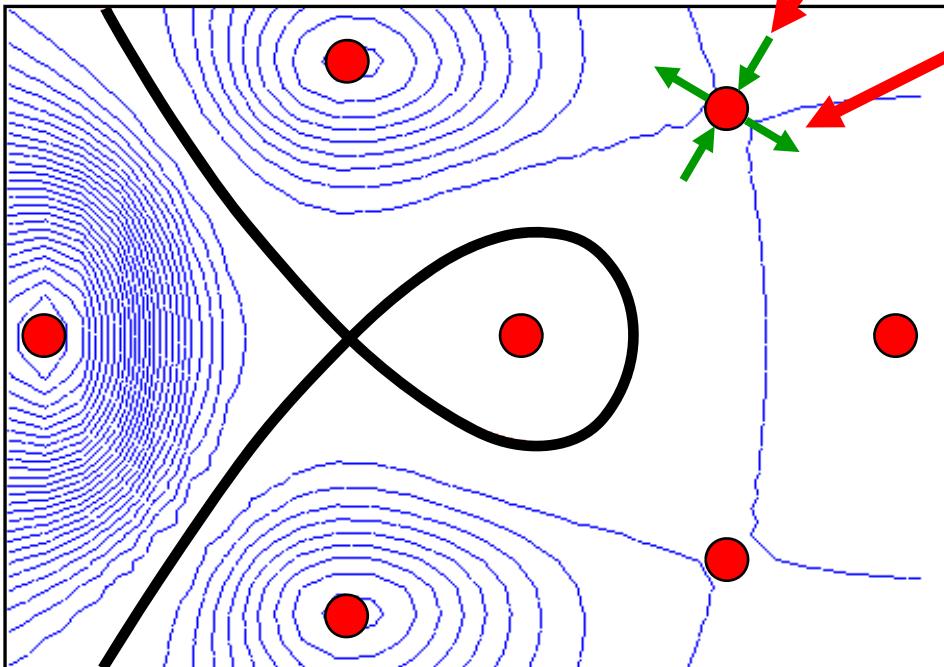
eigenvector of  $H$   
with positive eigenvalue

Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



eigenvector of  $H$   
with negative eigenvalue  
and

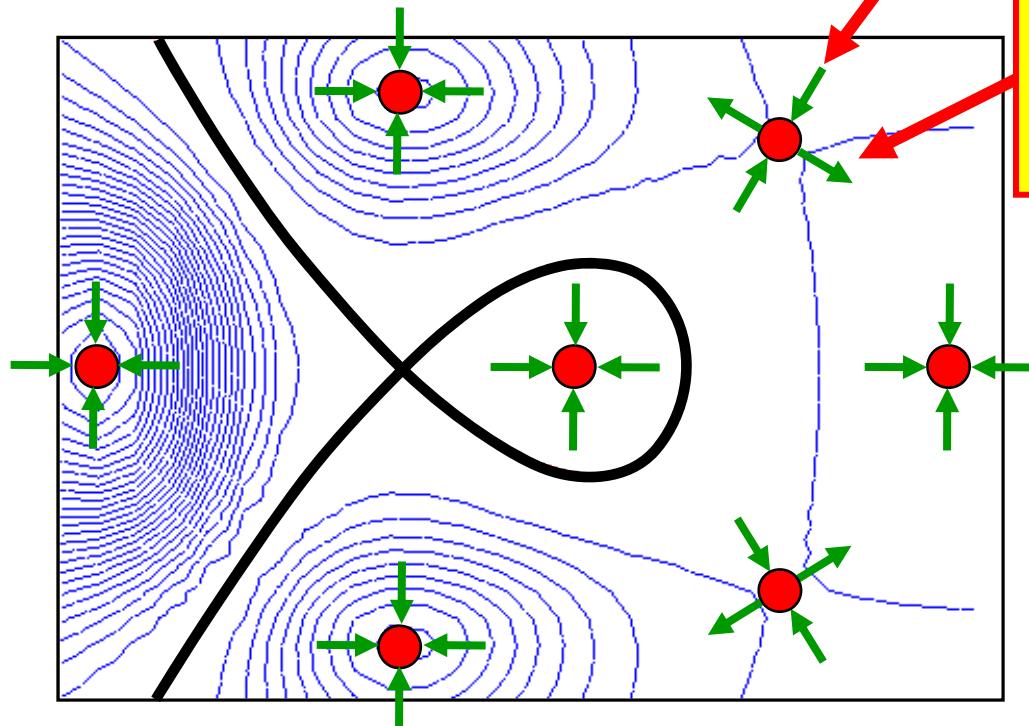
eigenvector of  $H$   
with positive eigenvalue

Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



eigenvector of  $H$   
with negative eigenvalue  
and

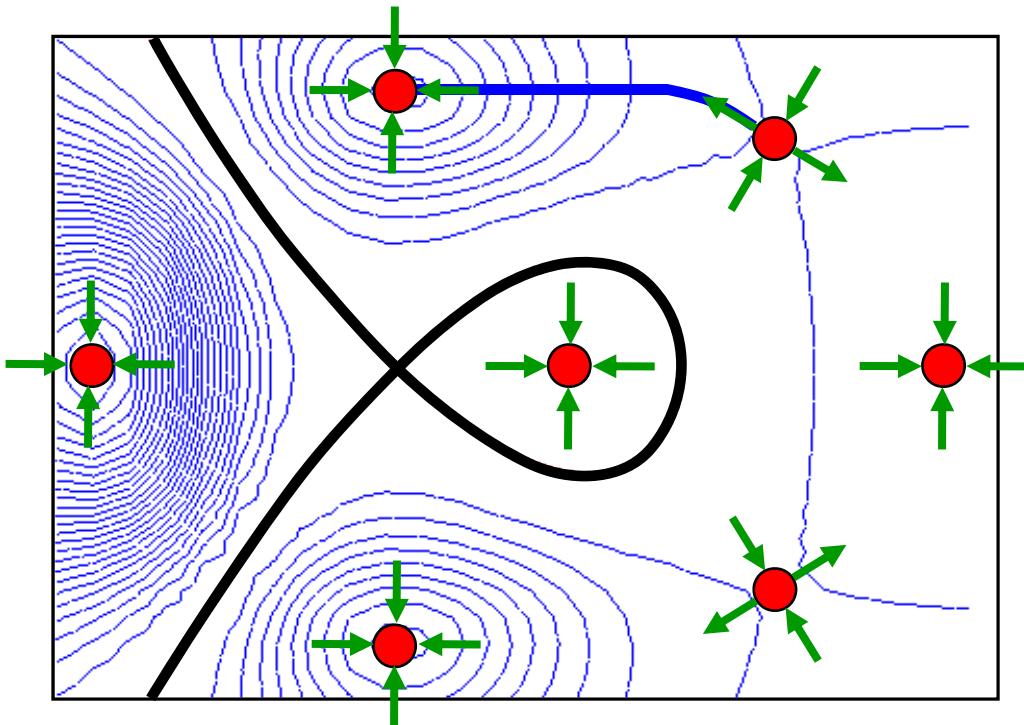
eigenvector of  $H$   
with positive eigenvalue

Hessian of  $g$

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

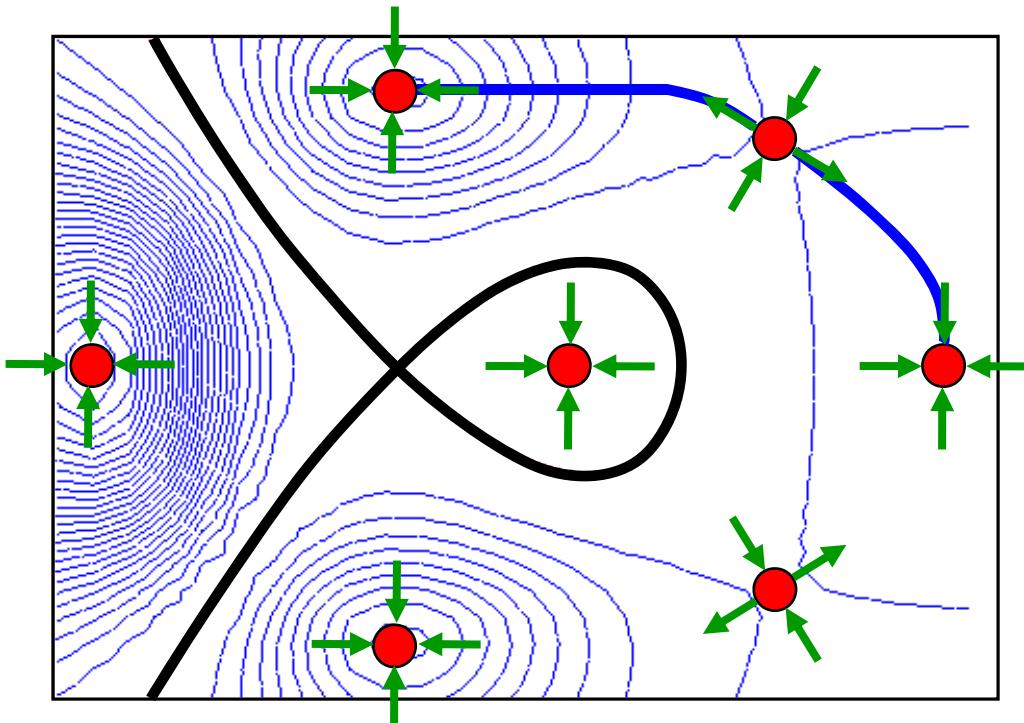
$$f \neq 0$$

Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

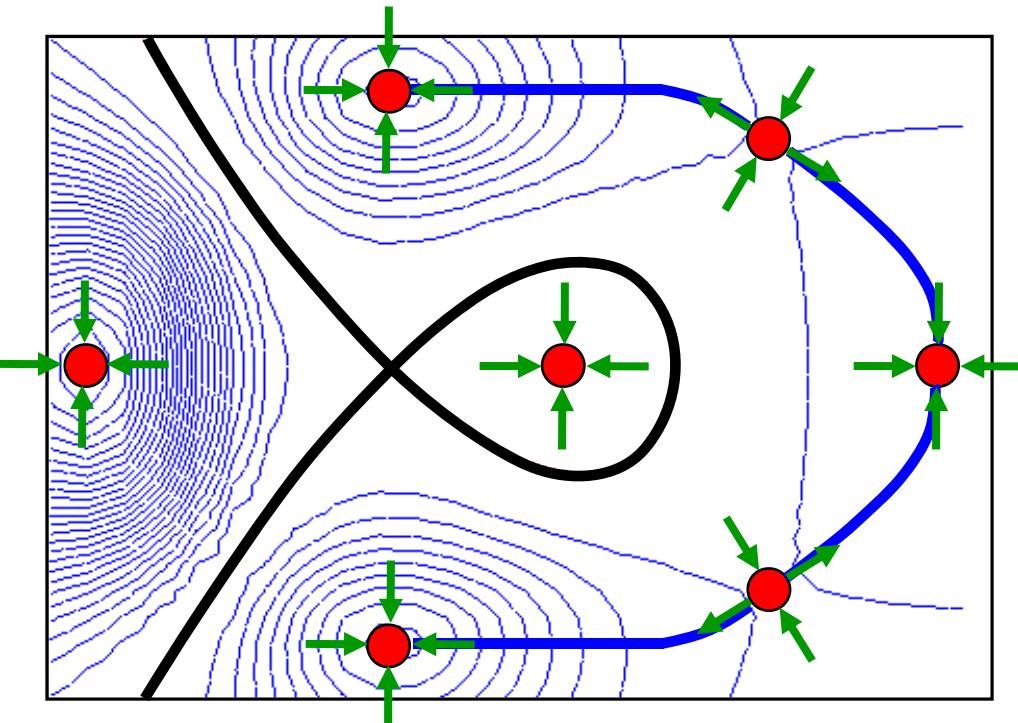
$$f \neq 0$$

Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

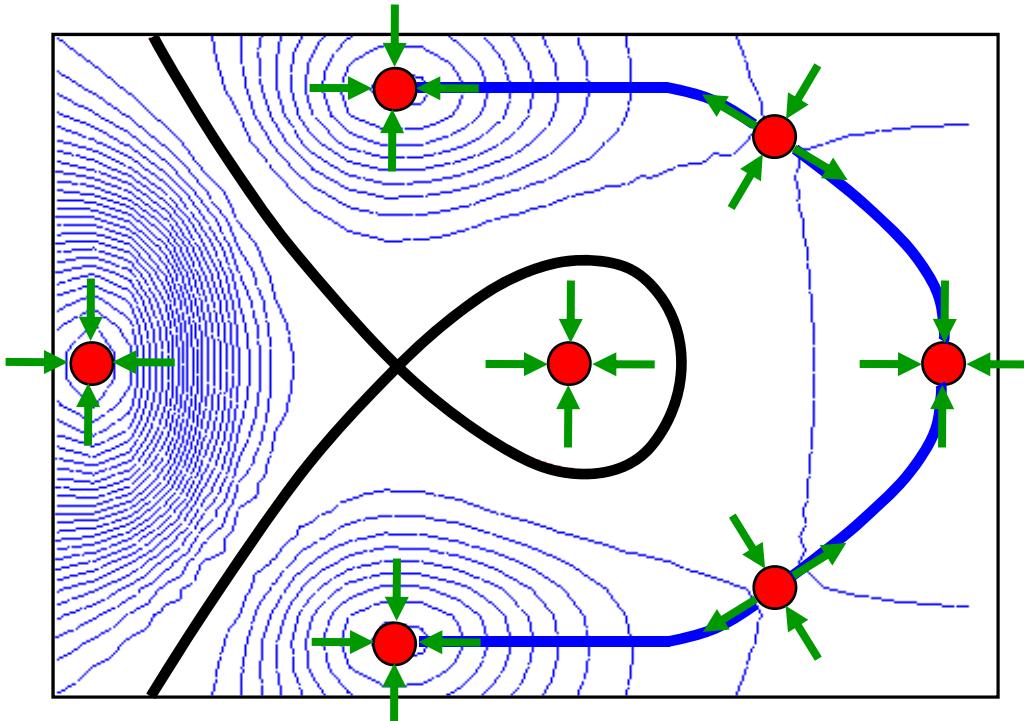
$$f \neq 0$$

Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

$$f \neq 0$$

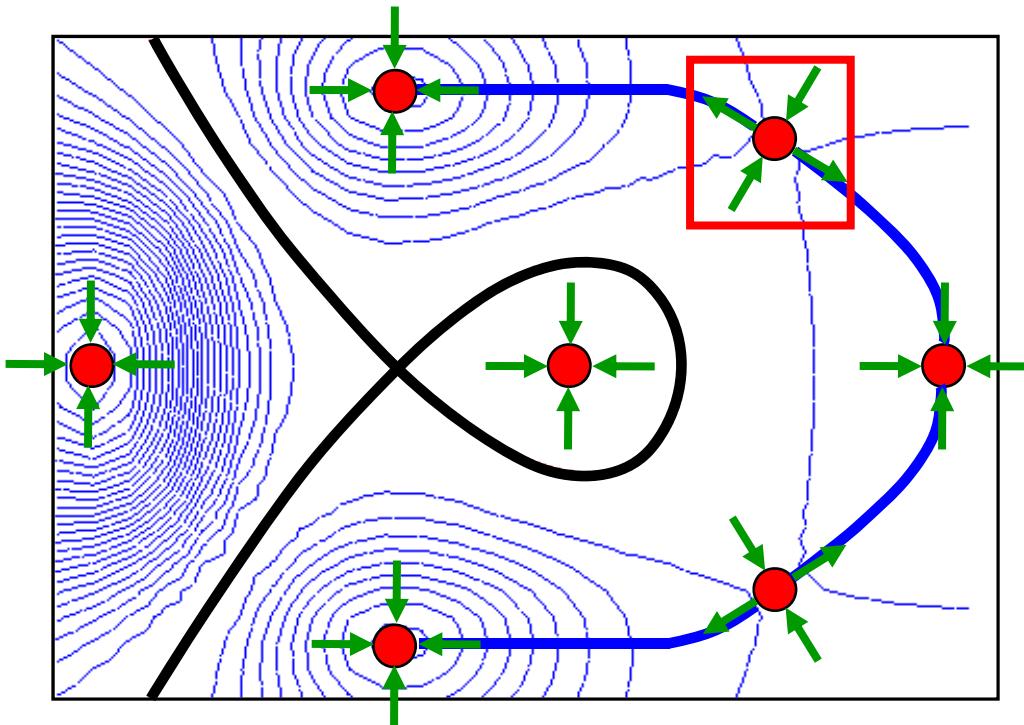
Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

Interval ODE solver

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

$$f \neq 0$$

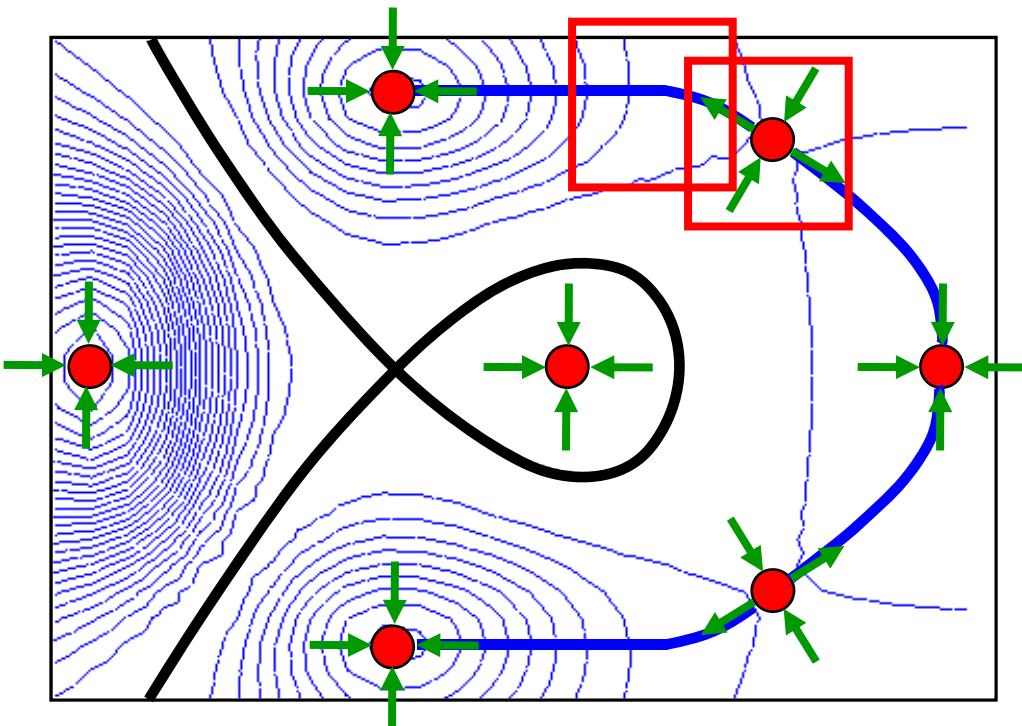
Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

Interval ODE solver

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

$$f \neq 0$$

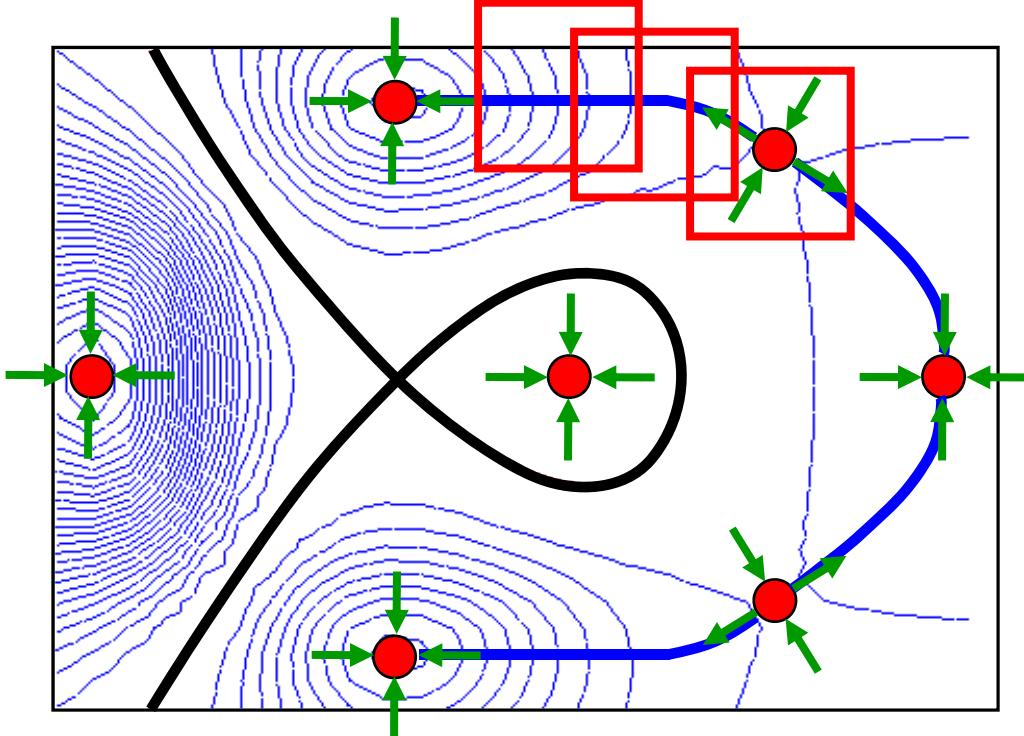
Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

