Informal Overview
Symbolic Computation (Quantifier Elimination)

Hoon Hong
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These slides were given during a joint lecture for two summer schools (SC2 and VTSA) held at MPI in 2017.

Many participants were from satisfiability checking and software verification. Some were from symbolic computation.

Thus

- Style : Informal / Selective
- Credit: Emphasis on the contributions of the participants from symbolic computation

Symbolic Computation

"Sociological Overview"

Symbolic:

Symbolic: Countable sets

Symbolic: Countable sets

- Finite fields

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- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc

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- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc
- Vector, Matrices, Tensors, etc
- Logical formulas

Symbolic: Countable sets

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- Integers, Rational numbers, Algebraic numbers, Computable real numbers, Infinitesimals, etc
- Polynomials, Differential equations, Computable power series, etc
- Vector, Matrices, Tensors, etc
- Logical formulas
- …….

Computation: Simplifications

Computation: Simplifications

$$x^2 + 2x + 1$$

$$\downarrow$$

$$(x+1)^2$$

Compute

Computation: Simplifications

$$x^2 + 2x + 1 \longrightarrow (x+1)^2$$

$$x + 1 = 0 \longrightarrow x = -1$$

Compute   Solve

Computation: Simplifications

$x^2 + 2x + 1$	$x + 1 = 0$	$\forall x\; x^2 \geq 0$
$\downarrow$	$\downarrow$	$\downarrow$
$(x+1)^2$	$x = -1$	$true$
Compute	Solve	Prove

Computation: Simplifications

x^2+2x+1	$x+1=0$	$\forall x\ x^2 \geq 0$	*Object*
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
$(x+1)^2$	$x=-1$	*true*	*Object*
Compute	Solve	Prove	Simplify

Main Achievements during 50 yrs

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- **Computing**
  - Polynomial Multiplication, Factorization, GCD, etc
    - Subresultant theory
    - Homomorphism and Lifting
    - …

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  - First order over reals
    - CAD
    - Block QE (Deformation and Optimization)
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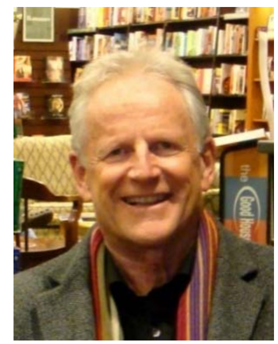

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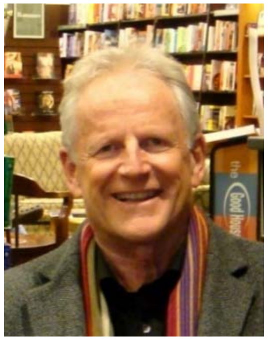

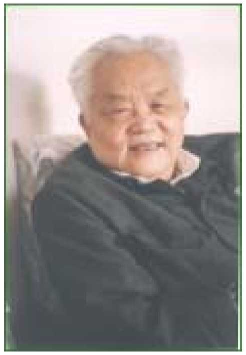

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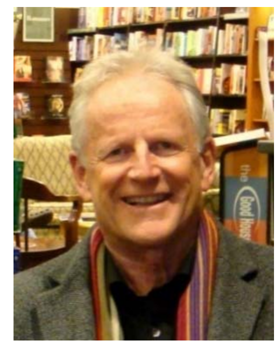

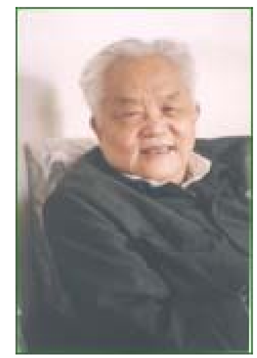

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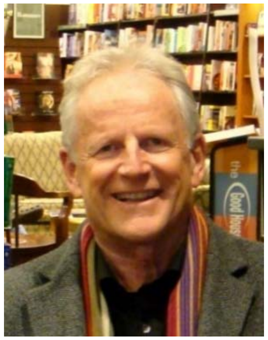