Interval ODE solver

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

$$f \neq 0$$

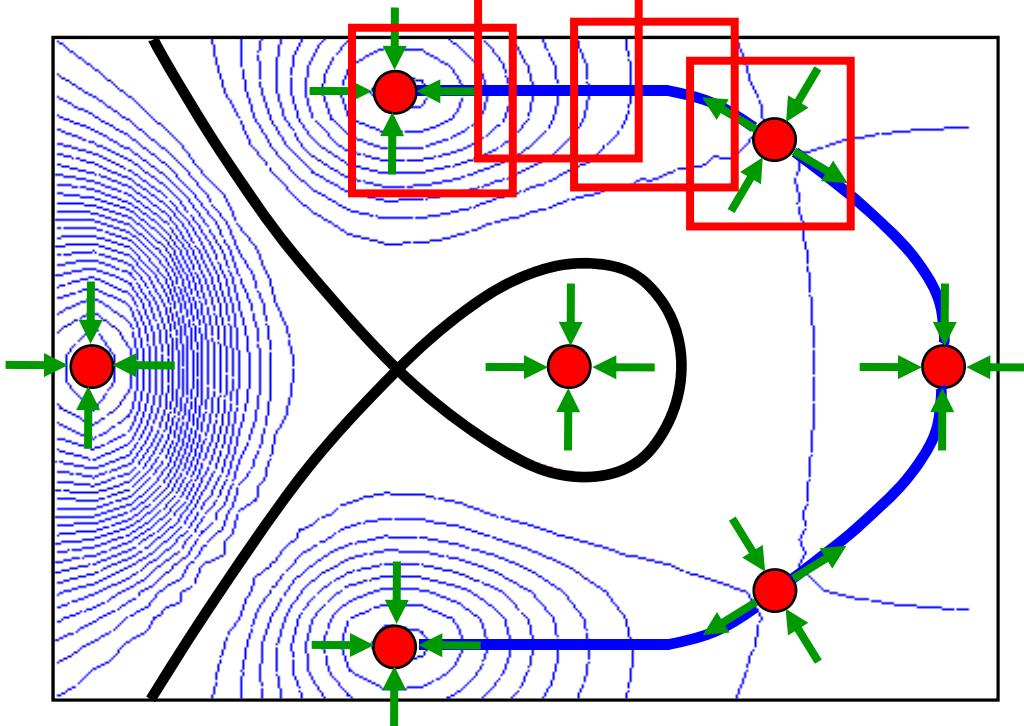
Hessian of  $g$

$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

Interval ODE solver

# Idea

$$g = \frac{f^2}{(x^2 + y^2 + 1)^{d+1}}$$



$$g_x = g_y = 0$$

and

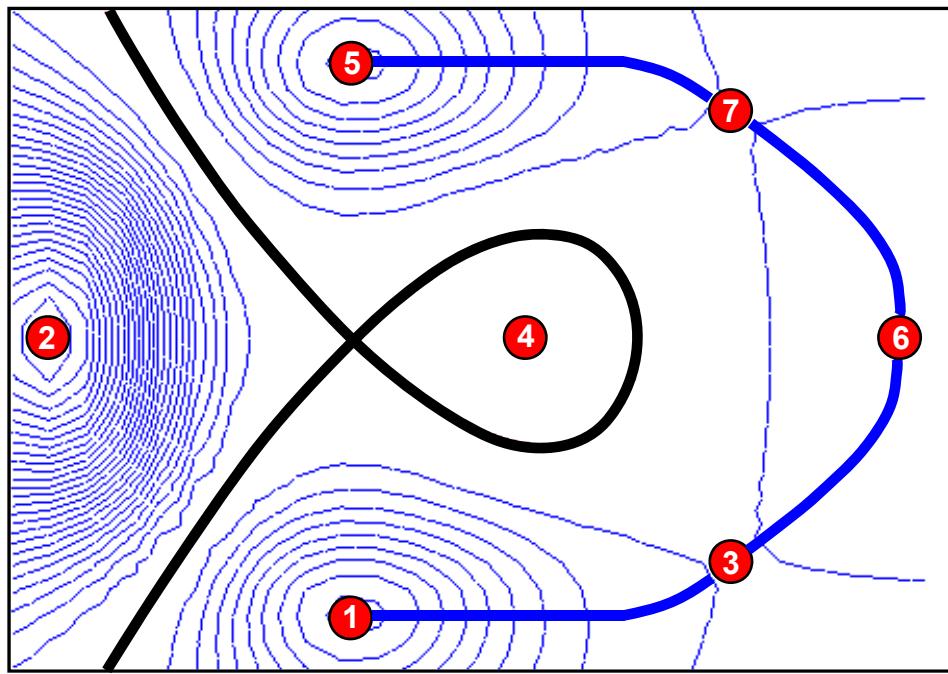
$$f \neq 0$$

Hessian of  $g$

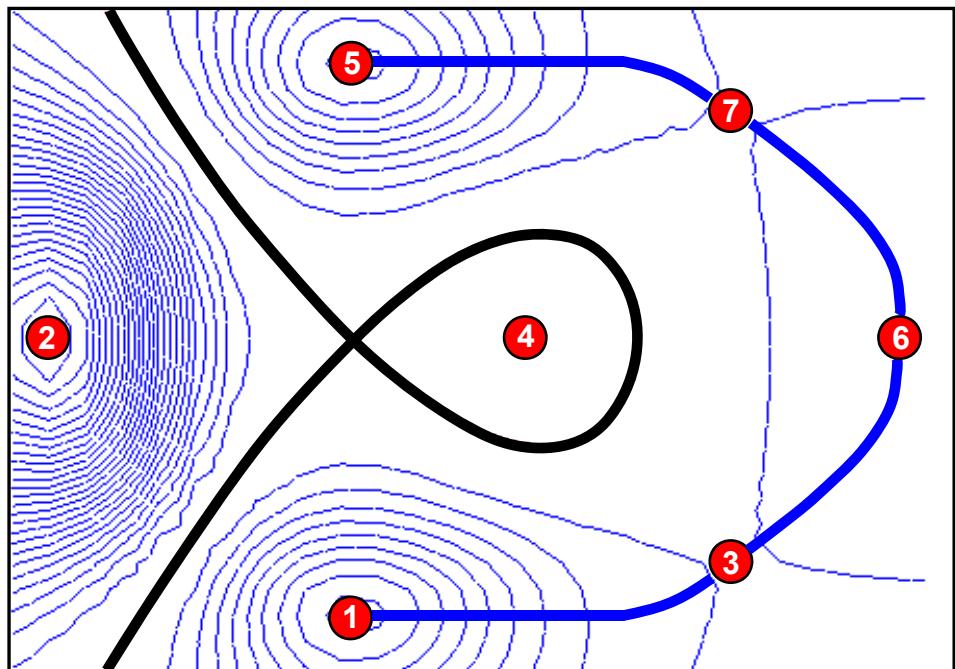
$$\begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

Interval ODE solver

# Idea



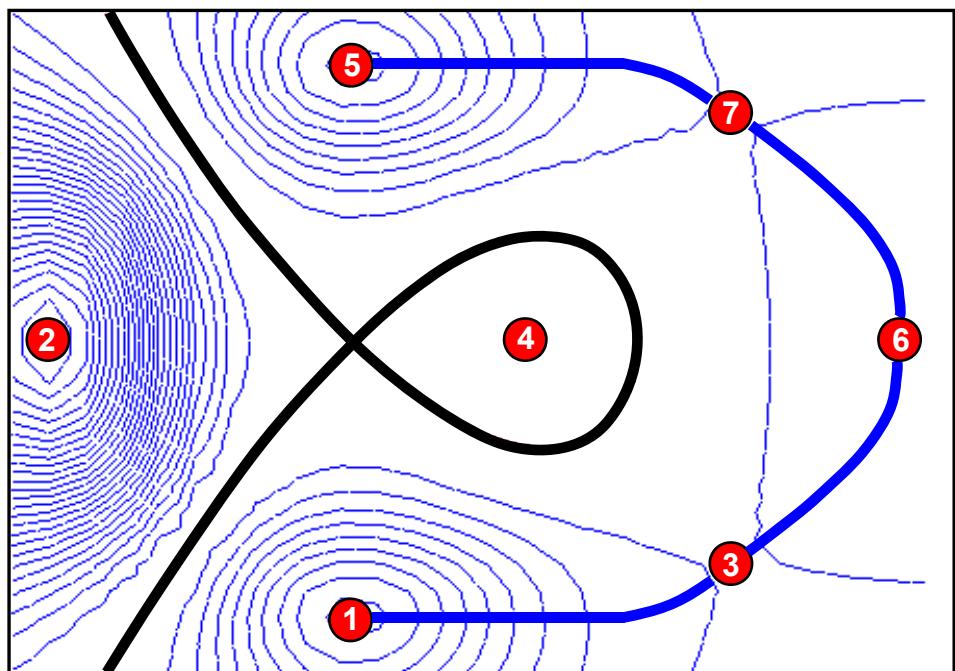
# Idea



## Adjacency

	1	2	3	4	5	6	7
1	0	0	1	0	0	0	0
2	0	0	0	0	0	0	0
3	1	0	0	0	0	1	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	1
6	0	0	1	0	0	0	1
7	0	0	0	0	1	1	0

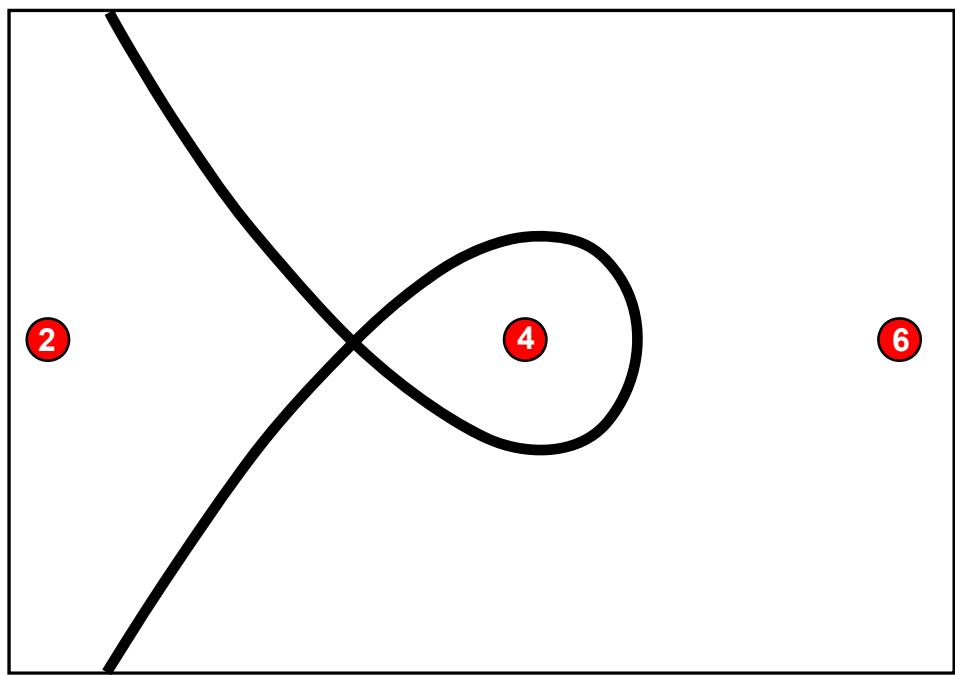
# Idea



# Closure

	1	2	3	4	5	6	7
1	1	0	1	0	1	1	1
2	0	1	0	0	0	0	0
3	1	0	1	0	1	1	1
4	0	0	0	1	0	0	0
5	1	0	1	0	1	1	1
6	1	0	1	0	1	1	1
7	1	0	1	0	1	1	1

# Idea



# Closure

	1	2	3	4	5	6	7
1	1	0	1	0	1	1	1
2	0	1	0	0	0	0	0
3	1	0	1	0	1	1	1
4	0	0	0	1	0	0	0
5	1	0	1	0	1	1	1
6	1	0	1	0	1	1	1
7	1	0	1	0	1	1	1

# Error function

# Problem

Input:  $f(x)$

Output:  $\exists x \text{ s.t. } f(x) > 0$

$$x = (x_1, \dots, x_n)$$

$$f = (f_1, \dots, f_m)$$

Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$

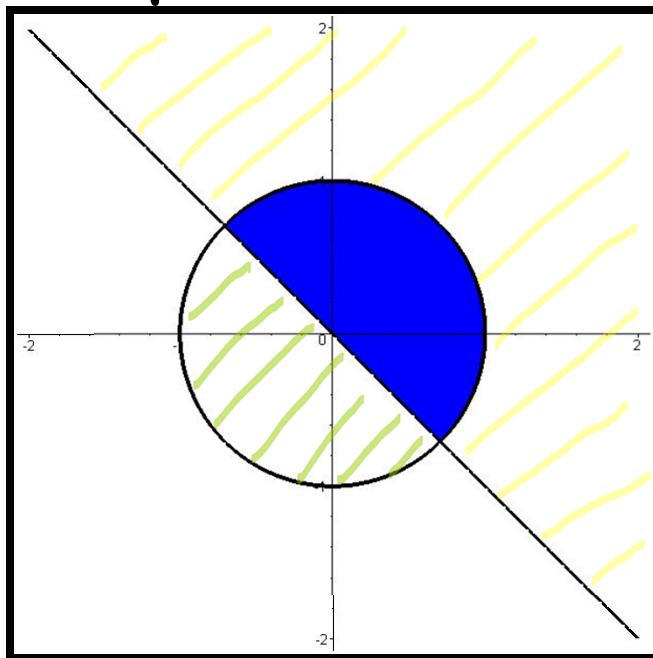
$$x_1 + x_2$$

Output :

Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2$

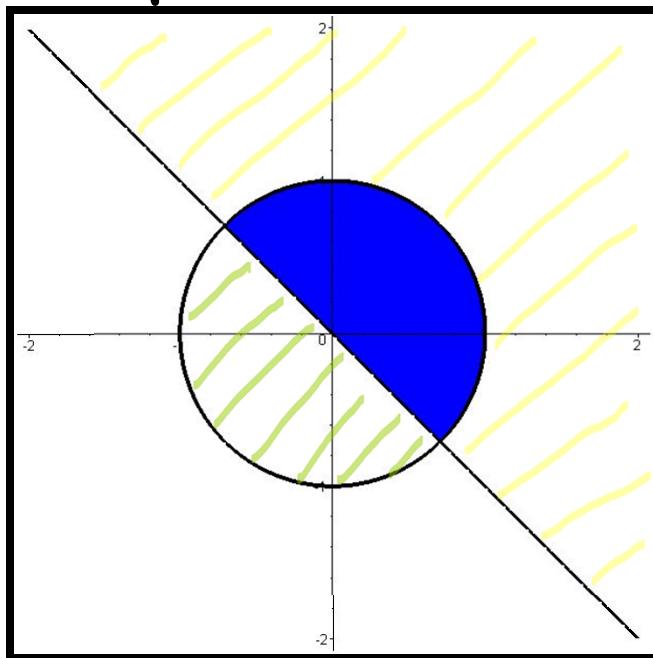
Output :



Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2$

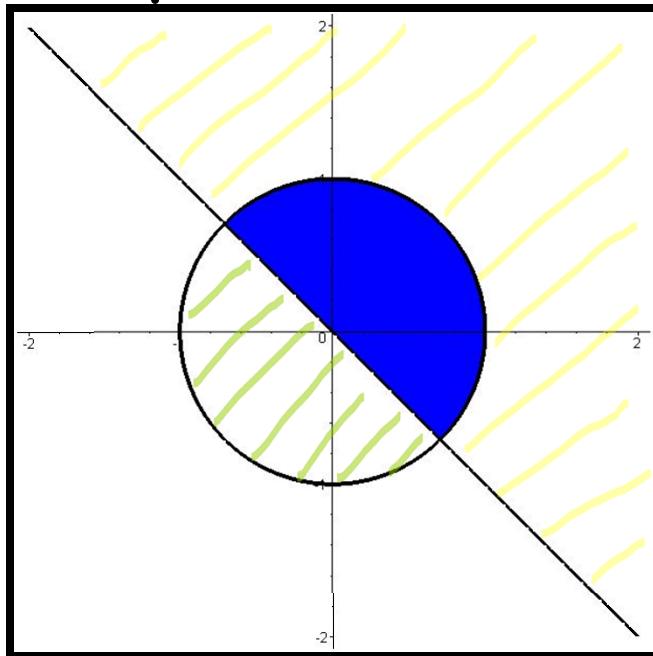
Output : True



Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2$

Output : True



Ex. 2

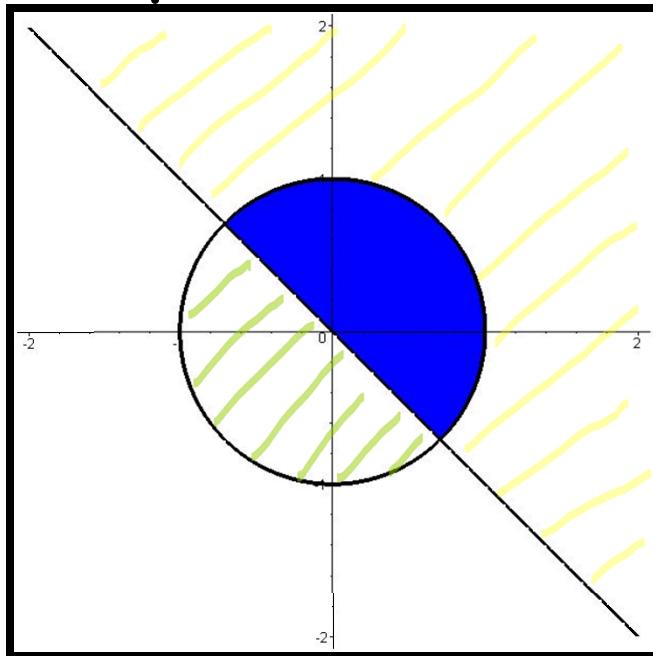
Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2 - 2$

Output:

Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2$

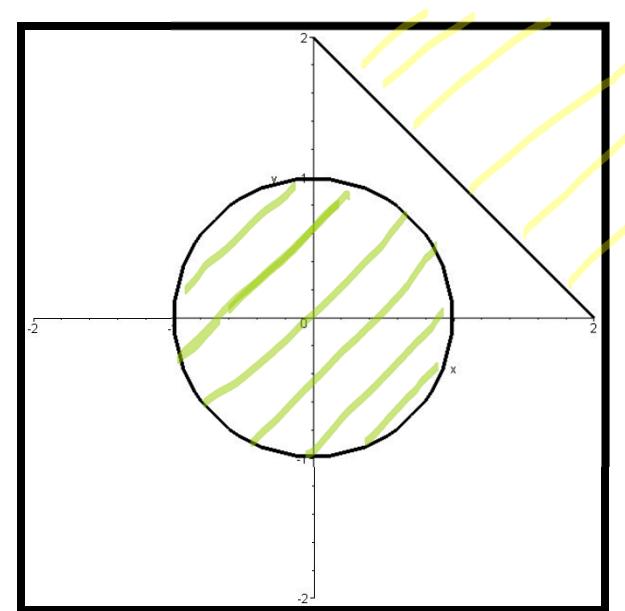
Output : True



Ex. 2

Input :  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2 - 2$

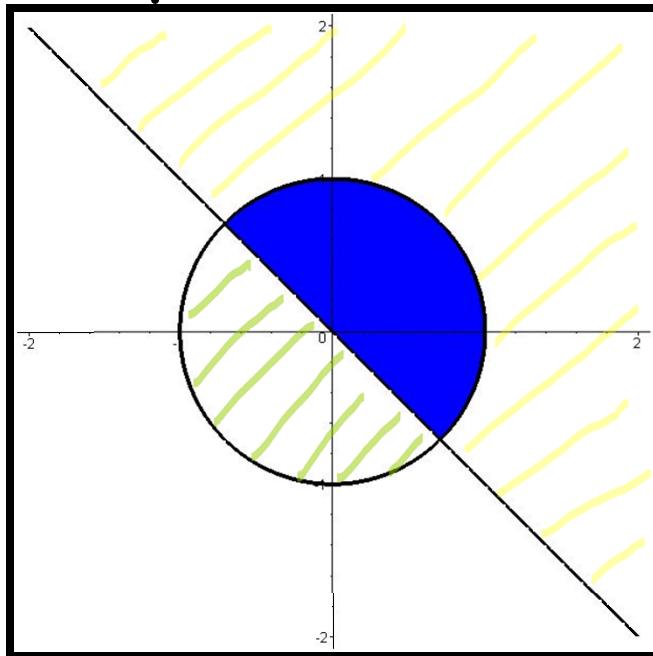
Output:



Ex. 1

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2$

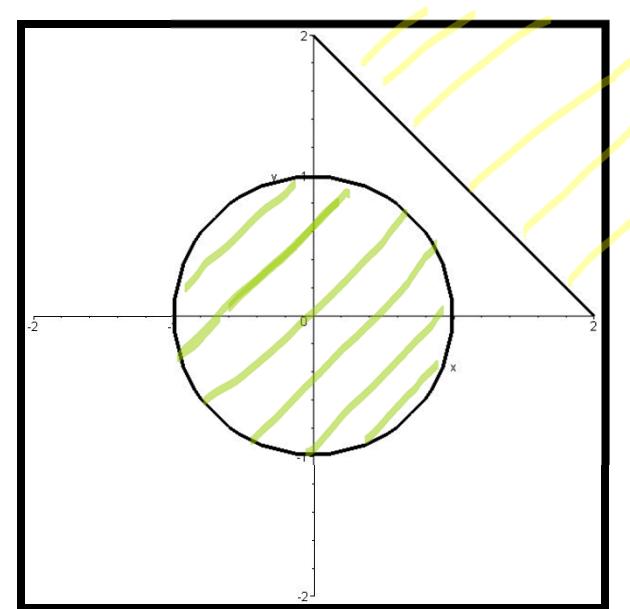
Output : True



Ex. 2

Input:  $-x_1^2 - x_2^2 + 1$   
 $x_1 + x_2 - 2$

Output: False



Ex. 3

Input :  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$

$$x_1^2 + x_2^2 - 3$$
$$-x_1 - x_2$$

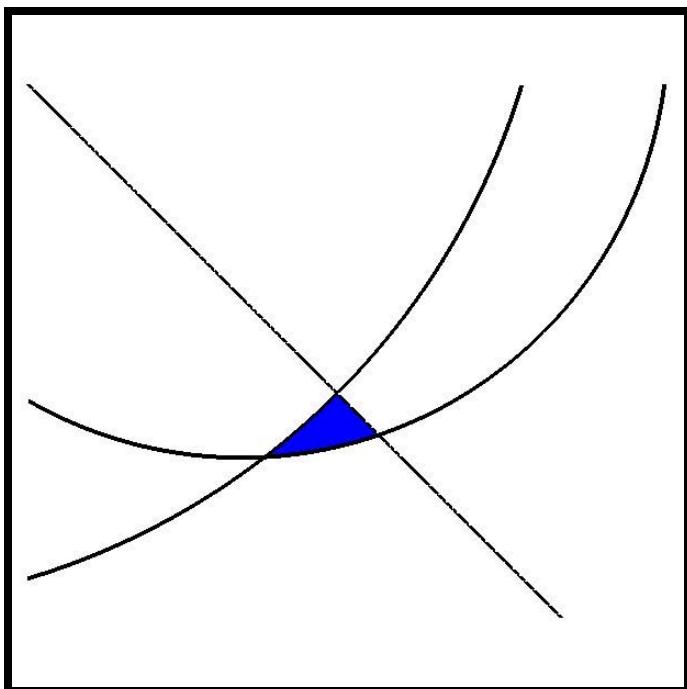
Output :

Ex. 3

Input :-  $(x_1 - 1)^2 + (x_2 - \frac{3}{8})^2 + 1$

$$x_1^2 + x_2^2 - 3$$
$$-x_1 - x_2$$

Output :

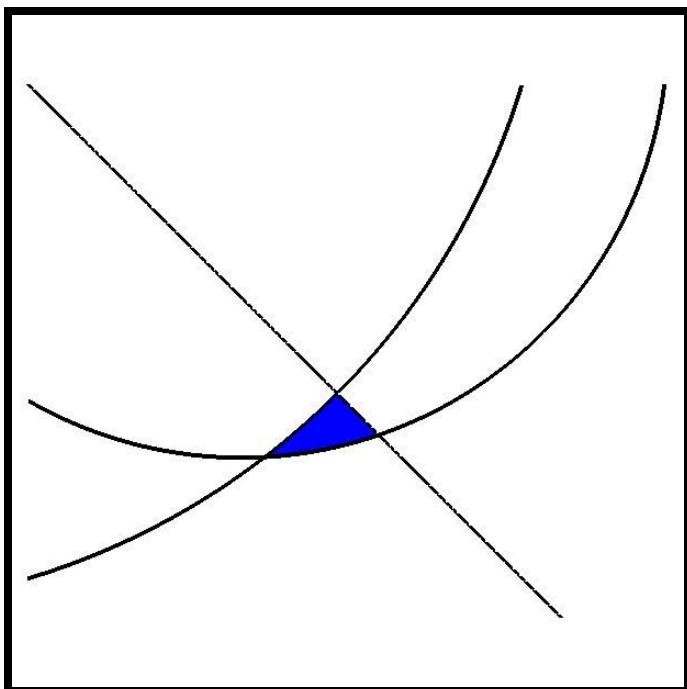


Ex. 3

Input:  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$

$$x_1^2 + x_2^2 - 3$$
$$-x_1 - x_2$$

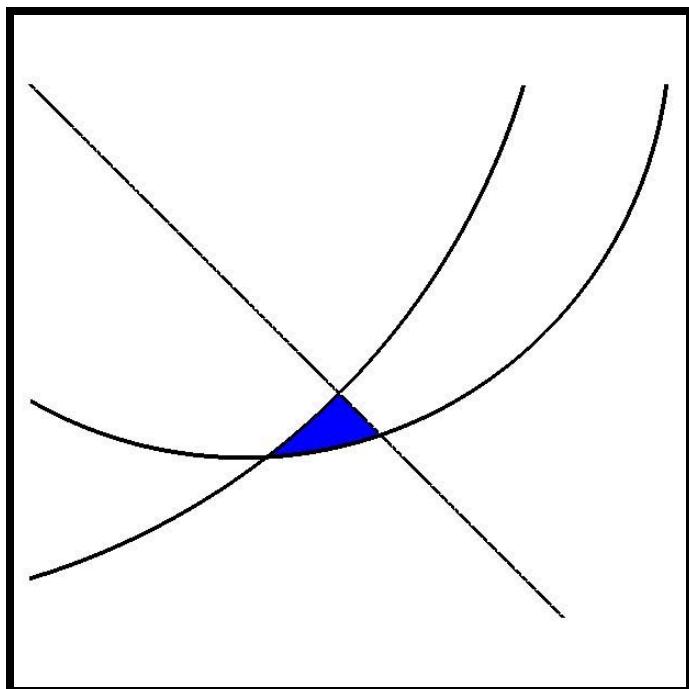
Output : True



Ex. 3

Input :  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$   
 $x_1^2 + x_2^2 - 3$   
 $-x_1 - x_2$

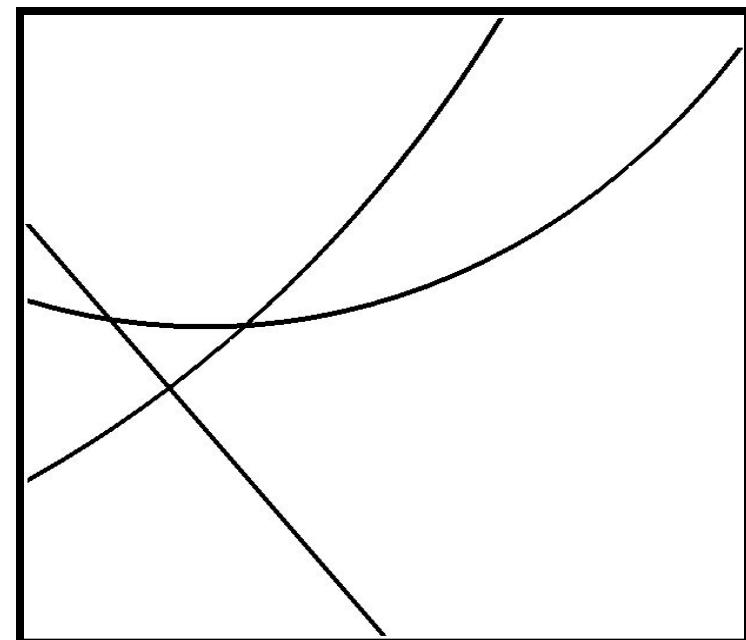
Output : True



Ex. 4

Input :  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$   
 $x_1^2 + x_2^2 - 3$   
 $-x_1 - x_2 - \frac{1}{2}$

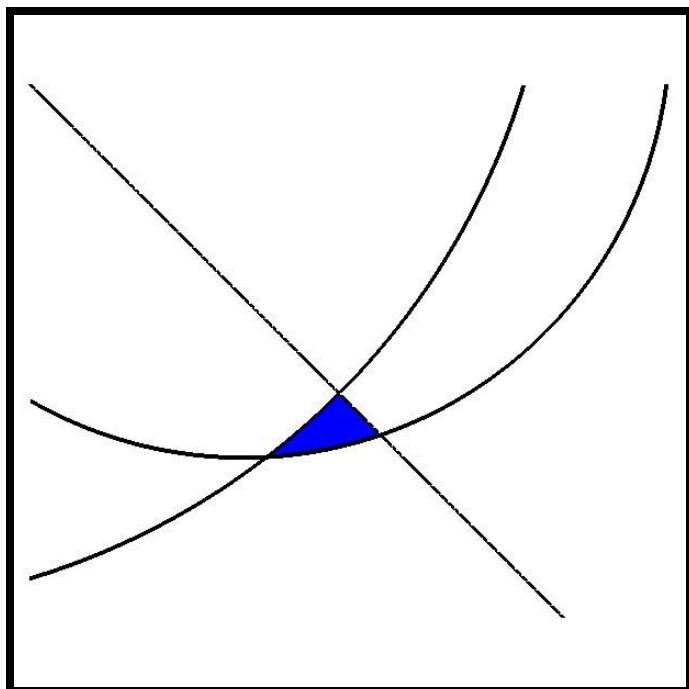
Output:



Ex. 3

Input:  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$   
 $x_1^2 + x_2^2 - 3$   
 $-x_1 - x_2$

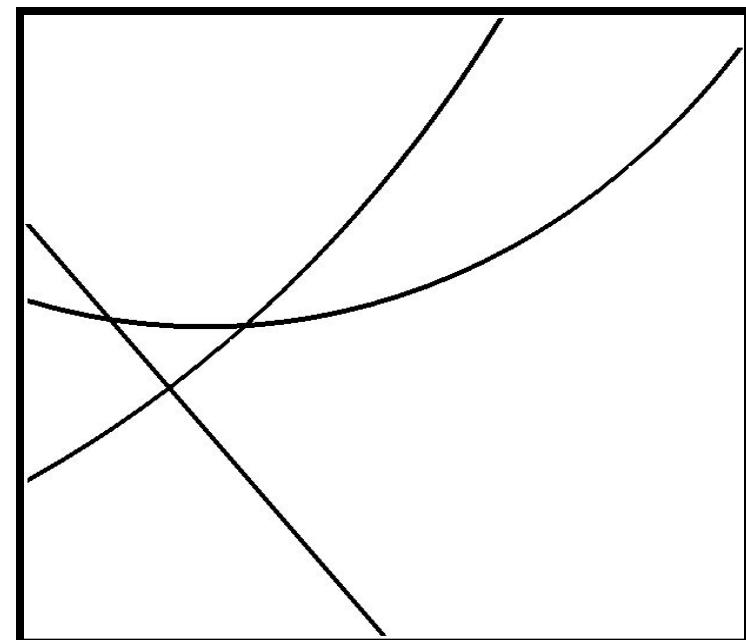
Output : True



Ex. 4

Input:  $-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1$   
 $x_1^2 + x_2^2 - 3$   
 $-x_1 - x_2 - \frac{1}{2}$

Output: False



Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3  
x3^2 + 2 x 1 x3^4 x4 - 2  
x1 x3 x4^4 - 8  
  
4 x1 x4 + 3 x2 x3 x4 + 5  
x1^2 x3 x4 + 7 x1 x2 x3^3  
- 1  
  
5 x1 x2 x3 - 5 x1 x2 x3^2 +  
7 x1^3 x3^2 - 5
```

# vars = 4

# polys = 3

degree = 6

Output:

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3  
x3^2 + 2 x 1 x3^4 x4 - 2  
x1 x3 x4^4 - 8  
  
4 x1 x4 + 3 x2 x3 x4 + 5  
x1^2 x3 x4 + 7 x1 x2 x3^3  
- 1  
  
5 x1 x2 x3 - 5 x1 x2 x3^2 +  
7 x1^3 x3^2 - 5
```

# vars = 4

# polys = 3

degree = 6

Output: **True**

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3  
x3^2 + 2 x 1 x3^4 x4 - 2  
x1 x3 x4^4 - 8  
  
4 x1 x4 + 3 x2 x3 x4 + 5  
x1^2 x3 x4 + 7 x1 x2 x3^3  
- 1  
  
5 x1 x2 x3 - 5 x1 x2 x3^2 +  
7 x1^3 x3^2 - 5
```

# vars = 4

# polys = 3  
degree = 6

Output: **True**

Ex. 6

Input:

```
5 x3^4 + x1^2 x3^3 - 7 x1 x2  
x3^2 x4 - 5 x1^3 x3^2 x4 + 4  
x1 x2^2 x4^3 - 3  
  
- 6 x2 x3 - 2 x1^3 x2^2 - 8 x2^4  
x3 + x2^2 x3^3 + x1 x2^3 x3^2  
- 4  
  
x3^3 - x1^2 x2^2 - 7 x1^3 x4^2  
+ 8 x2 x3^4 + 7 x1^2 x2^2 x4^2  
- 9
```

# vars = 4

# polys = 3  
degree = 6

Output:

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3  
x3^2 + 2 x 1 x3^4 x4 - 2  
x1 x3 x4^4 - 8  
  
4 x1 x4 + 3 x2 x3 x4 + 5  
x1^2 x3 x4 + 7 x1 x2 x3^3  
- 1  
  
5 x1 x2 x3 - 5 x1 x2 x3^2 +  
7 x1^3 x3^2 - 5
```

# vars = 4

# polys = 3

degree = 6

Output: True

Ex. 6

Input:

```
5 x3^4 + x1^2 x3^3 - 7 x1 x2  
x3^2 x4 - 5 x1^3 x3^2 x4 + 4  
x1 x2^2 x4^3 - 3  
  
- 6 x2 x3 - 2 x1^3 x2^2 - 8 x2^4  
x3 + x2^2 x3^3 + x1 x2^3 x3^2  
- 4  
  
x3^3 - x1^2 x2^2 - 7 x1^3 x4^2  
+ 8 x2 x3^4 + 7 x1^2 x2^2 x4^2  
- 9
```

# vars = 4

# polys = 3

degree = 6

Output: False

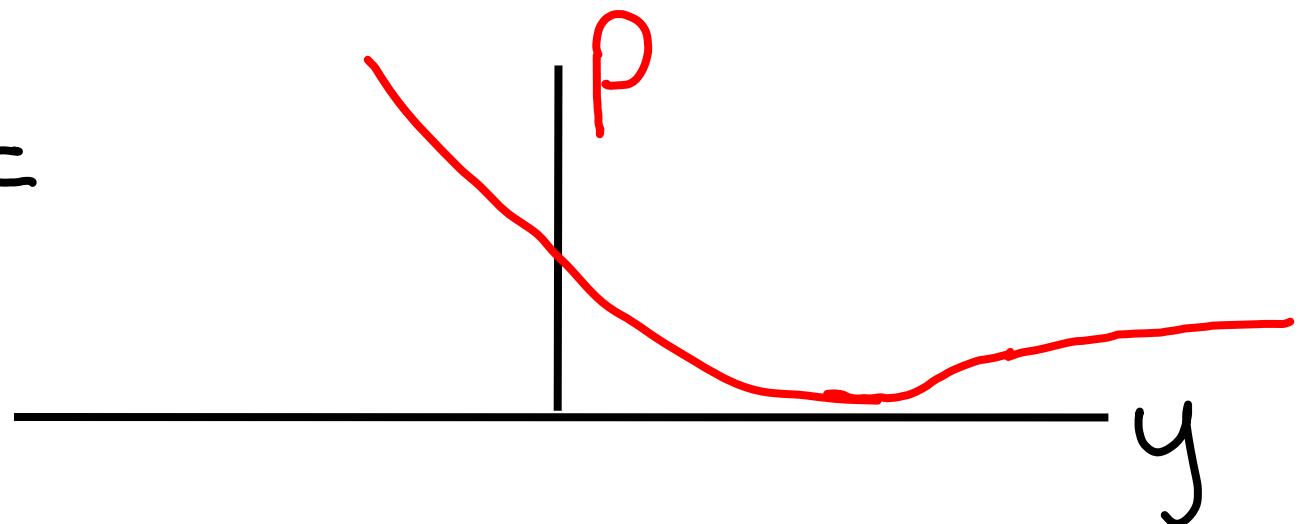
# Error function

$$e(x) = p(f_1(x)) + \dots + P(f_m(x))$$

# Error function

$$c(x) = p(f_1(x)) + \dots + p(f_m(x))$$

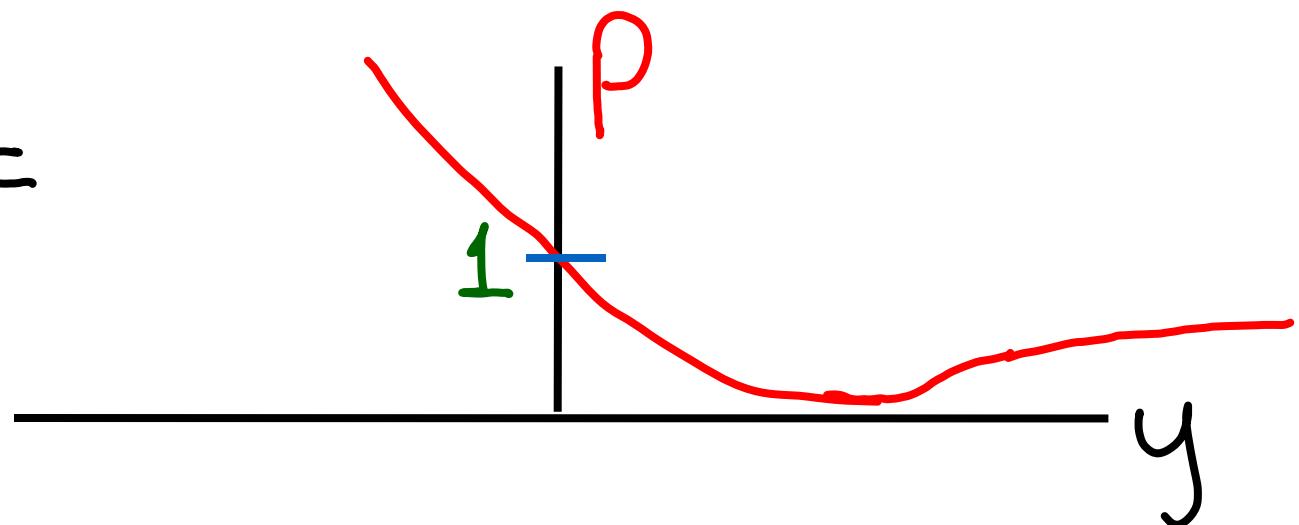
$$p(y) =$$



# Error function

$$c(x) = p(f_1(x)) + \dots + p(f_m(x))$$

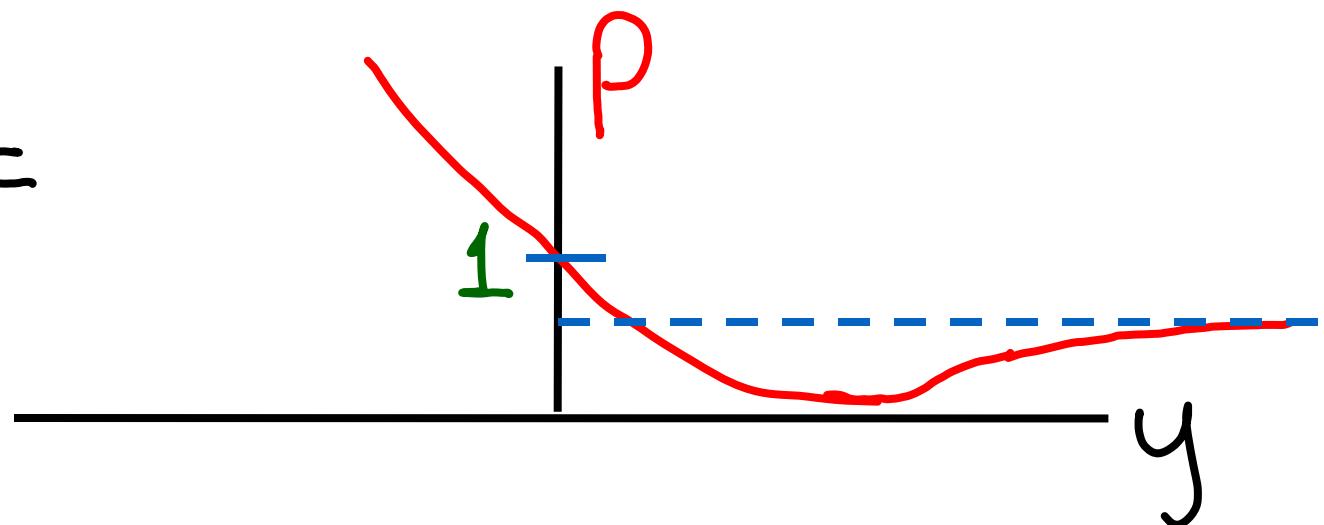
$$p(y) =$$



# Error function

$$c(x) = p(f_1(x)) + \dots + p(f_m(x))$$

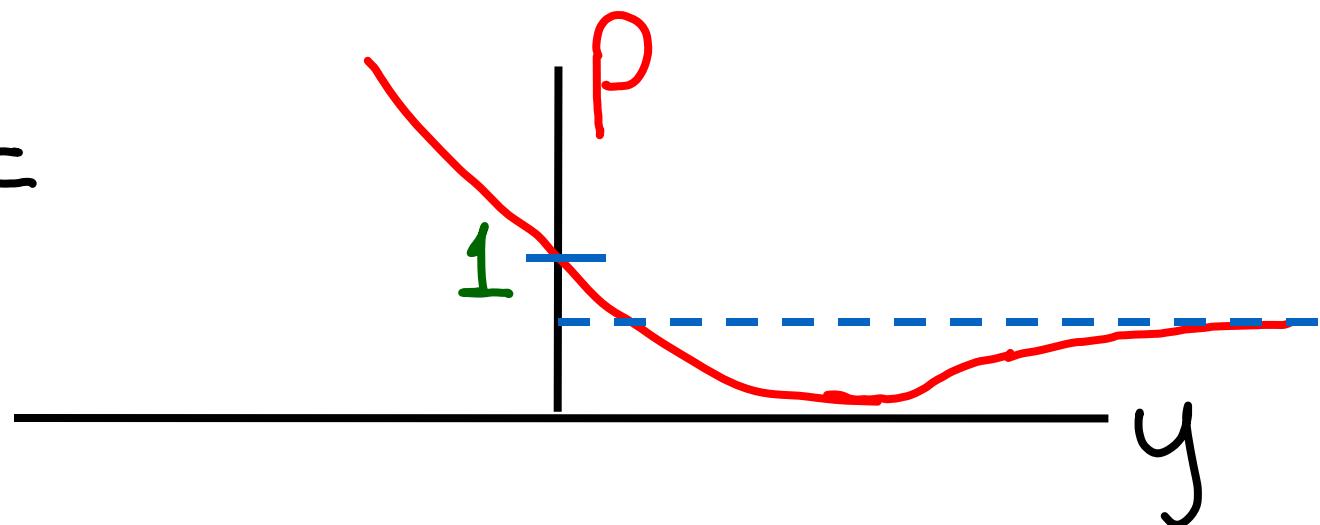
$$p(y) =$$



# Error function

$$c(x) = p(f_1(x)) + \dots + p(f_m(x))$$

$$p(y) =$$



# Current Choice

$$p(y) = -\frac{y + \sqrt{y^2 + 4}}{2} - \left( -\frac{0.05y + \sqrt{0.05y^2 + 4}}{2m} \right) + \frac{1}{m}$$