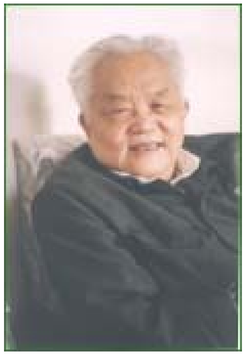

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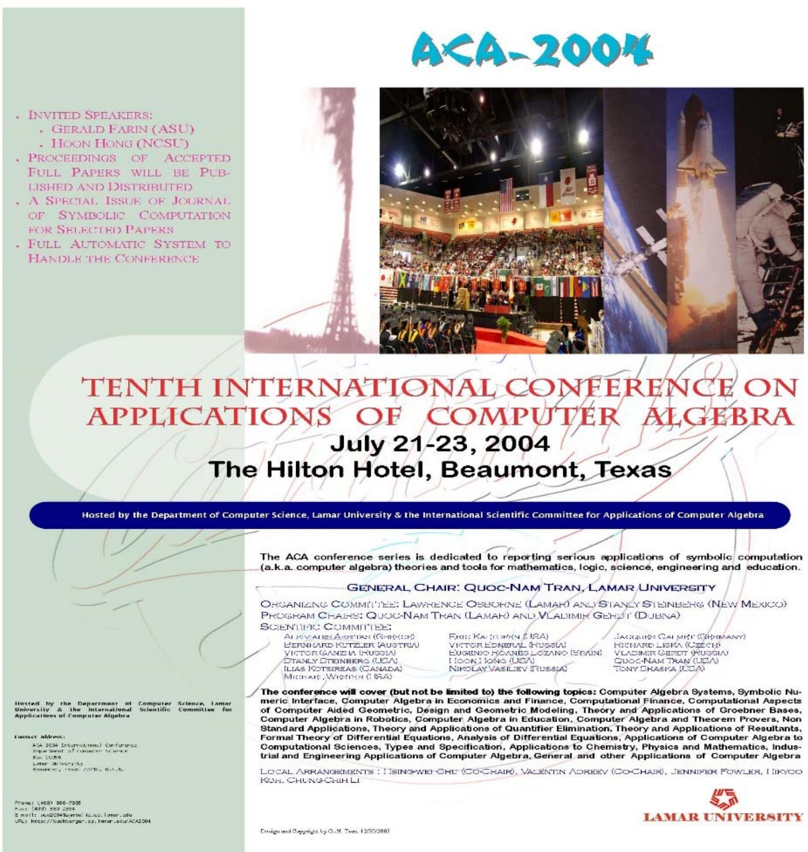
Emphasis:
Foundations and theories

.

Main Conferences

Applications of Computer Algebra (ACA)