# Error function

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$

# Error function

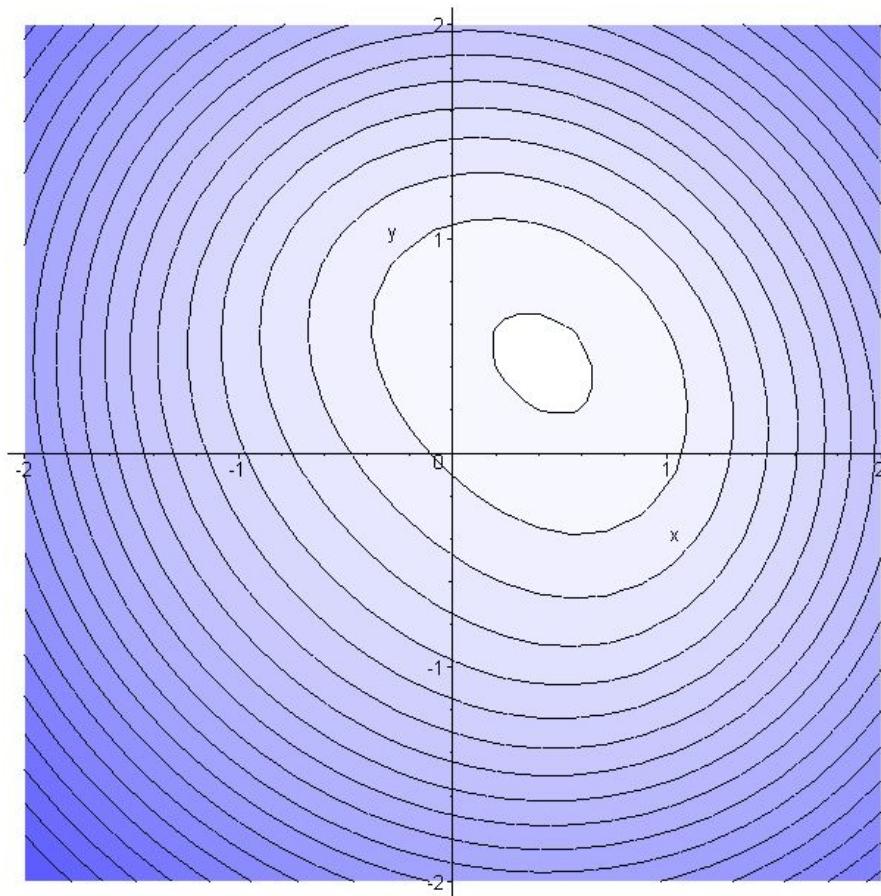
$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$

$$e(x) =$$

# Error function

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$

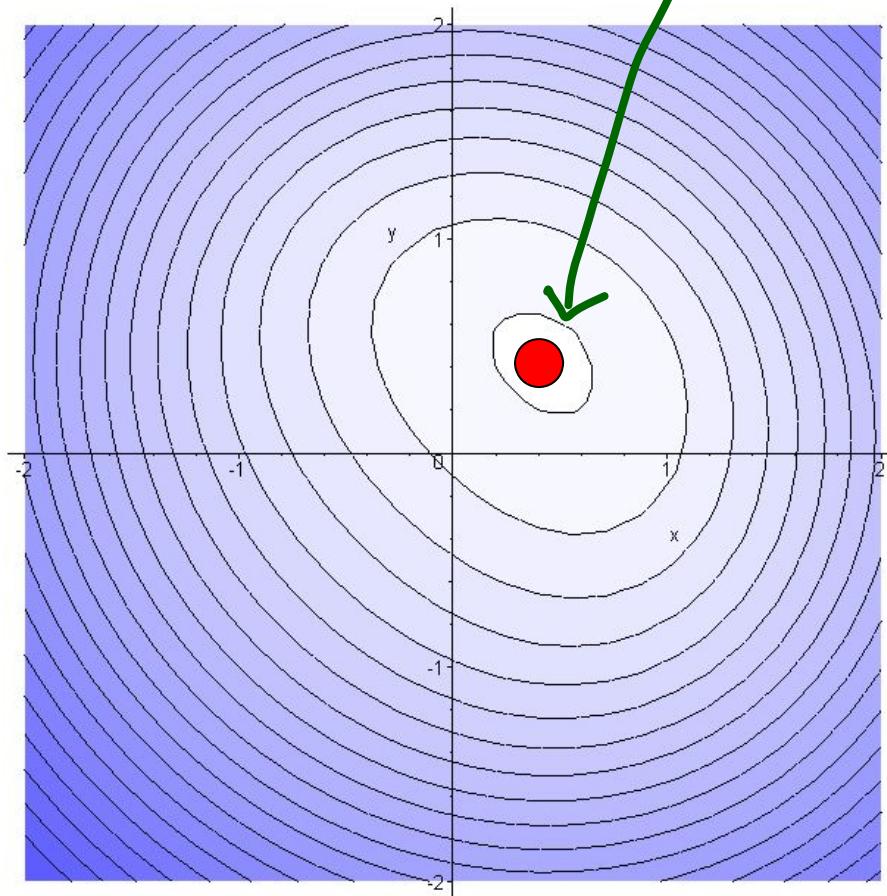
$$e(x) =$$



# Error function

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$

$$e(x) =$$



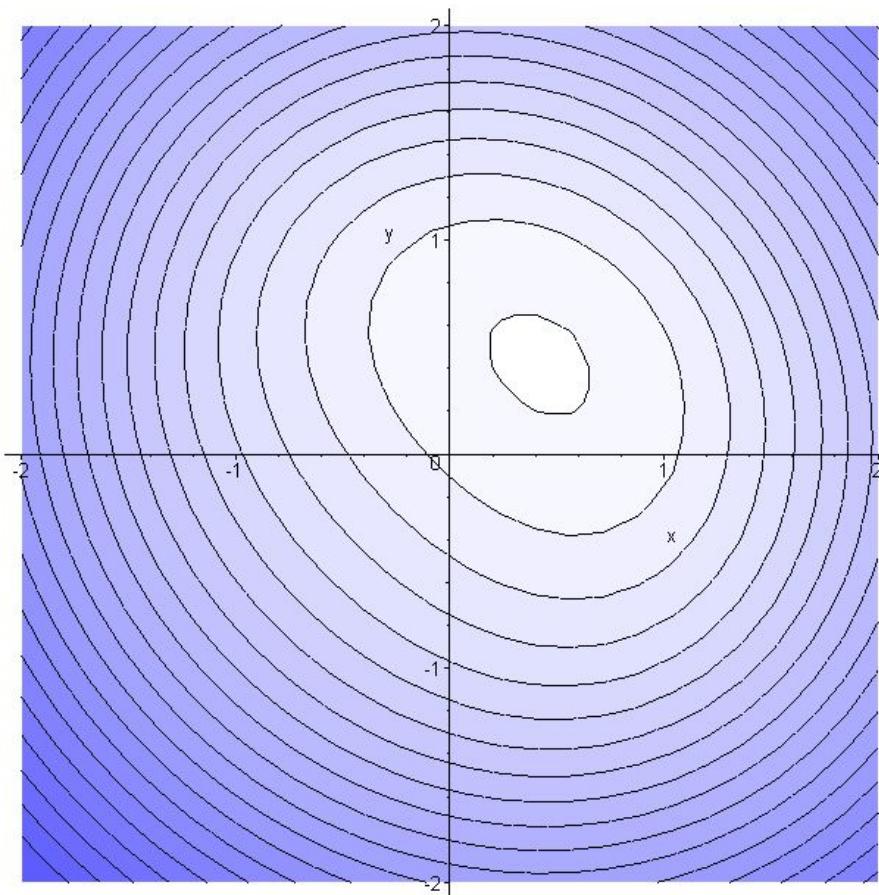
Sample and Test

# Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$

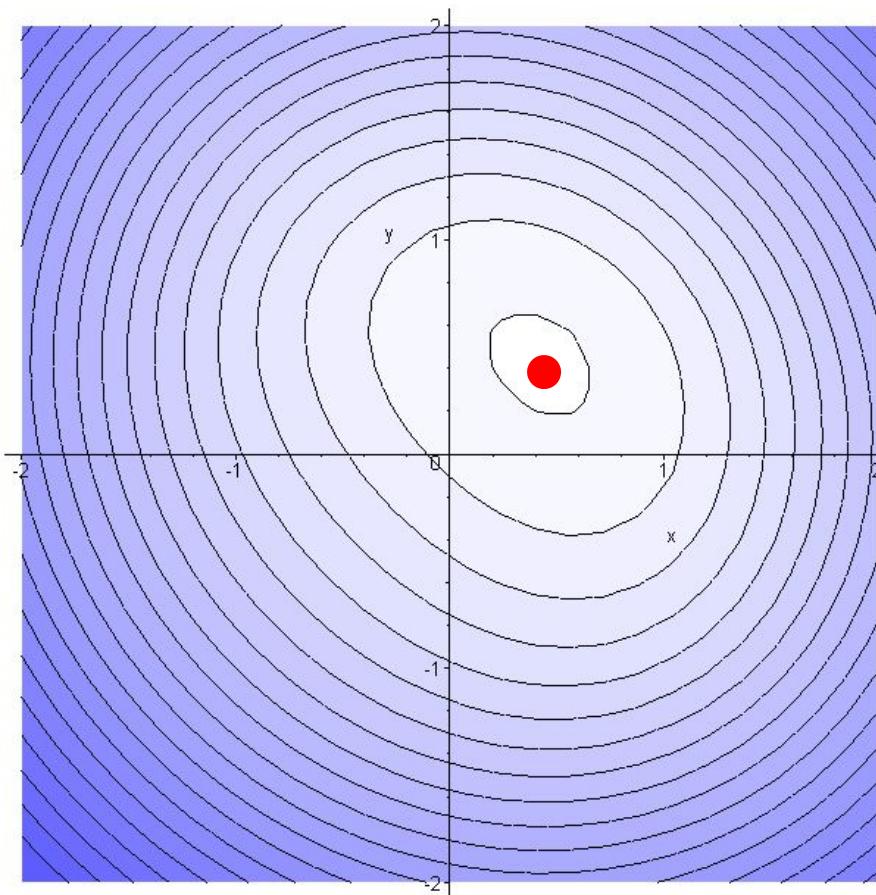
# Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



# Sample and Test Ex. 1

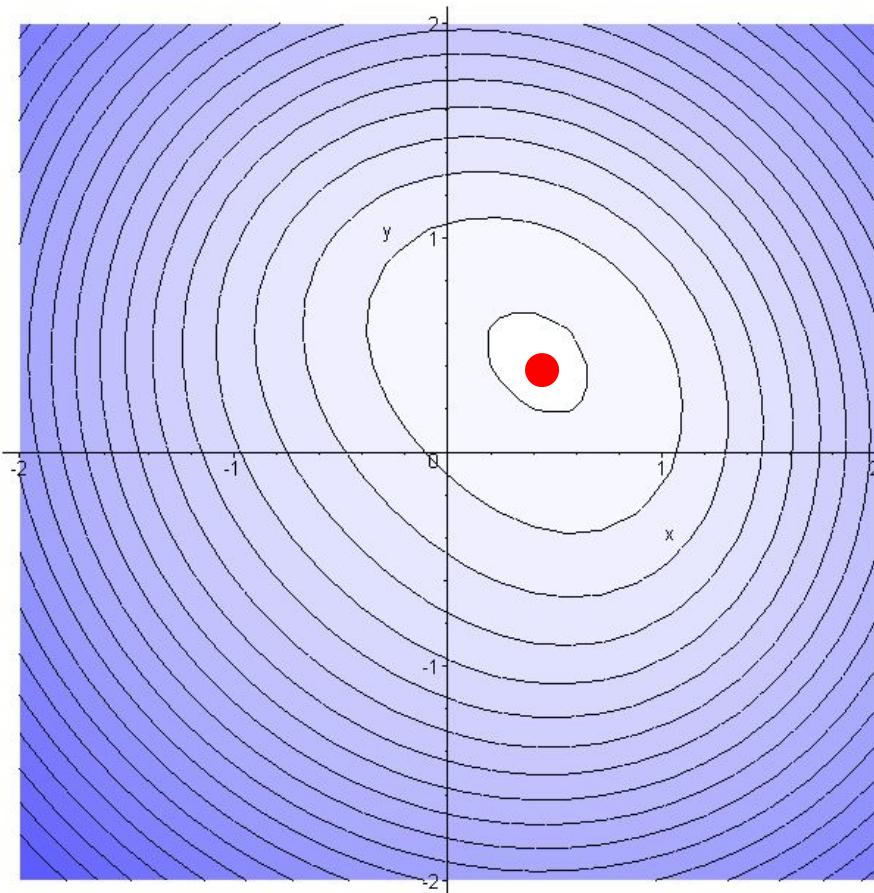
$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



# Sample and Test Ex. 1

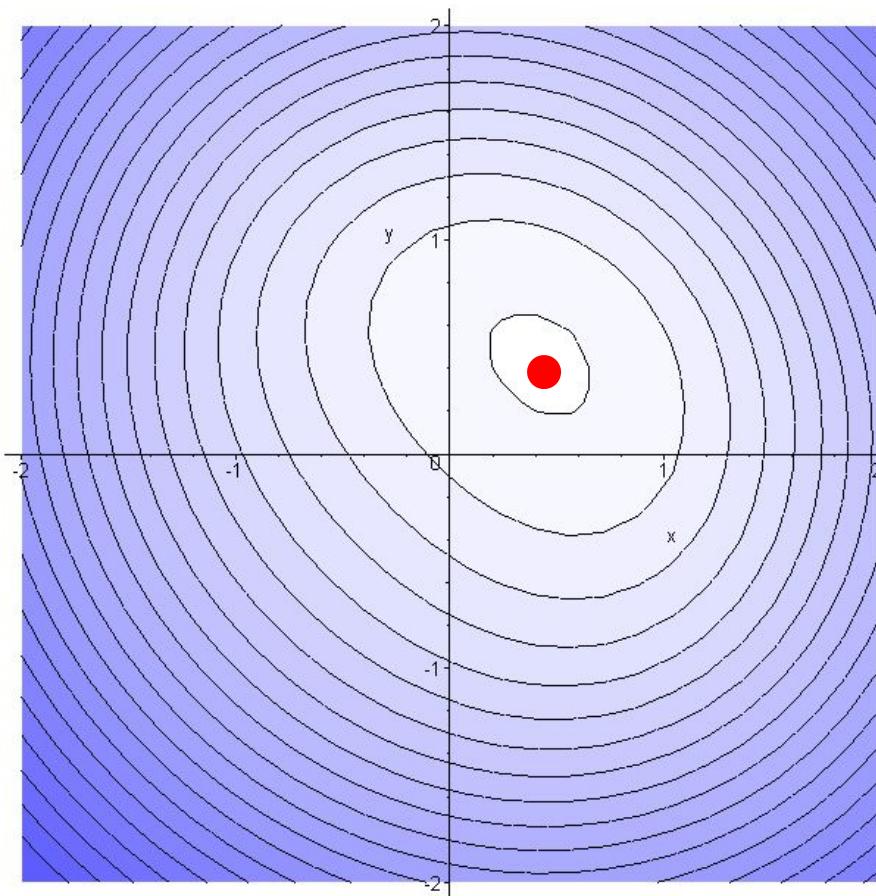
+

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



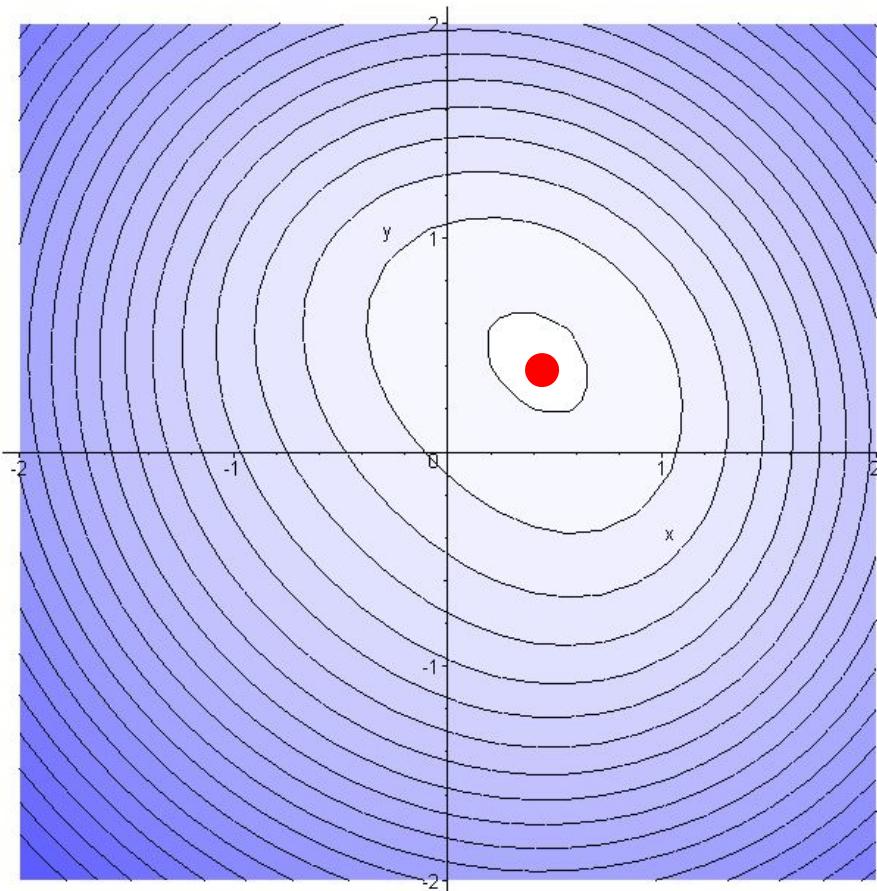
# Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



# Sample and Test Ex. 1

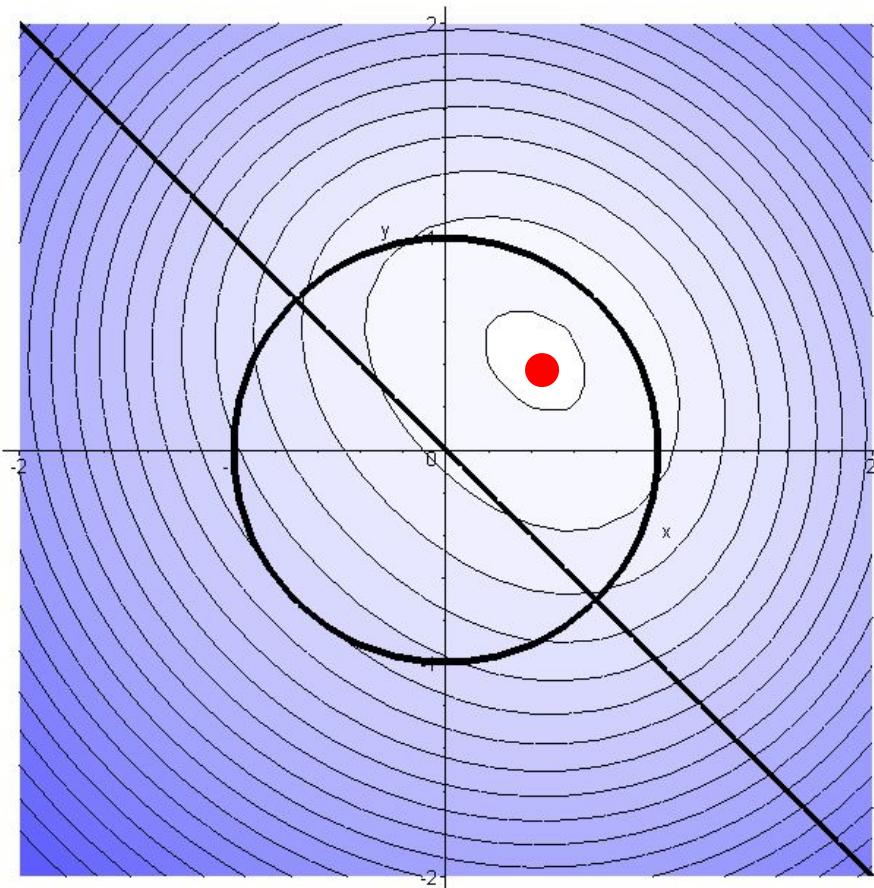
$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



True!

# Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2)$$



True!

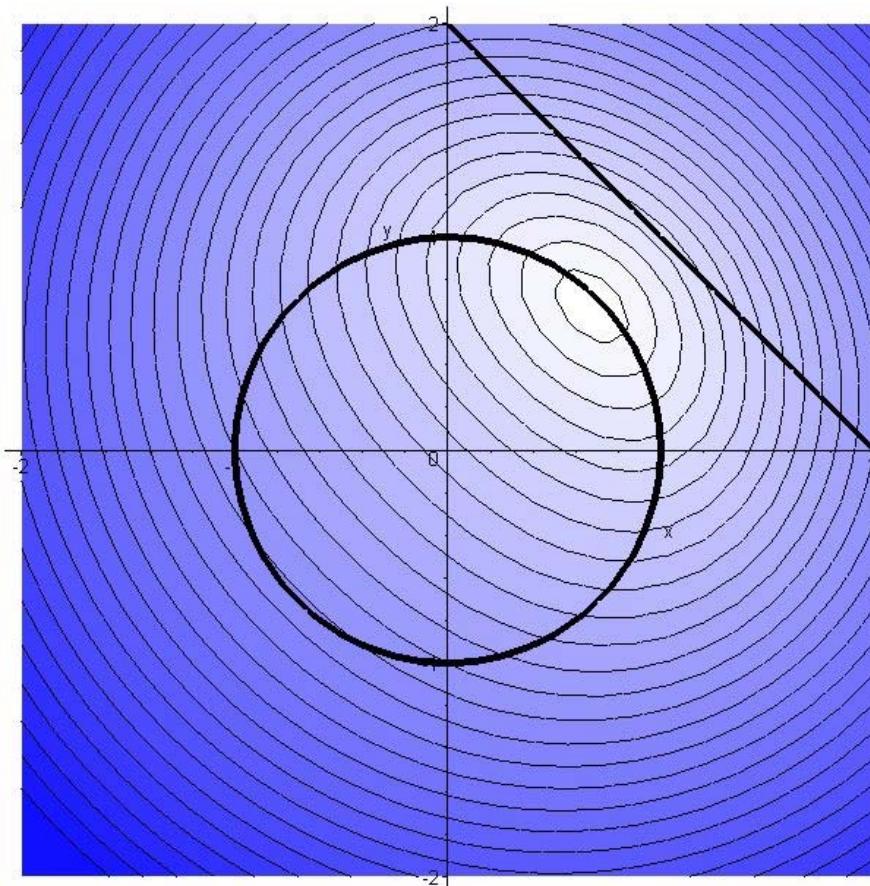
Sample and Test

# Sample and Test Ex. 2

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$

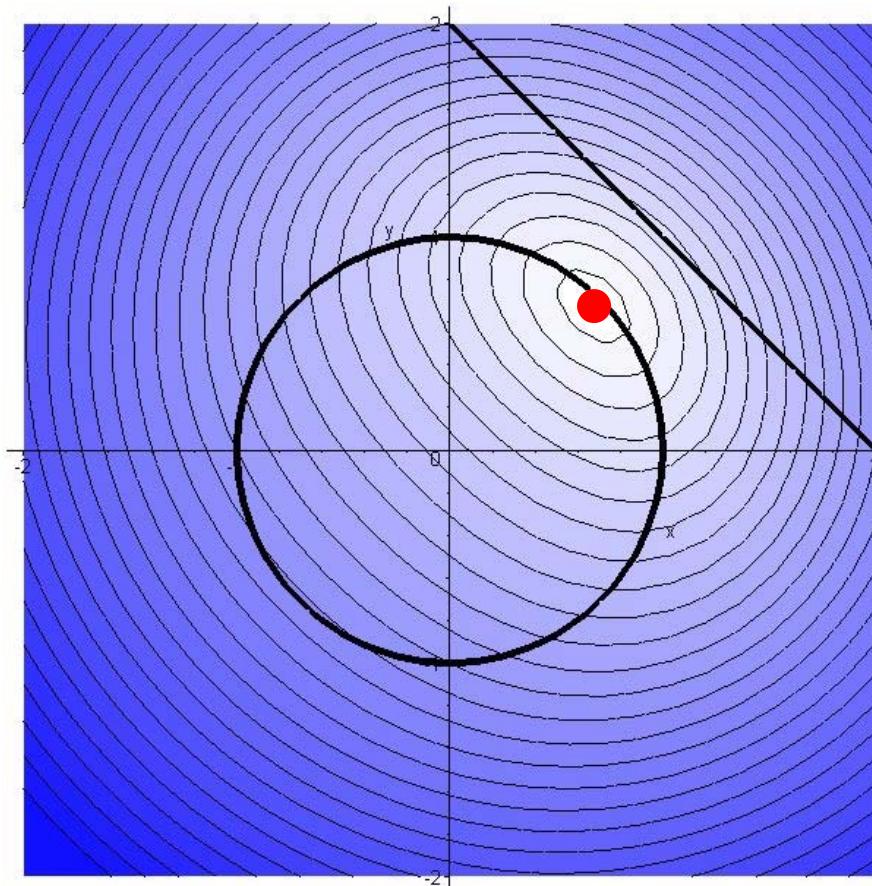
# Sample and Test Ex. 2

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



# Sample and Test Ex. 2

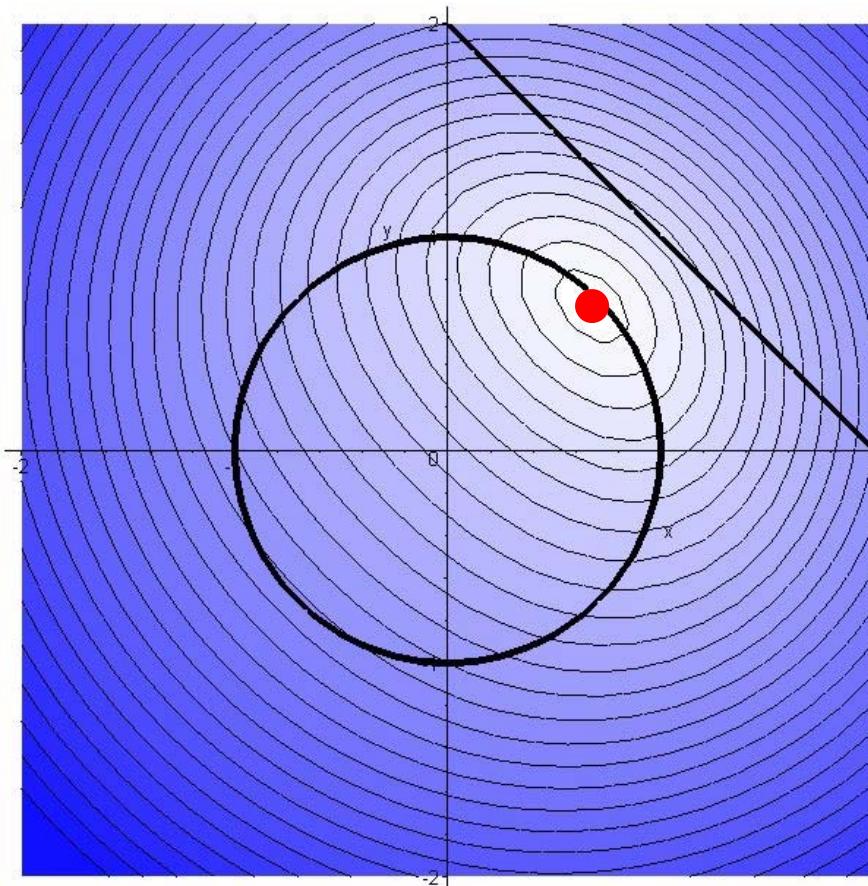
$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



# Sample and Test Ex. 2

+

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$

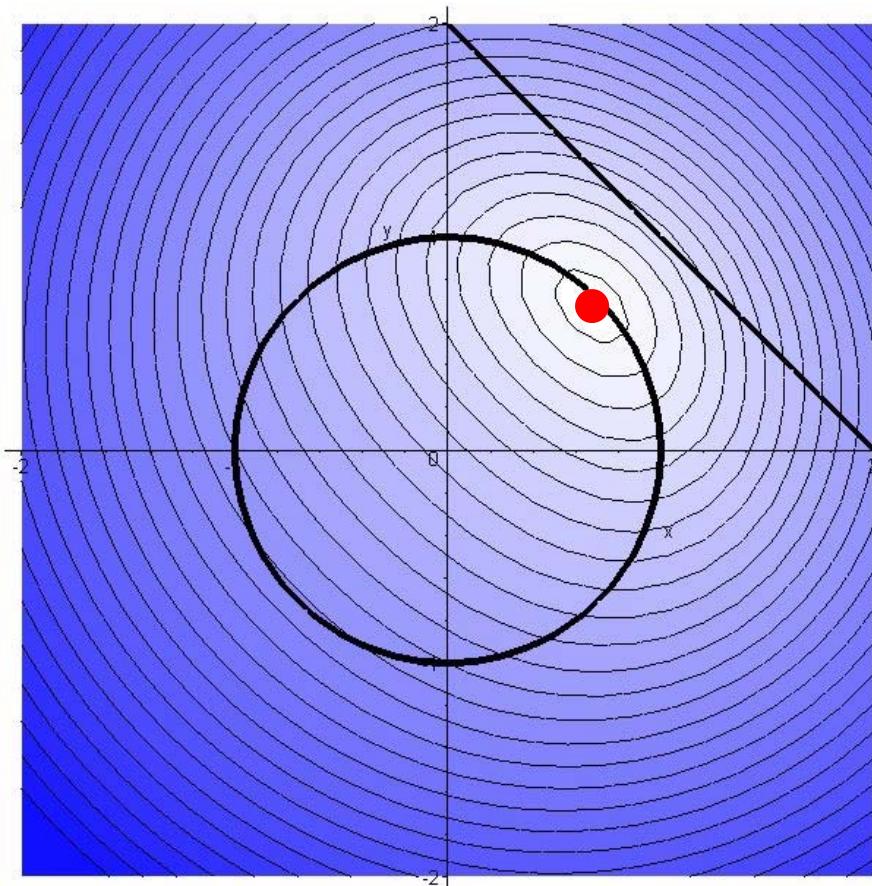


# Sample and Test Ex. 2

+

-

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$

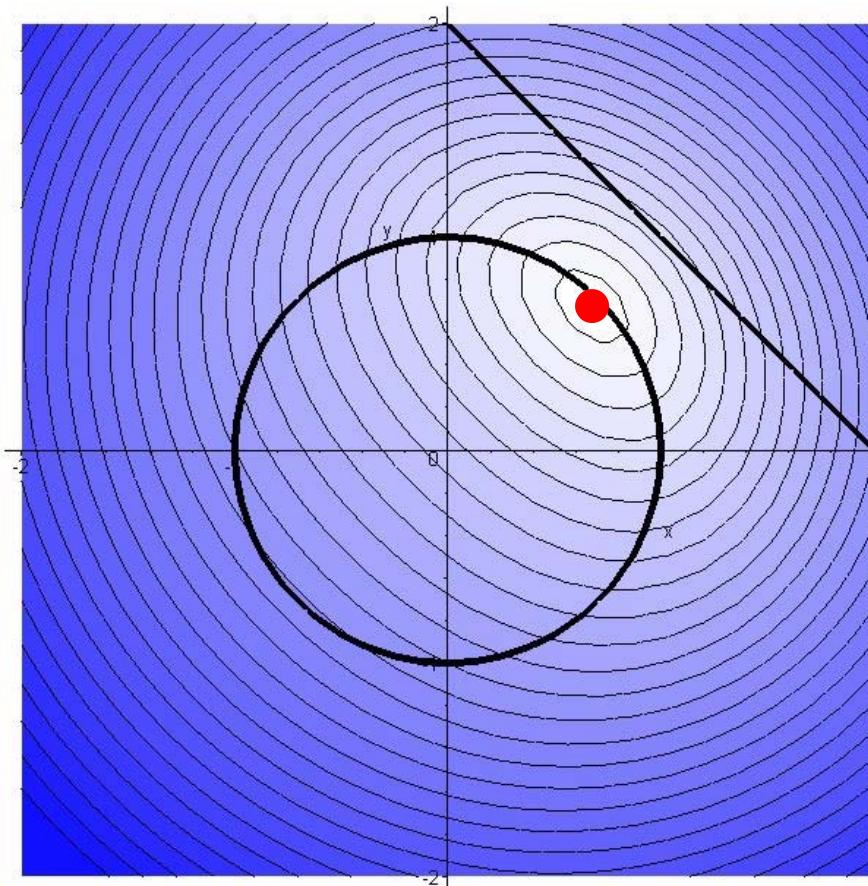


# Sample and Test Ex. 2

+

-

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



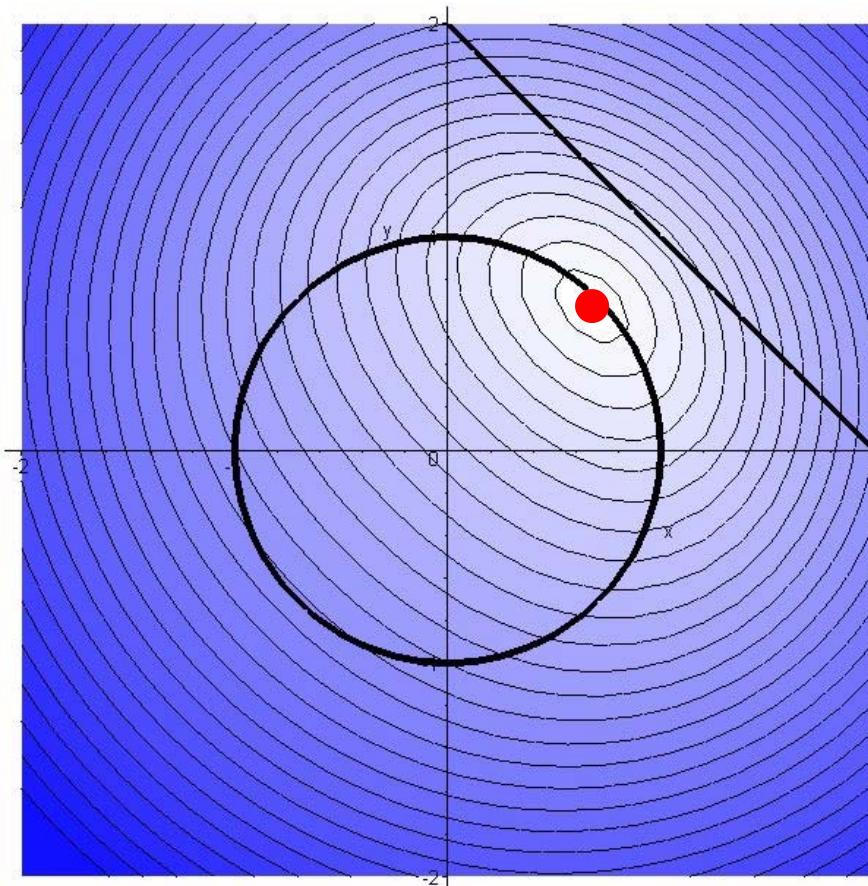
e

# Sample and Test Ex. 2

+

-

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



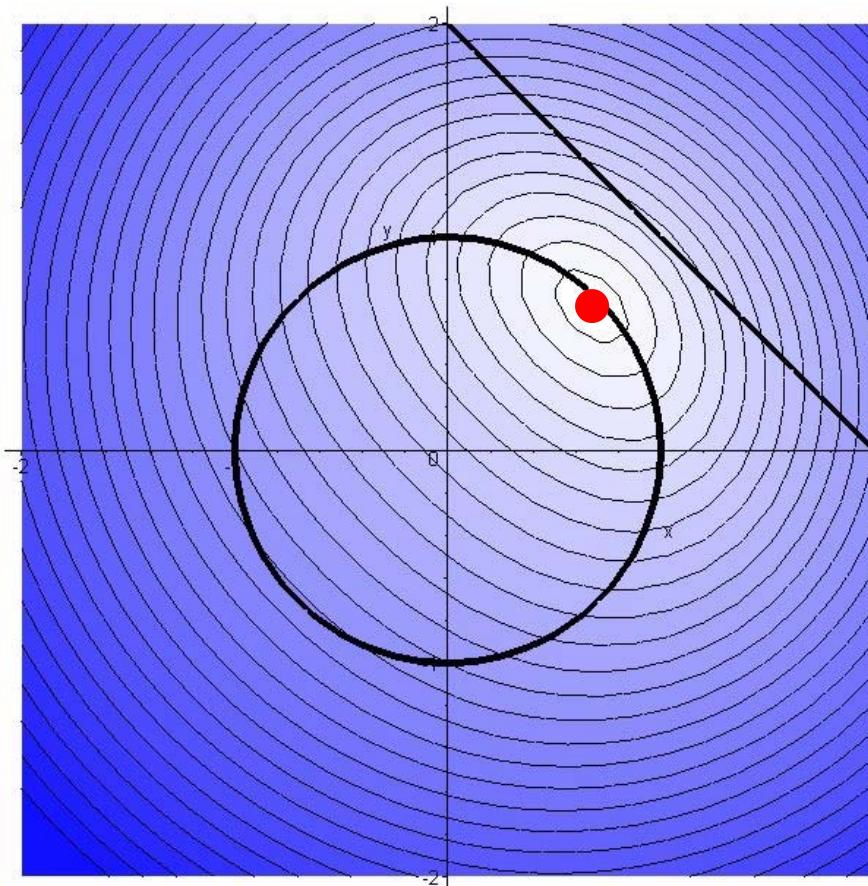
$$e = 2.3$$

# Sample and Test Ex. 2

+

-

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



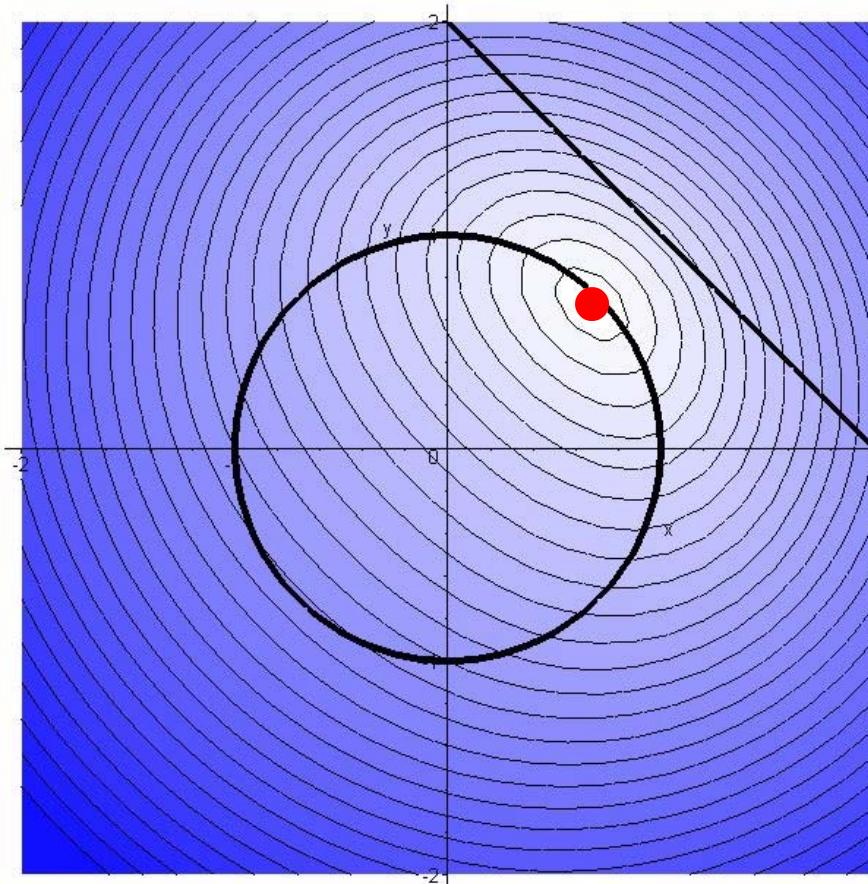
$$e = 2.3 > 2$$

# Sample and Test Ex. 2

+

-

$$f = (-x_1^2 - x_2^2 + 1, x_1 + x_2 - 2)$$



$$e = 2.3 > 2$$

False!

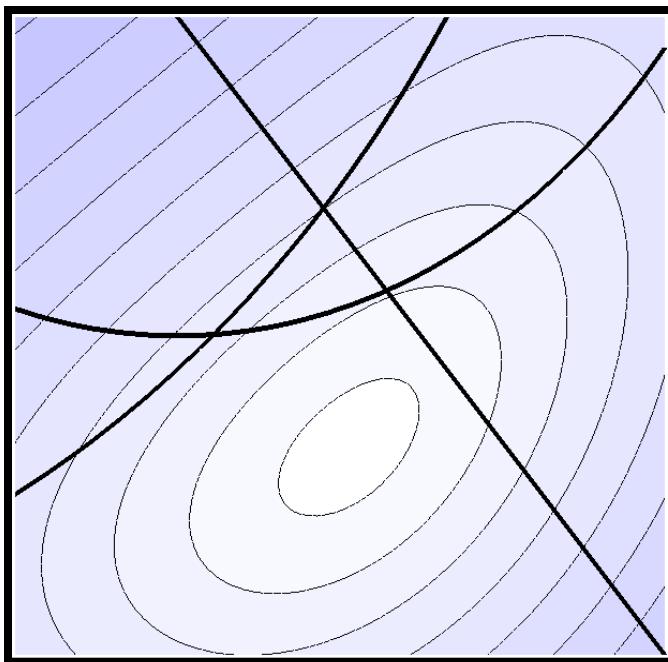
Sample and Test

# Sample and Test Ex. 3

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

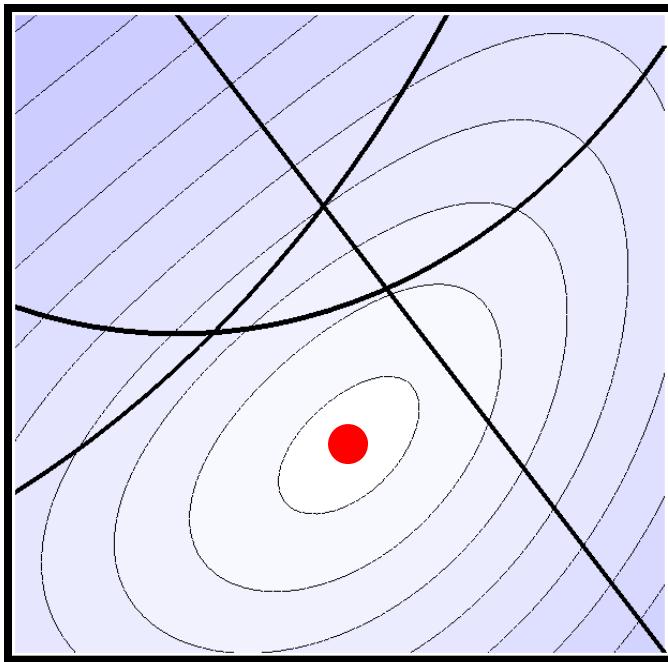
# Sample and Test Ex. 3

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



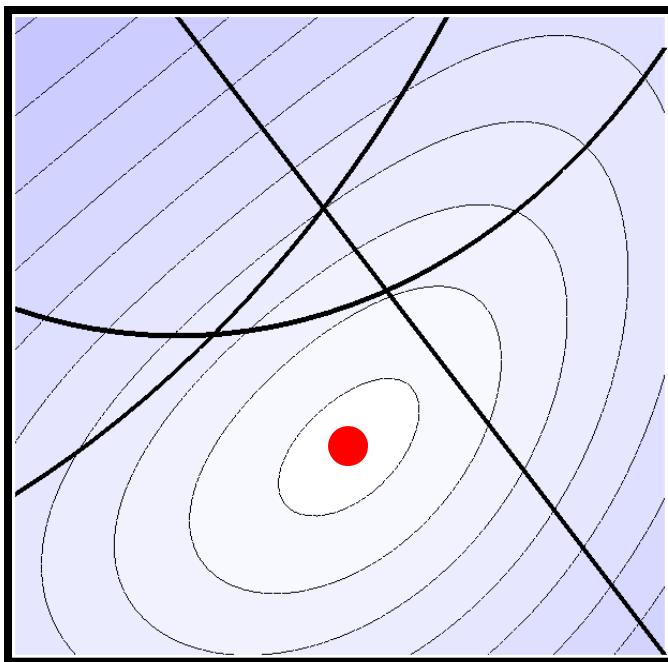
# Sample and Test Ex. 3

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



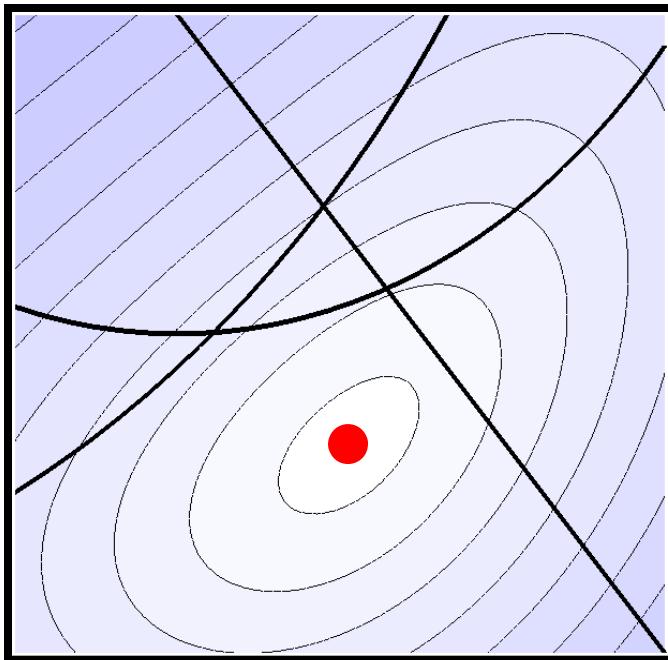
# Sample and Test Ex. 3

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



# Sample and Test Ex. 3

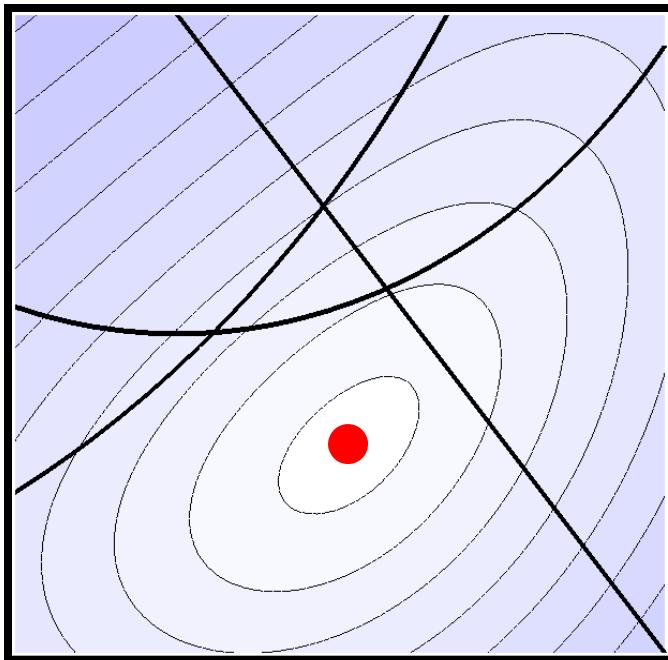
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



$$e = 2.7$$

# Sample and Test Ex. 3

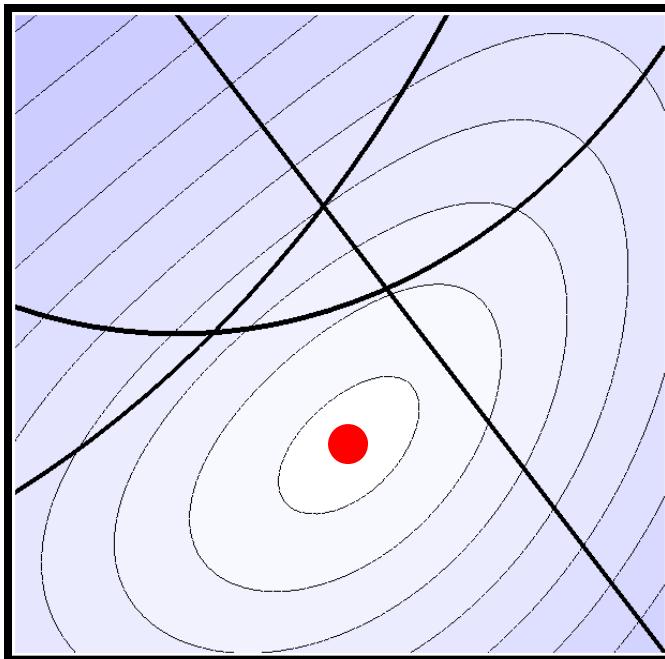
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



$$e = 2.7 < 3$$

# Sample and Test Ex. 3

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



$$e = 2.7 < 3$$

?