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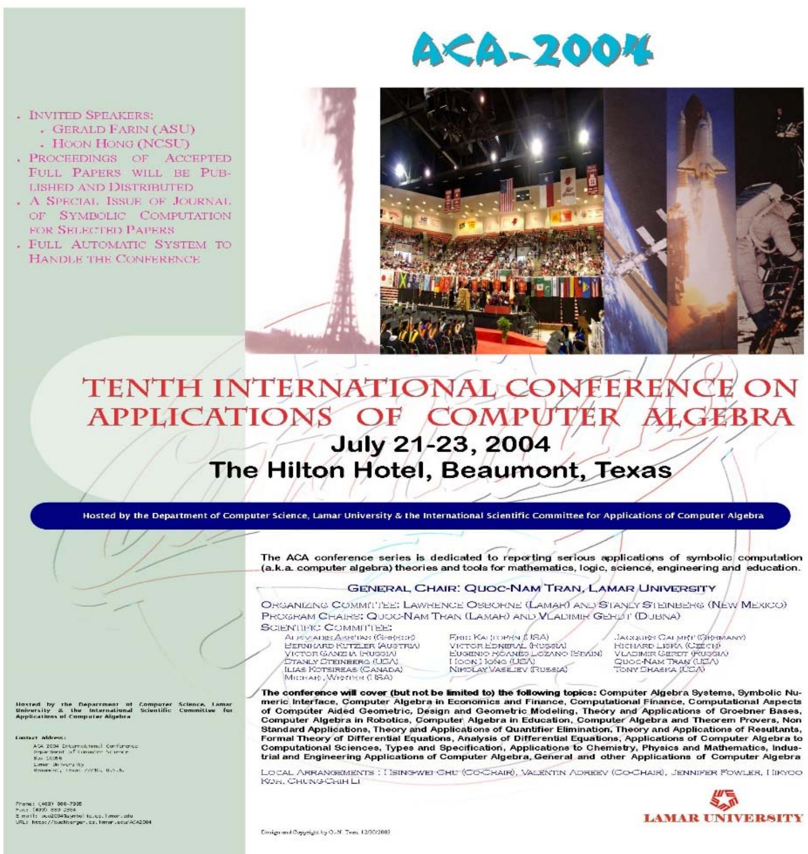
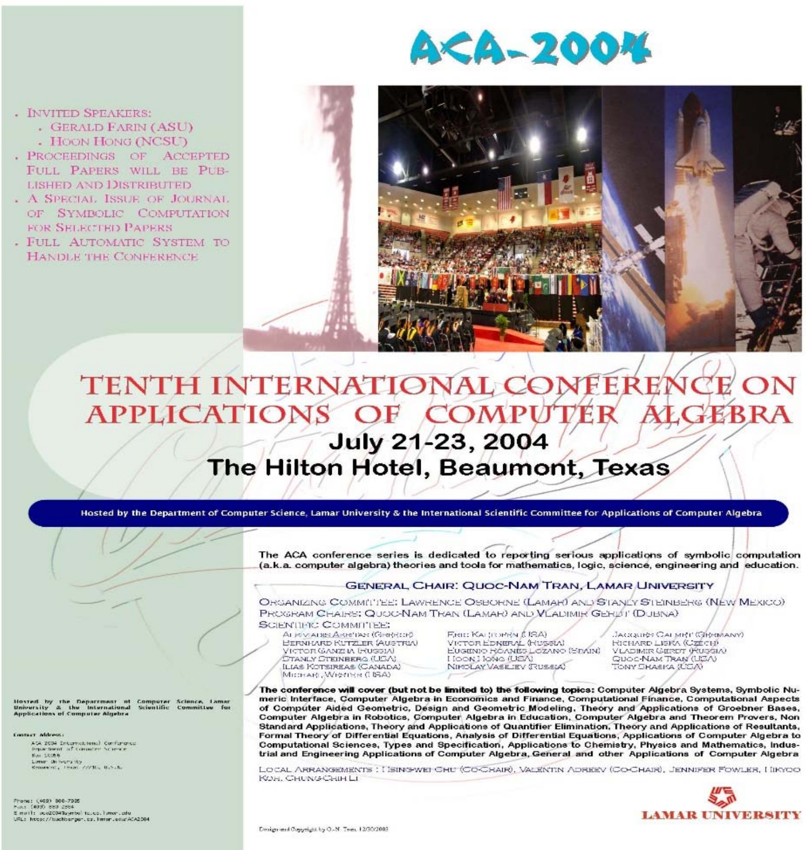


Started in:
1995

Main Conferences

Applications of Computer Algebra (ACA)



Started in:
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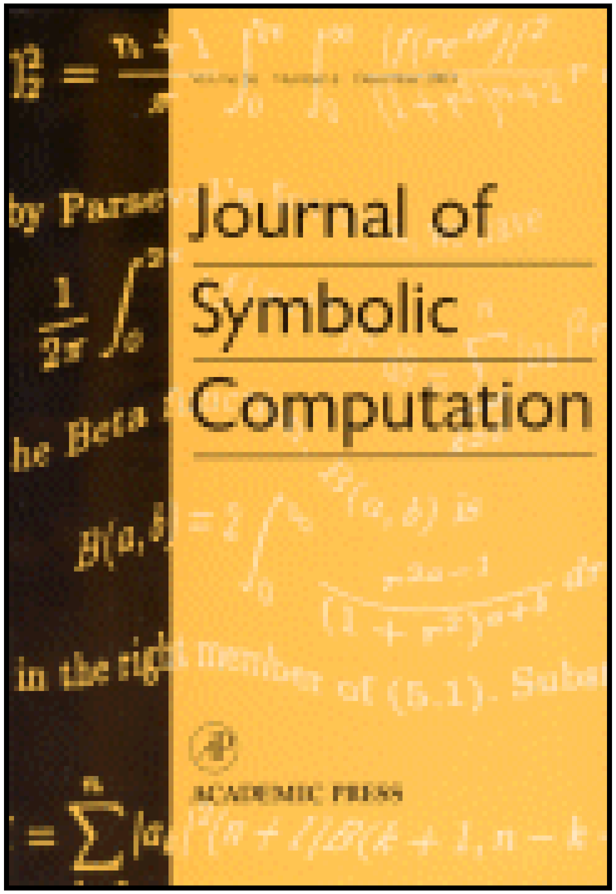
Main Journals

Main Journals

Journal of Symbolic Computation (JSC)