# Sample and Test Ex. 3

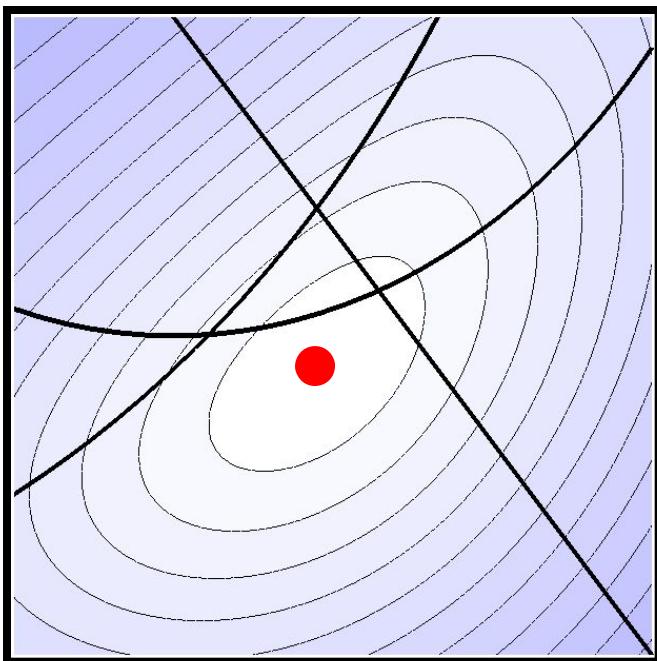
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, & x_1^2 + x_2^2 - 3, & -x_1 - x_2 \end{pmatrix}$$

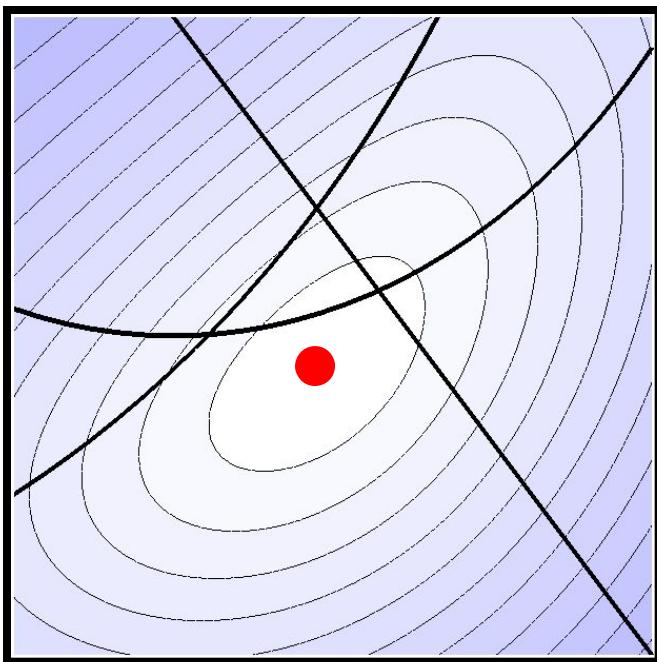
# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, \\ x_1^2 + x_2^2 - 3, \\ -x_1 - x_2 \end{pmatrix}$$



# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1 & - \\ x_1^2 + x_2^2 - 3 & + \\ -x_1 - x_2 & + \end{pmatrix}$$



$$e = 2.6 < 3$$

?

Sample and Test Ex. 3

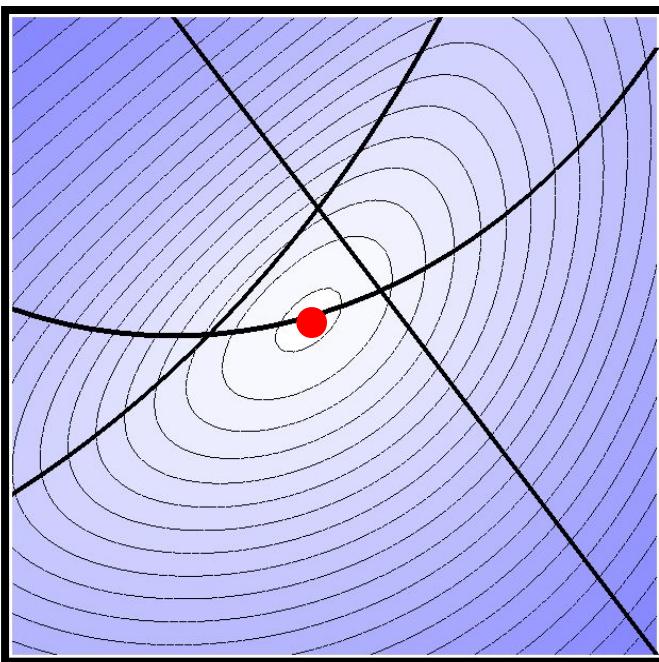
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, \\ x_1^2 + x_2^2 - 3, \\ -x_1 - x_2 \end{pmatrix}$$

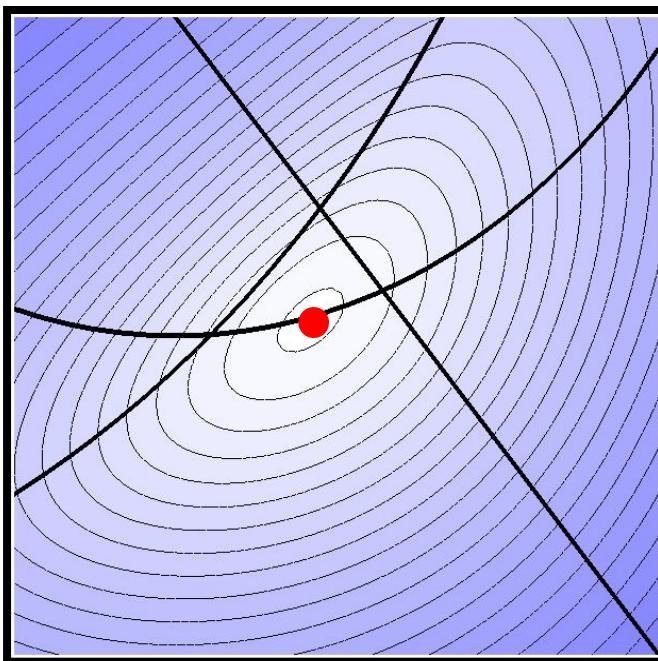
# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, \\ x_1^2 + x_2^2 - 3, \\ -x_1 - x_2 \end{pmatrix}$$



# Sample and Test Ex. 3

$$f = 4 \left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



$$e = 2.3 < 3$$

?

# Sample and Test Ex. 3

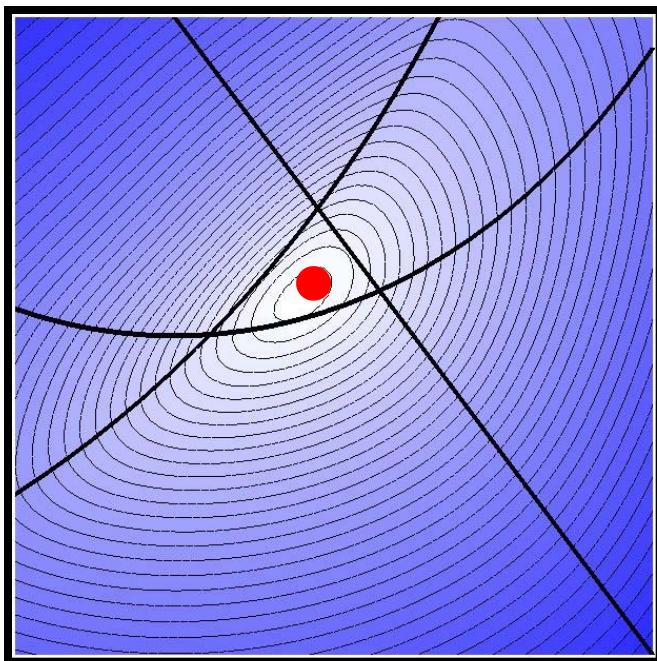
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, & x_1^2 + x_2^2 - 3, & -x_1 - x_2 \end{pmatrix}$$

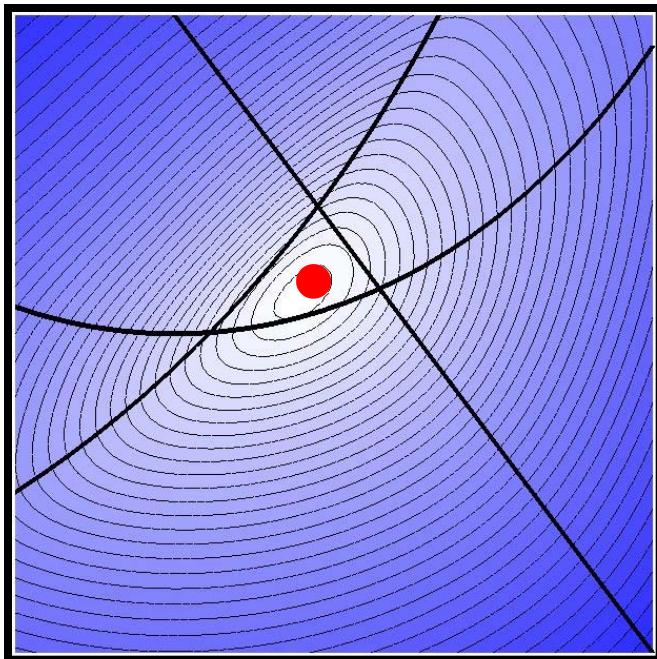
# Sample and Test Ex. 3

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, \\ x_1^2 + x_2^2 - 3, \\ -x_1 - x_2 \end{pmatrix}$$



# Sample and Test Ex. 3

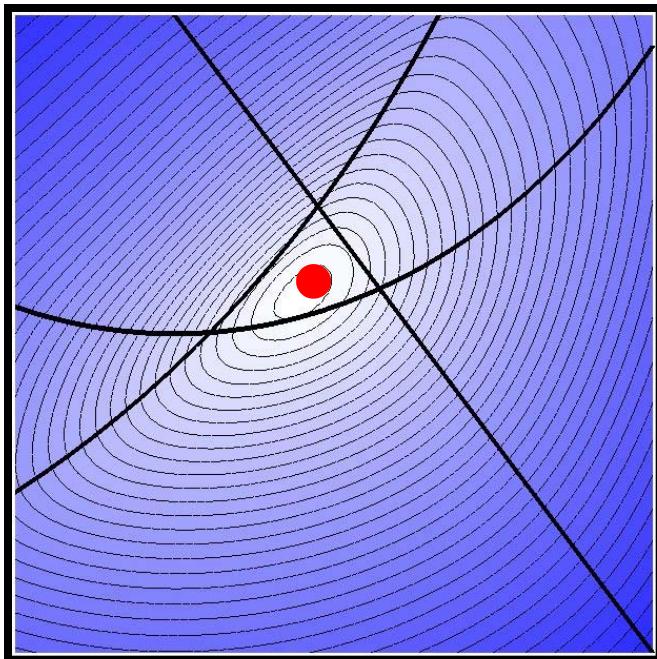
$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1 \\ x_1^2 + x_2^2 - 3 \\ -x_1 - x_2 \end{pmatrix}$$



$$e = 1.9 < 3$$

# Sample and Test Ex. 3

$$f = \left\langle -\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 - \frac{3}{2})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right\rangle$$

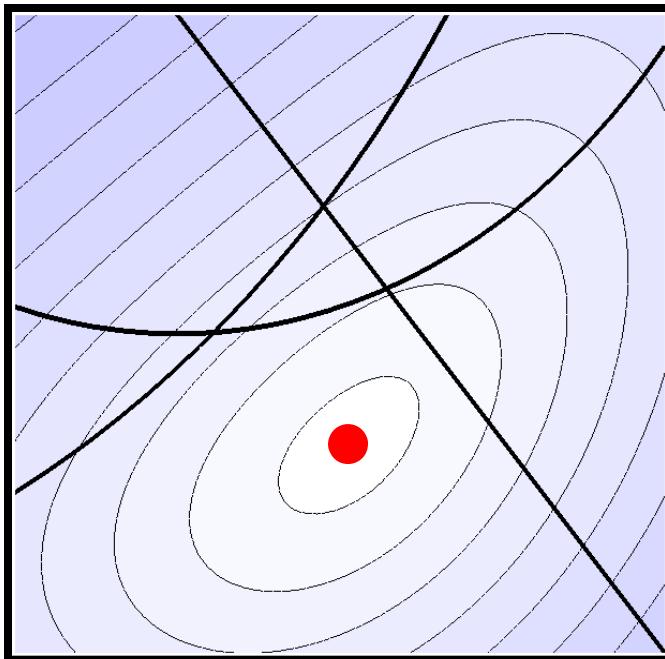


$$\epsilon = 1.9 < 3$$

True!

# Sample and Test Ex. 3 Movie

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

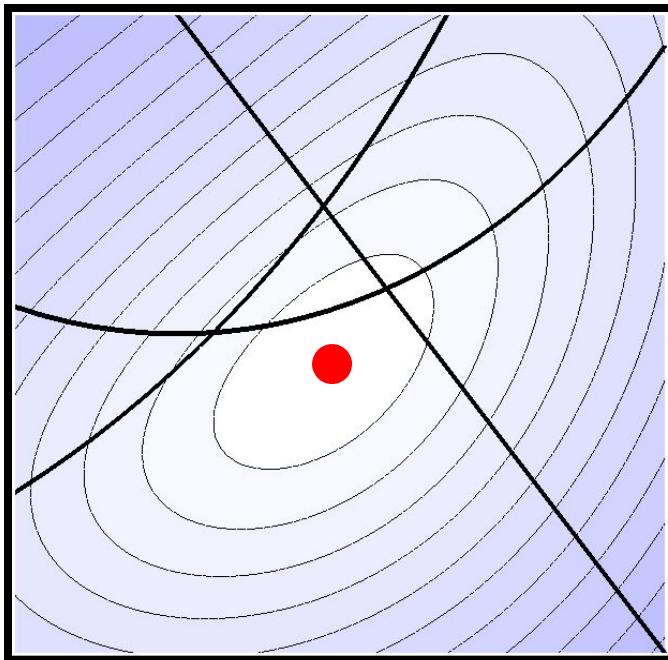


$$e = 2.7 < 3$$

?

# Sample and Test Ex. 3 Movie

$$f = \begin{pmatrix} -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, \\ x_1^2 + x_2^2 - 3, \\ -x_1 - x_2 \end{pmatrix}$$

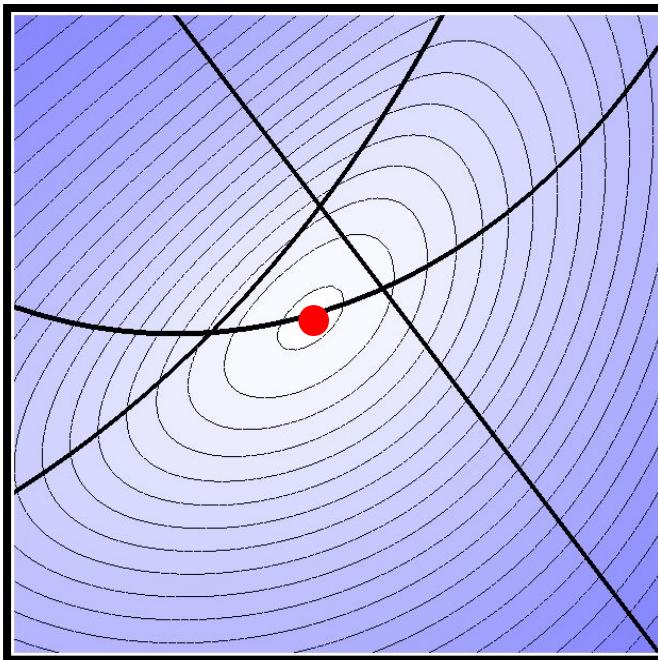


$$e = 2.6 < 3$$

?

# Sample and Test Ex. 3 Movie

$$f = 4 \left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

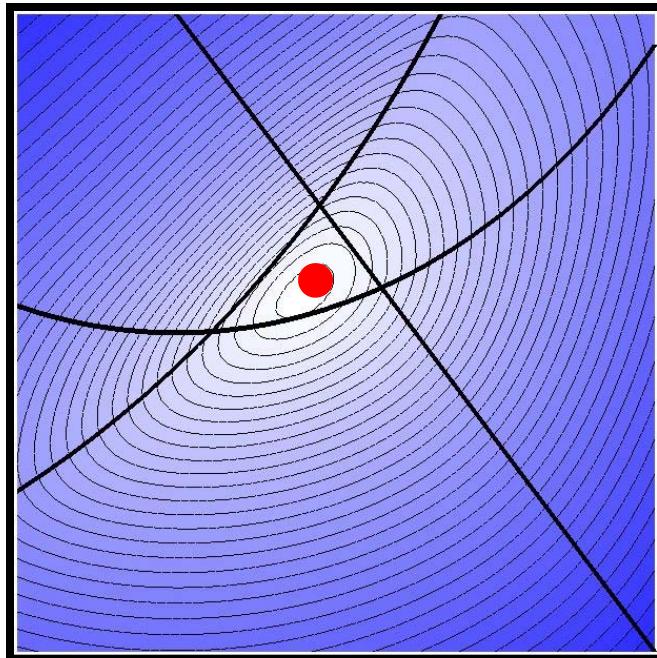


$$e = 2.3 < 3$$

?

# Sample and Test Ex. 3 Movie

$$f = \left( -\frac{1}{2}(x_1 - 1)^2 - \frac{1}{2}(x_2 - \frac{3}{2})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$

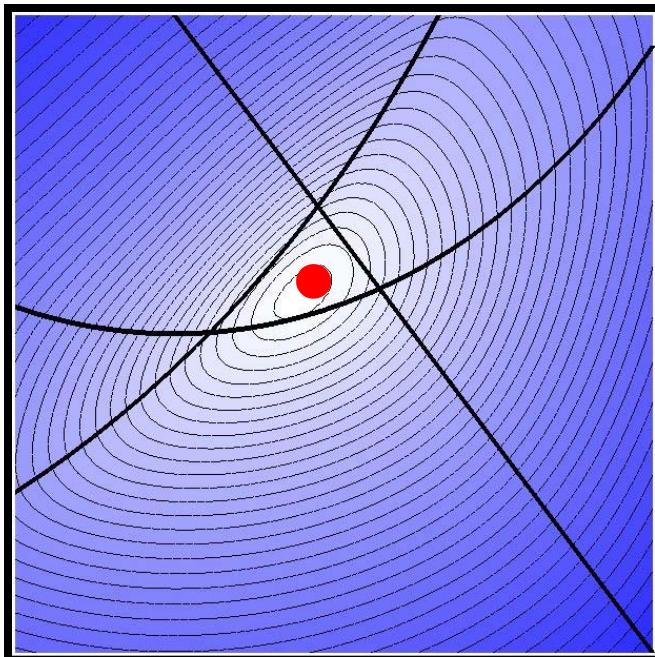


$$e = 1.9 < 3$$

True!

# Sample and Test Ex. 3 Movie

$$f = g\left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 \right)$$



$$e = 1.9 < 3$$

True!