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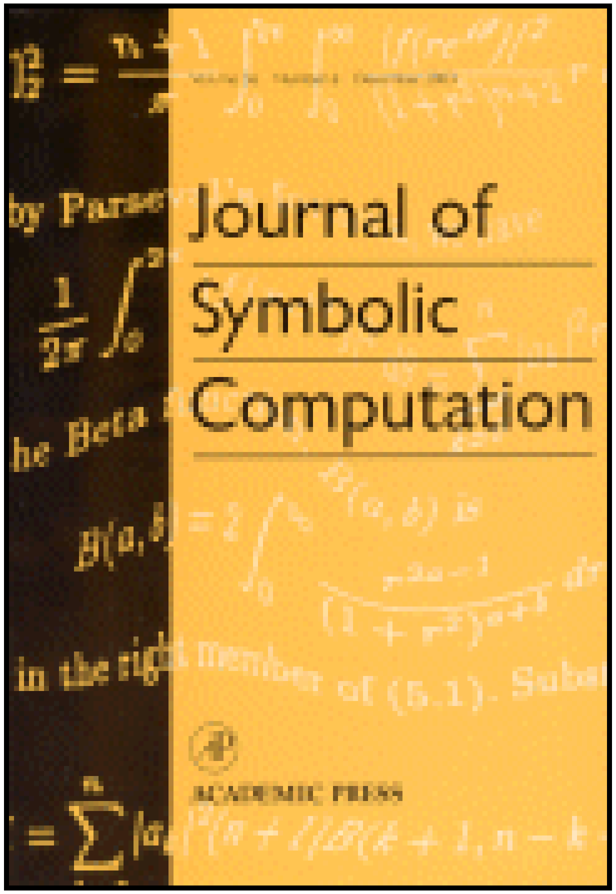

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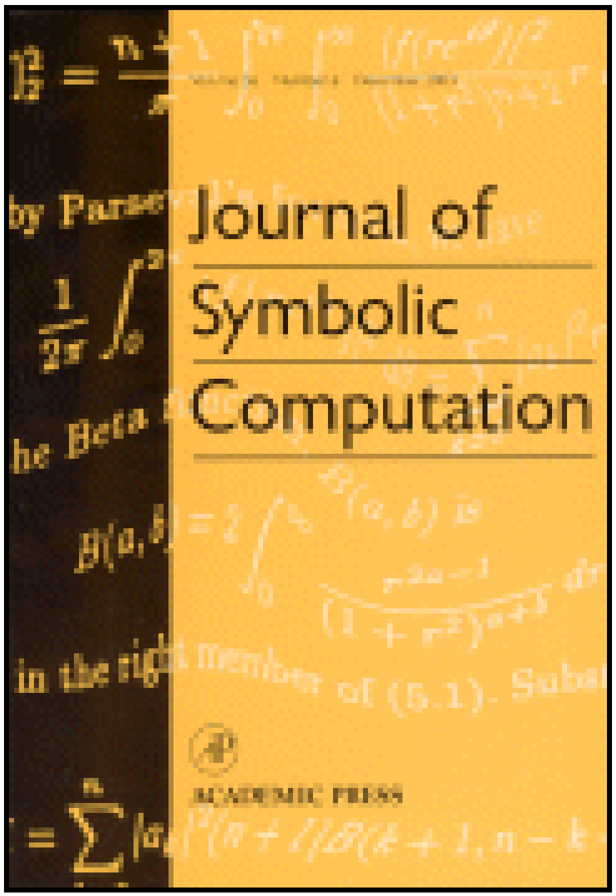

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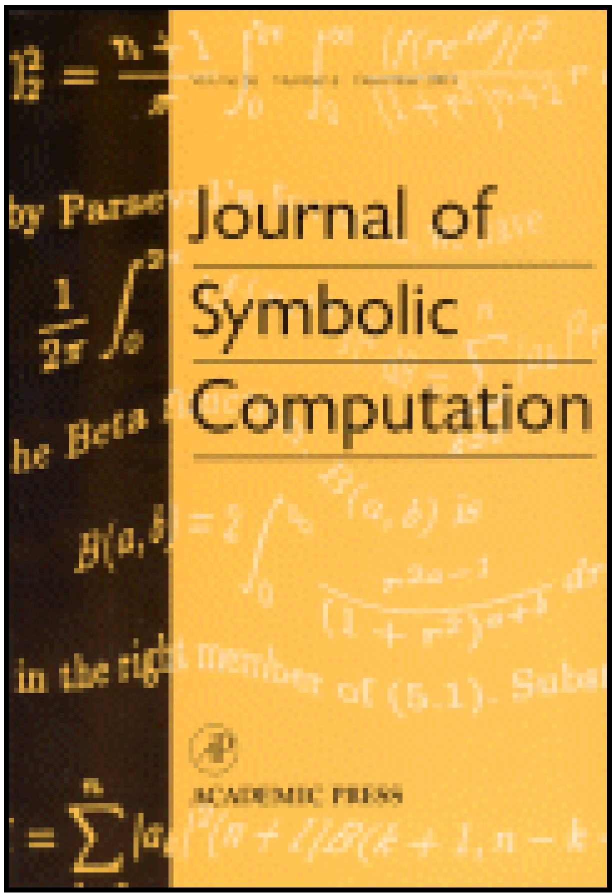

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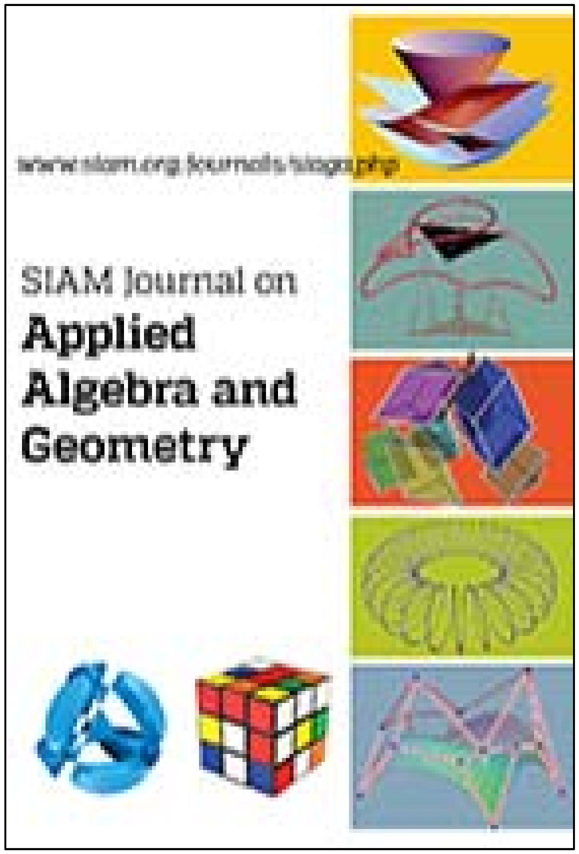
Foundations and Theories

Main Journals

Applied Algebra and Geometry



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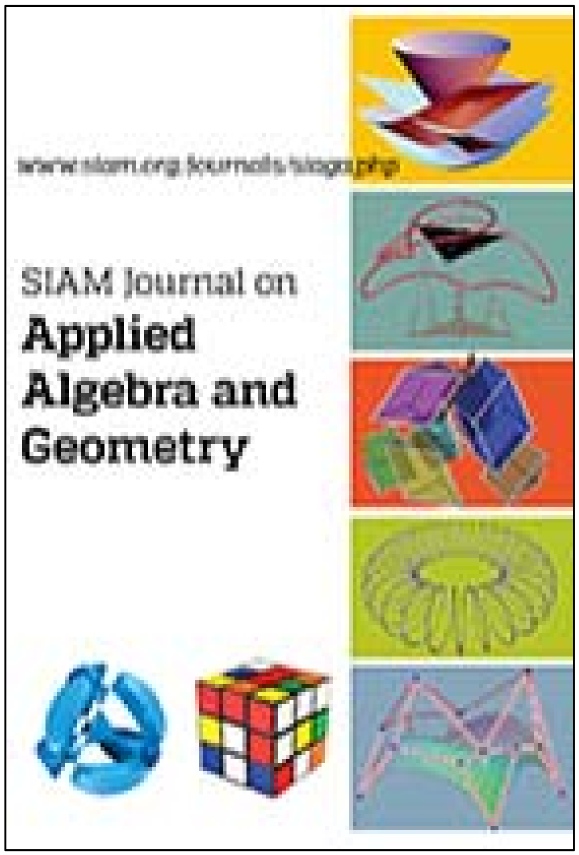