Theorem

$\forall$  feasible  $f \nmid$  suff. large  $C \quad f(x^*) > 0.$

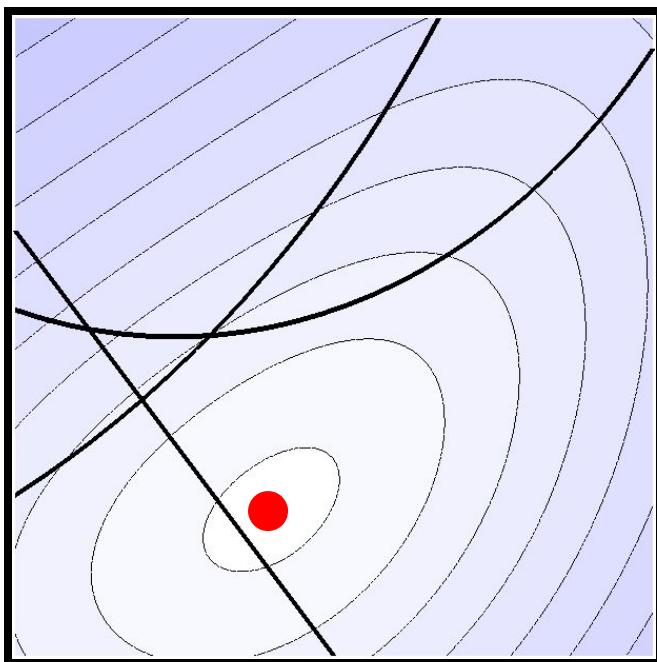
Sample and Test

# Sample and Test Ex. 4

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$

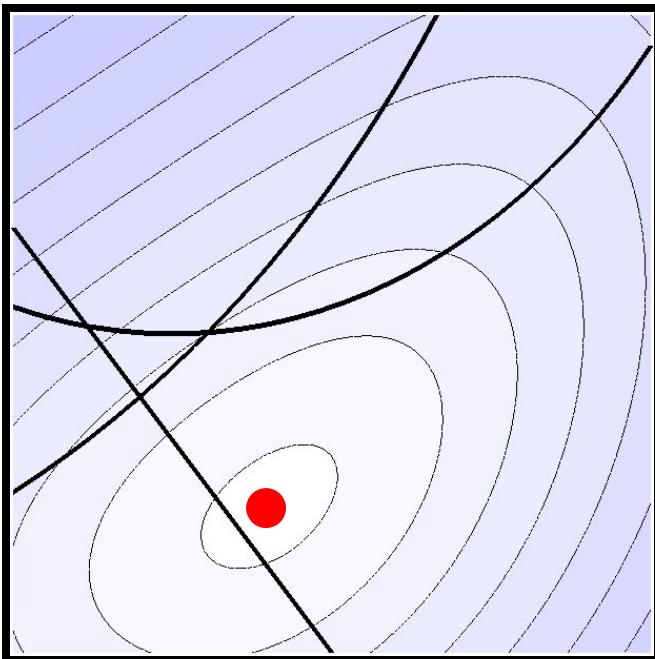
# Sample and Test Ex. 4

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



# Sample and Test Ex. 4

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



$$e = 2.9 < 3$$

?

# Sample and Test Ex. 4

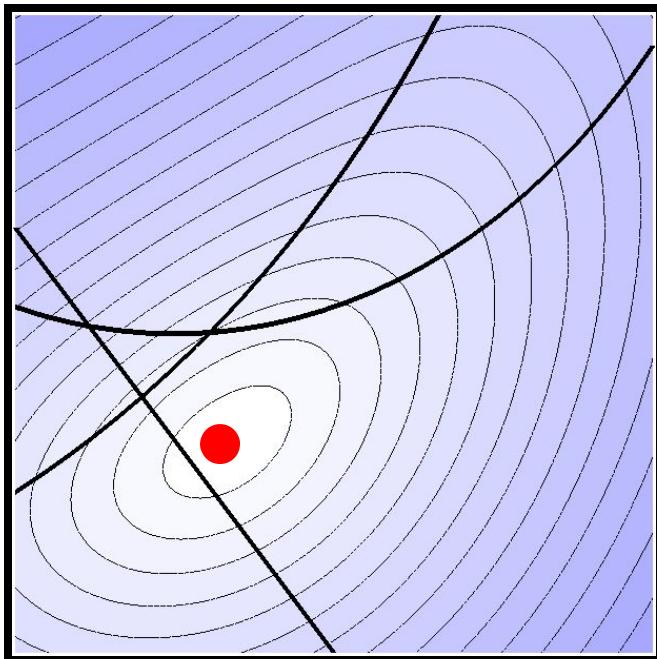
$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$

# Sample and Test Ex. 4

$$f = 2 \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$

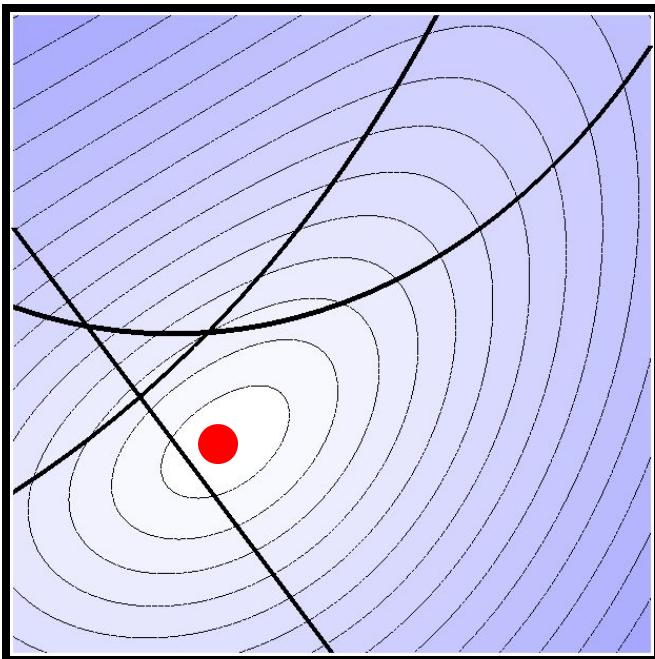
# Sample and Test Ex. 4

$$f = 2 \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



# Sample and Test Ex. 4

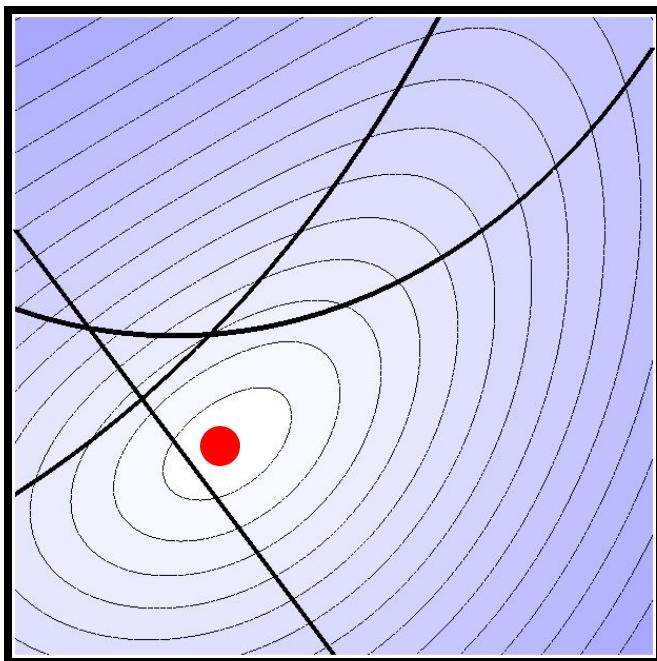
$$f = 2 \left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



$$e = 3.1$$

# Sample and Test Ex. 4

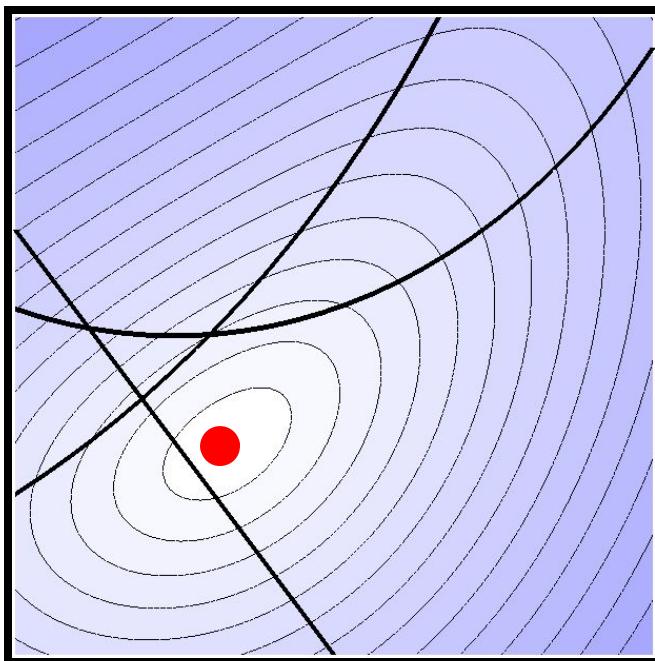
$$f = 2 \left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



$$e = 3.1 > 3$$

# Sample and Test Ex. 4

$$f = 2 \left( - (x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



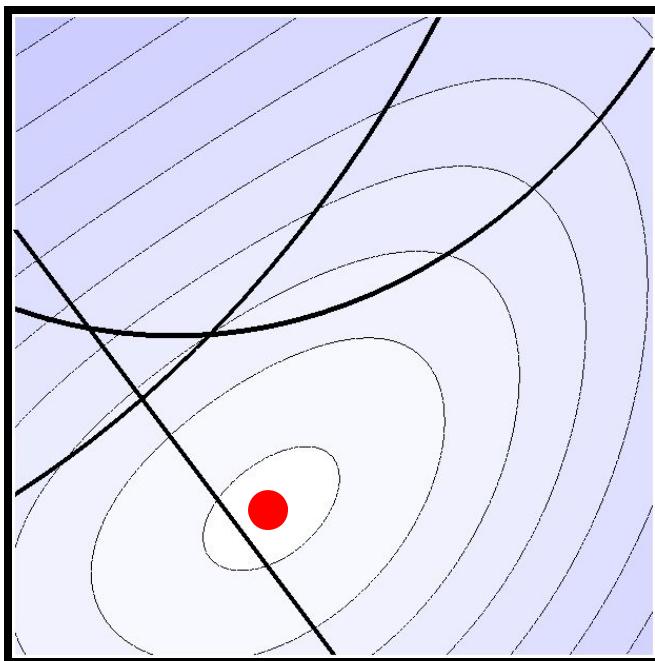
$$e = 3.1 > 3$$

False!

# Sample and Test Ex. 4

Movie

$$f = \left( -(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2} \right)$$



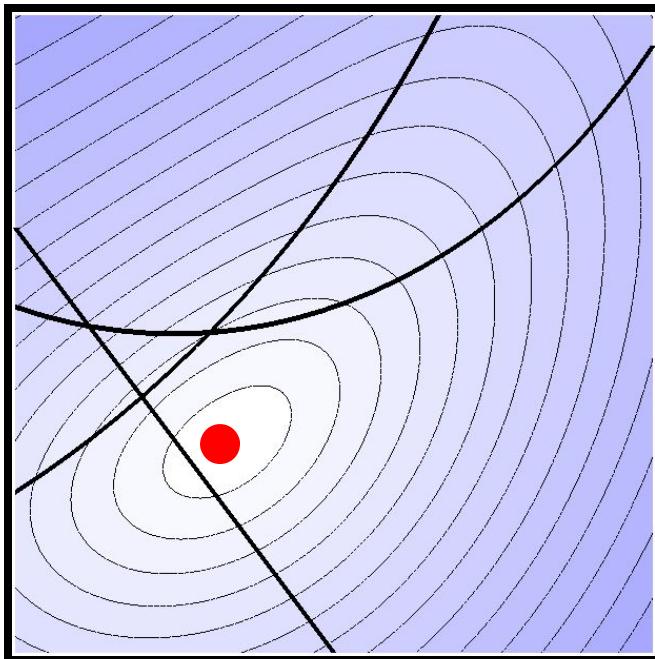
$$e = 2.9 < 3$$

?

# Sample and Test Ex. 4

Movie

$$f = 2(-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2})$$



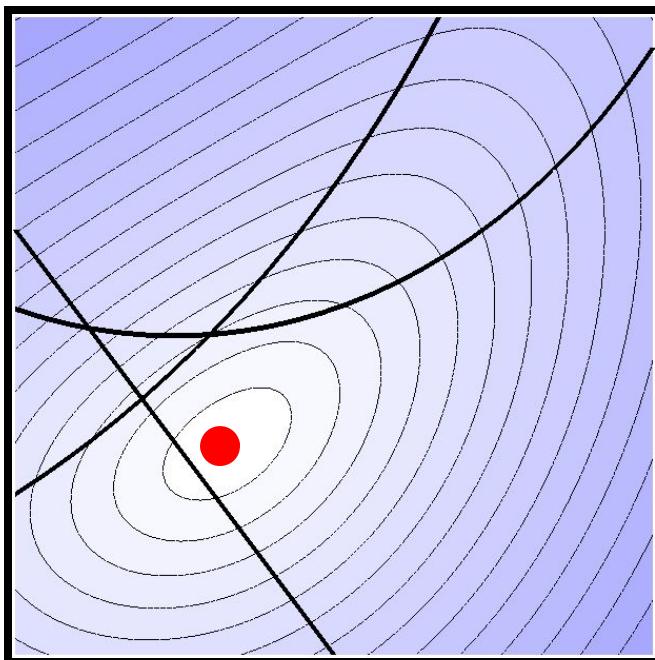
$$e = 3.1 > 3$$

False!

# Sample and Test Ex. 4

Movie

$$f = 2(-(x_1 - 1)^2 - (x_2 - \frac{3}{8})^2 + 1, x_1^2 + x_2^2 - 3, -x_1 - x_2 - \frac{1}{2})$$



$$e = 3.1 > 3$$

False!

Theorem

$\forall$  infeasible  $f \nvdash$  suff. large  $C$   $e(x^*) > m$

# Theorems

$\forall$  feasible  $f \nmid$  suff. large  $c$   $f(x^*) > 0$

$\dot{\forall}$  infeasible  $f \nmid$  suff. large  $c$   $e(x^*) > m$

# Algorithm

$C = 1$

loop "few" times

$x^*, e^* = \min(e_c)$

if  $f(x^*) > 0$  return true

if  $e^* > m$  return false

"increase" C

call symbolic

$$e_c = \rho(c \cdot f_1(x)) + \dots + \rho(c \cdot f_m(x))$$

# Algorithm

$C=1$

loop "few" times

$x^*, e^* = \min(e_c)$   $\leftarrow$  use intervals

if  $f(x^*) > 0$  return true

if  $e^* > m$  return false

"increase" C

call symbolic

$$e_c = \rho(c \cdot f_1(x)) + \dots + \rho(c \cdot f_m(x))$$

# Algorithm

$C=1$

loop "few" times

$x^*, e^* = \min(e_c) \leftarrow$  use intervals

if  $f(x^*) > 0$  return true

if  $e^* > m$  return false

"increase"  $C$

call symbolic  $\leftarrow$  use local geom.

$$e_c = \rho(c \cdot f_1(x)) + \dots + \rho(c \cdot f_m(x))$$

# Algorithm

$C=1$

loop "few" times

$x^*, e^* = \min(e_c) \leftarrow$  use intervals

if  $f(x^*) > 0$  return true

if  $e^* > m$  return false

"increase"  $C$

call symbolic  $\leftarrow$  use local geom.

$\nwarrow$  only when "unstable"

$$e_c = \rho(c \cdot f_1(x)) + \dots + \rho(c \cdot f_m(x))$$

# Timing

# Var	# Poly	Deg	Previous	New
3	6	4	2 min	1 sec
4	8	5	>12 hour	5 sec
5	12	8	>72 hour	40 sec

```
degree = 3
```

```
# polys = 3
```

```
# vars = 1 - 20
```

time(sec)

```
degree    = 3
# polys   = 3
# vars    = 1 - 20
```

# vars

20

time(sec)

40

degree = 3  
# polys = 3  
# vars = 1 - 20

# vars

20

# vars	time(sec)
1	0.5
5	0.5
10	10
15	25
20	40

**time(sec)**

```
# polys = 3  
# vars  = 3  
degree = 1 - 20
```

**degree**  
**20**

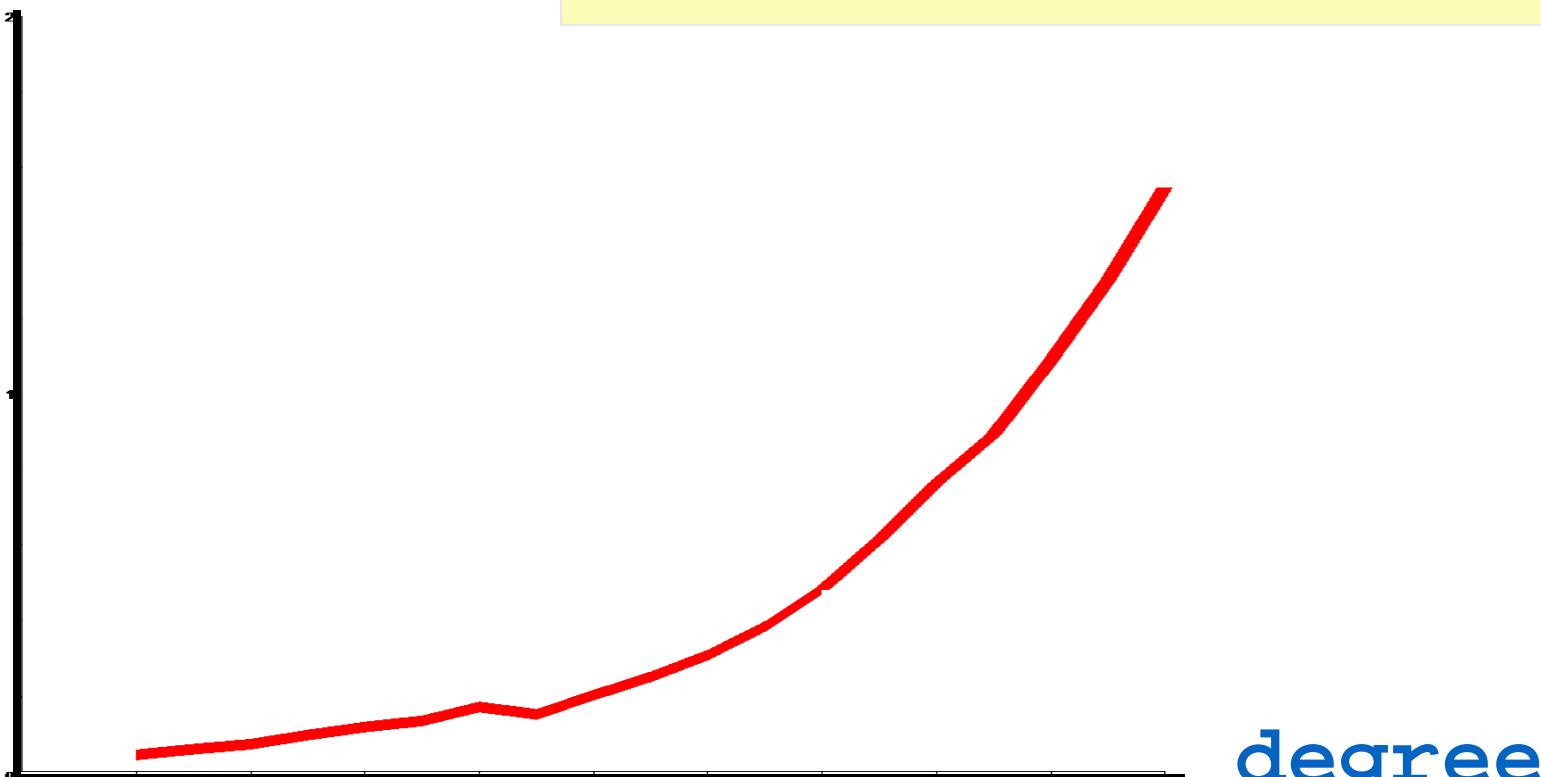
time(sec)

20

```
# polys = 3  
# vars = 3  
degree = 1 - 20
```

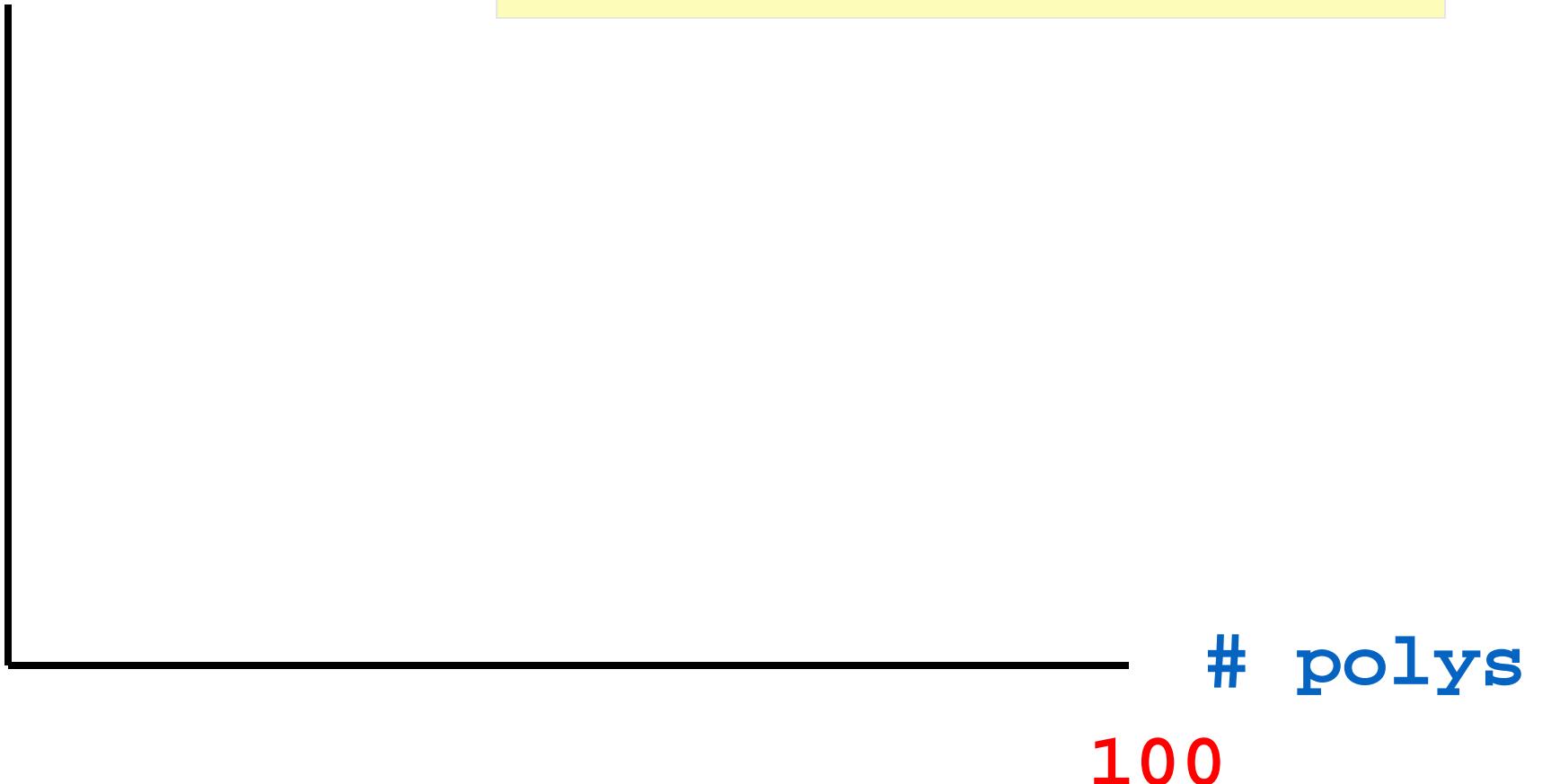
degree

20



**time(sec)**

```
degree      = 3
# vars      = 3
# polys = 1 - 100
```



time(sec)

2

2

1

1

0

# polys

100

degree = 3  
# vars = 3  
# polys = 1 - 100

time(sec)

2

2

1

1

0

0

# polys

100

degree = 3  
# vars = 3  
# polys = 1 - 100

# Numerical Projection

# Challenging Problem

In :  $\forall s, t \ f(a, b, s, t) < 0$

Out : Quantifier-free formula in  $a, b$

# Challenging Problem

In :  $\forall s, t \ f(a, b, s, t) < 0$

Out : Quantifier-free formula in  $a, b$

$$\begin{aligned} f = & 8*b^6*a^2*s^4*t^8 + 140*b^4*a^4*s^4*t^4 - 4*b^4*a^2*s^8*t^4 \\ & + 4*b^4*a^2*s^8*t^8 + 12*b^4*a^4*s^8*t^4 + 12*b^4*a^4*s^8*t^8 \\ & + 12*b^4*a^4*s^4*t^8 - 12*b^4*a^2*s^4*t^8 + 16*b^4*a^2*s^8*t^6 - \\ & 32*b^4*a^2*s^4*t^6 - 4*b^4*a^2*s^4*t^4 + 120*b^6*a^2*s^4*t^6 \\ & + 8*b^6*a^2*s^8*t^6 - 4*a^4*b^2*s^4*t^4 - 12*a^4*b^2*s^8*t^4 \\ & + 4*a^4*b^2*s^8*t^8 - 4*a^4*b^2*s^4*t^8 + 120*a^6*b^2*s^6*t^4 \\ & + 8*a^6*b^2*s^6*t^8 + 8*a^6*b^2*s^8*t^4 - 8*a^6*b^2*s^8*t^8 \\ & + 120*b^4*a^4*s^4*t^6 - 8*b^4*a^4*s^8*t^6 - 8*a^4*b^2*s^8*t^2 \\ & - 24*a^4*b^2*s^6*t^2 + 8*a^4*b^2*s^6*t^6 + 16*a^4*b^2*s^6*t^8 \\ & + 48*b^4*a^4*s^6*t^6 - 8*b^4*a^4*s^6*t^8 + 2*a^2*b^2*s^8*t^8 \\ & + 6*a^2*b^2*s^4*t^8 + 34*a^2*b^2*s^4*t^4 + 6*a^2*b^2*s^8*t^4 \\ & + 72*b^6*a^2*s^6*t^6 + 8*b^6*a^2*s^2*t^8 - 8*b^6*a^2*s^6*t^8 \\ & + 56*b^6*a^2*s^2*t^6 + 120*b^4*a^4*s^6*t^4 - 8*a^6*b^2*s^8*t^6 \\ & + 8*a^6*b^2*s^8*t^2 + 56*a^6*b^2*s^6*t^2 + 72*a^6*b^2*s^6*t^6 \end{aligned}$$

$$\begin{aligned}
& -32*a^4*b^2*s^6*t^4 + 2*a^2*b^2*s^2*t^8 + 6*a^2*b^2*s^8*t^6 \\
& + 2*a^2*b^2*s^8*t^2 + 6*a^2*b^2*s^2*t^2 + 6*a^2*b^2*s^6*t^8 \\
& + 10*a^2*b^2*s^2*t^6 + 10*a^2*b^2*s^6*t^2 + 22*a^2*b^2*s^6*t^6 \\
& - 8*a^4*b^2*s^4*t^6 + 8*b^4*a^2*s^6*t^6 - 24*b^4*a^2*s^2*t^6 \\
& - 8*b^4*a^2*s^2*t^8 - 8*b^4*a^2*s^6*t^4 - 4*b^2*s^2*t^4 \\
& + 2*b^4*t^6 + 3*b^4*t^8 - 4*b^6*t^8 + 26*a^2*b^2*s^4*t^6 \\
& + 14*a^2*b^2*s^4*t^2 + 26*a^2*b^2*s^6*t^4 + 14*a^2*b^2*s^2*t^4 \\
& - 2*b^2*t^6 - b^2*t^8 + 2*b^8*t^8 + 2*a^8*s^8 - a^2*s^4 - a^2*s^8 \\
& + 3*a^4*s^8 + a^4*s^4 - 4*a^6*s^8 + 2*a^4*s^6 + 8*a^4*s^6*t^2 - \\
& 2*a^2*s^6 + 48*a^5*b^3*t^7*s^5 - 16*a^5*b^3*t^7*s^7 \\
& + 160*a^5*b^3*t^5*s^5 + 32*a^5*b^3*t^5*s^7 - 8*a^2*s^6*t^2 \\
& + 112*a^5*b^3*t^3*s^5 + 48*a^5*b^3*t^3*s^7 - 8*a^2*s^6*t^6 \\
& + 12*b^4*s^2*t^8 + 2*b^4*s^8*t^6 + 8*b^4*s^6*t^6 - b^2*t^4 \\
& + 12*b^4*s^4*t^6 + 8*b^4*s^2*t^6 + 48*a*b^7*t^7*s^3 \\
& + 48*a*b^7*t^7*s^5 + 16*a*b^7*t^7*s^7 + 16*a*b^7*t^7*s \\
& - 16*a^3*b^3*s^3*t^7 + 16*a^3*b^3*s^5*t^7 + 32*a^3*b^3*s^7*t^7 \\
& - 16*a^3*b^3*s^7*t^3 - 16*a^3*b^3*s^3*t^5 + 16*a^3*b^3*s^7*t^5 \\
& - 16*a^5*b*t^7*s^7 - 48*a^5*b*t^5*s^7 - 48*a^5*b*t^3*s^7 \\
& - 16*a^5*b*t*s^7 + b^4*t^4 - 16*b^5*a^3*t^7*s^7 + 48*b^5*a^3*t^5*s^7 \\
& + 48*b^5*a^3*t^7*s^3 + 32*b^5*a^3*t^7*s^5 + 112*b^5*a^3*t^5*s^3 \\
& + 160*b^5*a^3*t^5*s^5 - 16*b^6*s^2*t^8 - 24*b^6*s^4*t^8 - \\
& 4*b^6*s^8*t^8 - 16*b^6*s^6*t^8 - 48*b^5*a*t^7*s^3 \\
& - 48*b^5*a*t^7*s^5 - 16*b^5*a*t^7*s^7 - 16*b^5*a*t^7*s \\
& + 3*b^4*s^8*t^8 + 12*b^4*s^6*t^8 + 18*b^4*s^4*t^8 + 8*b^8*s^2*t^8 \\
& + 12*b^8*s^4*t^8 + 2*b^8*s^8*t^8 + 8*b^8*s^6*t^8 + 24*a^3*b*t^5*s^5
\end{aligned}$$

$$\begin{aligned}
& +12*a^3*b*t^5*s^7+4*a^3*b*t^7*s^3+8*a^3*b*t^7*s^5 \\
& +12*a^3*b*t^3*s^3+24*a^3*b*t^3*s^5+12*a^3*b*t^3*s^7 \\
& +12*a^3*b*t^5*s^3-4*a^2*s^4*t^2+4*a^3*b*t*s^3+8*a^3*b*t*s^5 \\
& +4*a^3*b*t*s^7+8*a^8*s^8*t^2+8*a^8*s^8*t^6+2*a^8*s^8*t^8 \\
& +12*a^8*s^8*t^4-b^2*s^8*t^8-4*b^2*s^6*t^8-6*b^2*s^4*t^8 \\
& -4*b^2*s^2*t^8-2*b^2*s^8*t^6-8*b^2*s^6*t^6-12*b^2*s^4*t^6 \\
& -8*b^2*s^2*t^6+48*a^7*b*t^3*s^7+48*a^7*b*t^5*s^7 \\
& +16*a^7*b*t^7*s^7+16*a^7*b*t*s^7-16*a^3*b^3*s^5*t^3 \\
& +12*a^4*s^8*t^2+12*a^4*s^8*t^6+4*a^4*s^4*t^6+a^4*s^4*t^8 \\
& +3*a^4*s^8*t^8+18*a^4*s^8*t^4+6*a^4*s^4*t^4+4*a^4*s^4*t^2 \\
& -16*a^6*s^8*t^2-16*a^6*s^8*t^6-4*a^6*s^8*t^8-24*a^6*s^8*t^4 \\
& -4*a^2*s^8*t^2-4*a^2*s^8*t^6-4*a^2*s^4*t^6-a^2*s^4*t^8 \\
& -a^2*s^8*t^8-6*a^2*s^8*t^4-6*a^2*s^4*t^4+4*a^3*b*t^7*s^7 \\
& +12*a^4*s^6*t^4+2*a^4*s^6*t^8+8*a^4*s^6*t^6+12*a*b^3*t^3*s^3 \\
& +12*a*b^3*t^3*s^5+4*a*b^3*t^3*s^7+8*a*b^3*t^5*s \\
& +24*a*b^3*t^5*s^3+12*a*b^3*t^7*s^3+12*a*b^3*t^7*s^5 \\
& +4*a*b^3*t^7*s^7+24*a*b^3*t^5*s^5+8*a*b^3*t^5*s^7 \\
& +4*a*b^3*t^7*s+4*a*b^3*t^3*s-8*b^6*a^2*s^8*t^8+b^4*s^8*t^4 \\
& +4*b^4*s^6*t^4+6*b^4*s^4*t^4+4*b^4*s^2*t^4-12*a^2*s^6*t^4 \\
& -2*a^2*s^6*t^8-b^2*s^8*t^4-4*b^2*s^6*t^4-6*b^2*s^4*t^4
\end{aligned}$$

Arose from

Stability analysis of  
Mc Cormack scheme for

$$\frac{\partial u}{\partial t} = P \frac{\partial u}{\partial x} + Q \frac{\partial u}{\partial y}$$

- No previous methods can solve it.

$\gg 2$  months QEPCAD

$$a^{2/3} + b^{2/3} \leq 1$$

- No previous methods can solve it.

» 2 months QEPCAD

- Human Ad-hoc solution  
~ 6 months

$$a^{2/3} + b^{2/3} \leq 1$$

Degree of  $f(a,b,s,t)$

$s,t : 8$

$a,b : 16$

Degree of  $f(a,b,s,t)$

$s,t : 8$

$a,b : 16$

• Degree of Resultant (Dense)

$a,b : \sim 7500$

$a : \sim 10^8$

Degree of  $f(a,b,s,t)$

$s,t : 8$

$a,b : 16$

- Degree of Resultant (Dense)

$a,b : \sim 7500$

$a : \sim 10^8$

- But human ad-hoc (6 month).

$a,b : 6$

Degree of  $f(a,b,s,t)$

$s,t : 8$

$a,b : 16$

- Degree of Resultant (Dense)

$a,b : \sim 7500$

$a : \sim 10^8$

- But human ad-hoc (6 month).

$a,b : 6$

Huge extraneous factor!

So,

We need to find ways.

to avoid resultant

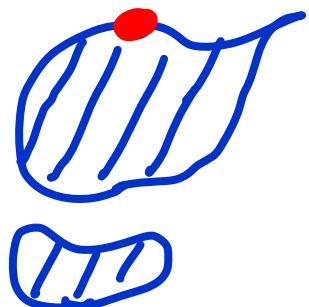
computation.

Theorem (Generic )

$$\forall \vec{y} \quad f_1(\vec{x}, \vec{y}) < 0 \quad \vee \dots \vee f_r(\vec{x}, \vec{y}) < 0$$

# Theorem (Generic)

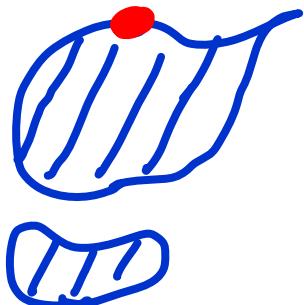
$$\forall \vec{y} \quad f_1(\vec{x}, \vec{y}) < 0 \quad \vee \dots \vee f_r(\vec{x}, \vec{y}) < 0$$



# Theorem (Generic)

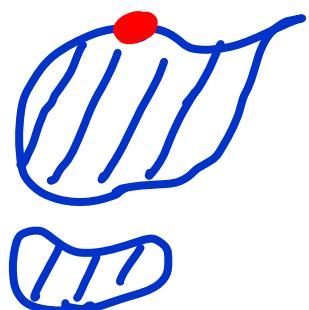
$$\forall \vec{y} \quad f_1(\vec{x}, \vec{y}) < 0 \quad \vee \dots \vee f_r(\vec{x}, \vec{y}) < 0$$

satisfies



# Theorem (Generic)

$$\forall \vec{y} \quad f_1(\vec{x}, \vec{y}) < 0 \quad \vee \dots \vee f_r(\vec{x}, \vec{y}) < 0$$



satisfies for some  $j_1, \dots, j_k$

$$f_1 = \dots = f_r = 0$$

and

$$\nabla f_{j_1}, \dots, \nabla f_{j_k} \text{ lin dep.}$$

Idea

# Idea

$$f(x, y, z) = 0$$

$$f_z(x, y, z) = 0$$

$\mathbb{R}^3$



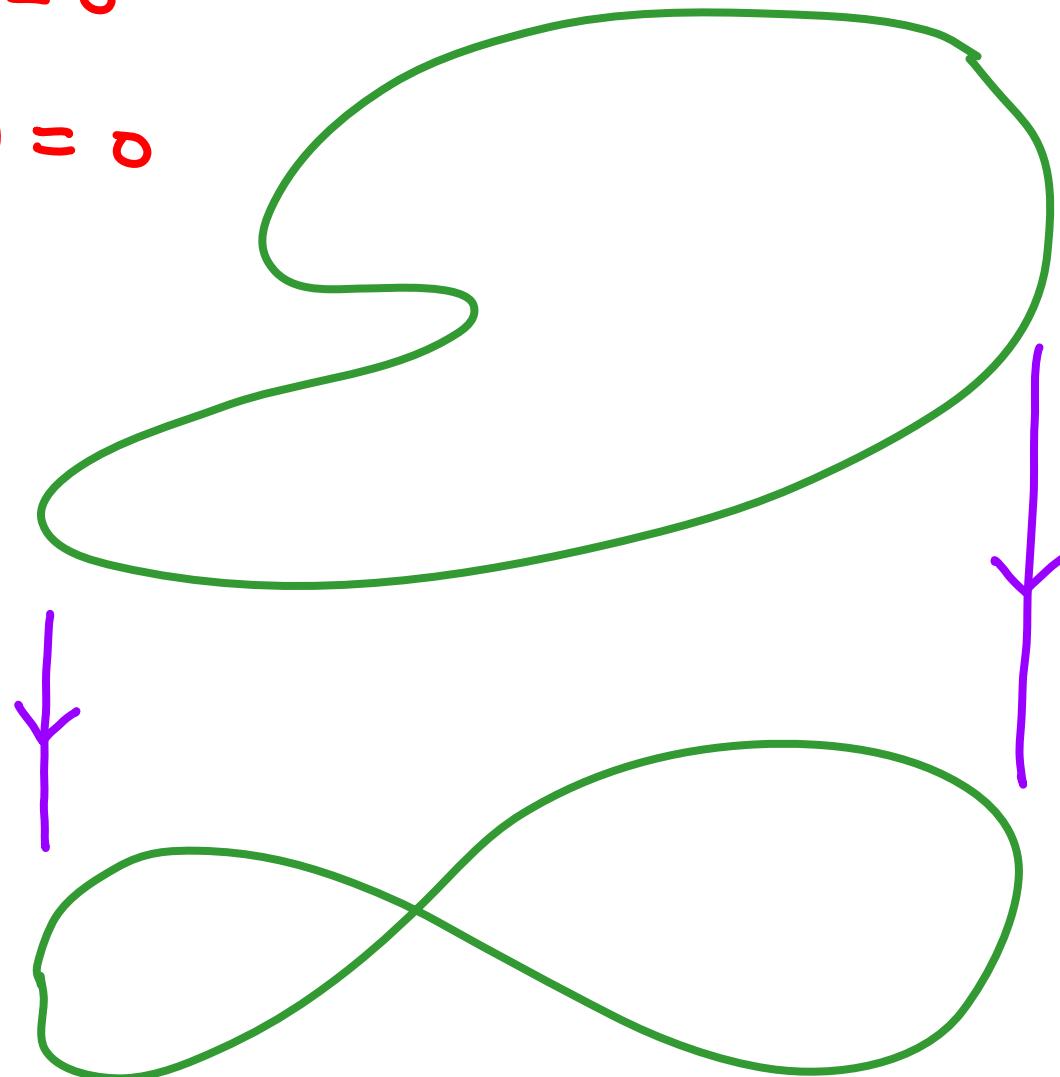
# Idea

$$f(x, y, z) = 0$$

$$f_z(x, y, z) = 0$$

$\mathbb{R}^3$

$\mathbb{R}^2$



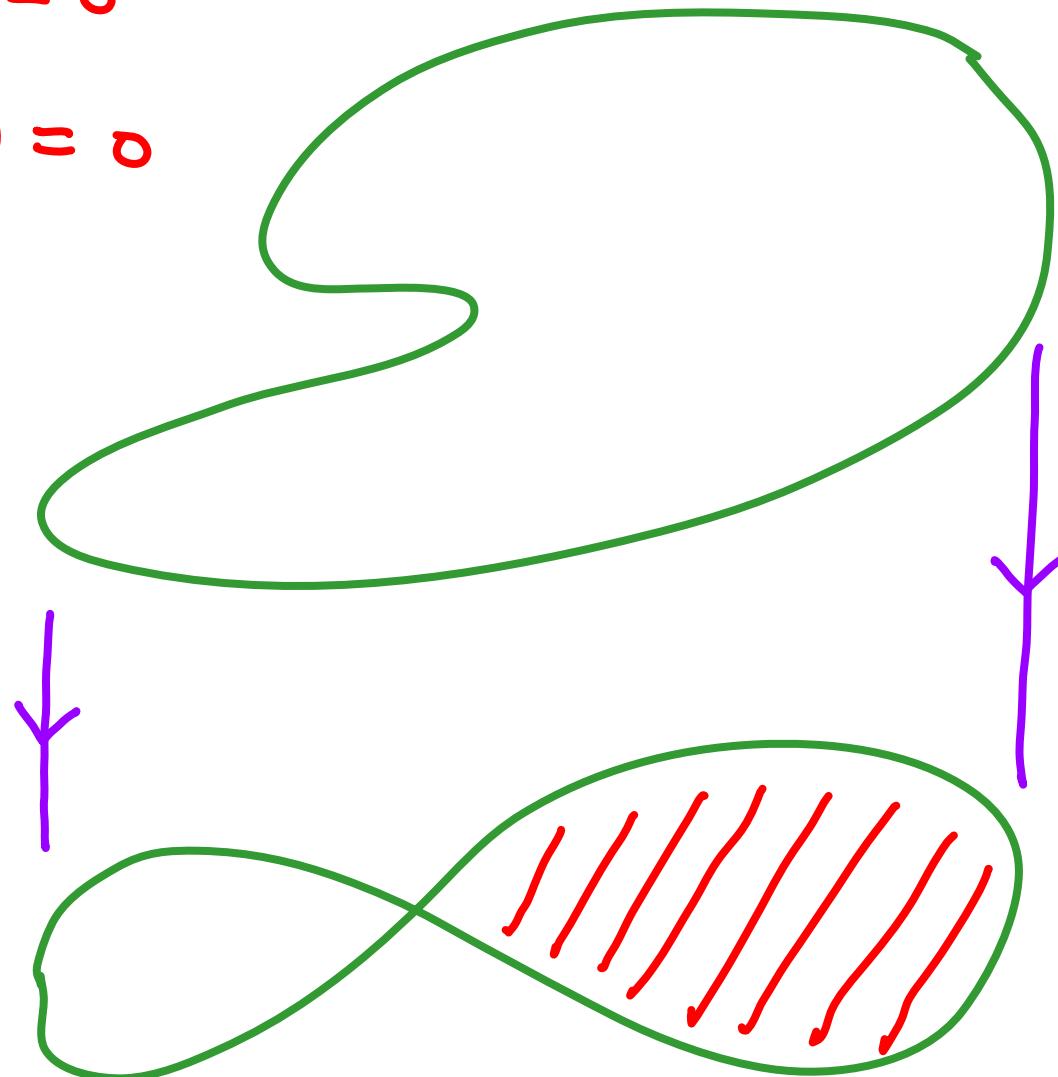
# Idea

$$f(x, y, z) = 0$$

$$f_z(x, y, z) = 0$$

$\mathbb{R}^3$

$\mathbb{R}^2$



Idea

$$\forall z \ f(x, y, z) > 0$$

# Idea

$$\forall z \ f(x, y, z) > 0$$

1. Trace the space curve

$$f = f_z = 0$$

numerically (Interval)

# Idea

$$\forall z \ f(x, y, z) > 0$$

1. Trace the space curve

$$f = f_z = 0$$

numerically (Interval)

2. Project it to the  $(x, y)$ -plane.

# Idea

$$\forall z \ f(x, y, z) > 0$$

1. Trace the space curve

$$f = f_z = 0$$

numerically (Interval)

2. Project it to the  $(x, y)$ -plane.

3. Determine the solution set  
by "sampling".

# Idea

$$\forall z \ f(x, y, z) > 0$$

1. Trace the space curve  
 $f = f_z = 0$   
numerically (Interval)
2. Project it to the  $(x, y)$ -plane.
3. Determine the solution set  
by "sampling".
- (4.) Estimate the boundary by curve fitting

Apply to the Challenging Problem,

$$\left\{ \begin{array}{l} f(a,b,s,t) = 0 \\ f_s(a,b,s,t) = 0 \\ f_t(a,b,s,t) = 0 \end{array} \right.$$

Apply to the Challenging Problem,

$$\left\{ \begin{array}{l} f(a,b,s,t) = 0 \\ f_s(a,b,s,t) = 0 \\ f_t(a,b,s,t) = 0 \end{array} \right.$$

$\Rightarrow$  Curve in 4-dim space.

Apply to the Challenging Problem,

$$\left\{ \begin{array}{l} f(a,b, s, t) = 0 \\ f_s(a,b, s, t) = 0 \\ f_t(a,b, s, t) = 0 \end{array} \right.$$

$\Rightarrow$  Curve in 4-dim space.

1. Curve Tracing
2. Project to  $(a,b)$  space

Finally - - -



to be filled in by

to be filled in by

You !