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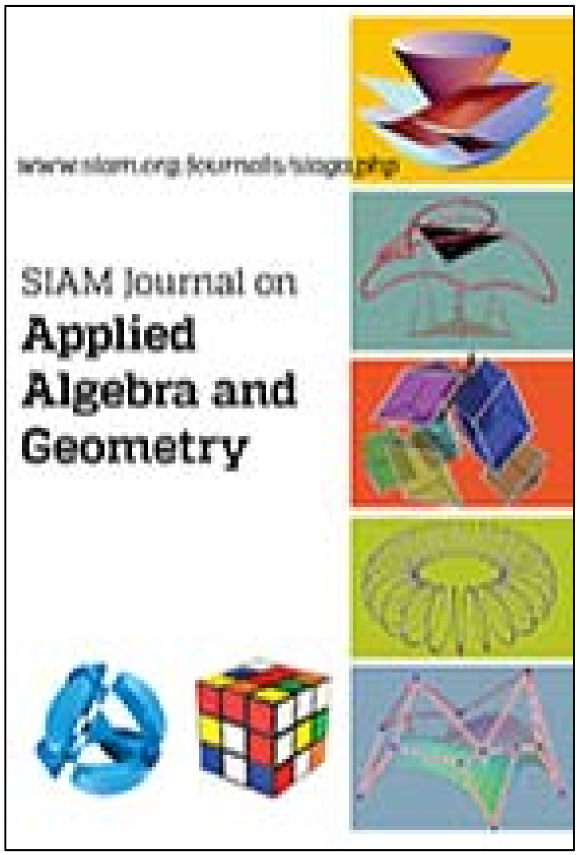

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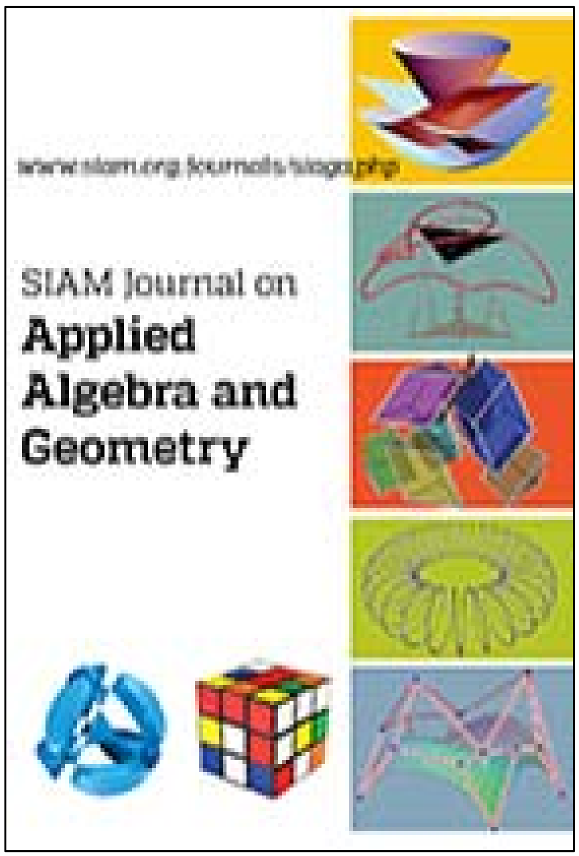

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Emphasis

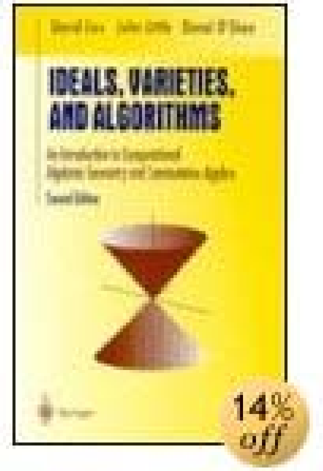
Applications

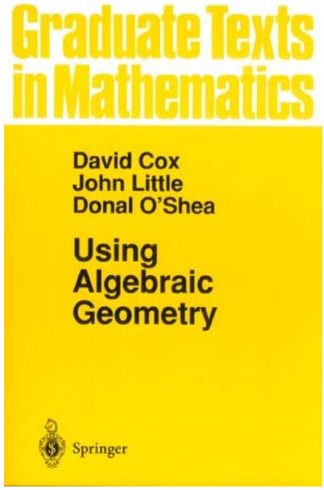
Textbooks

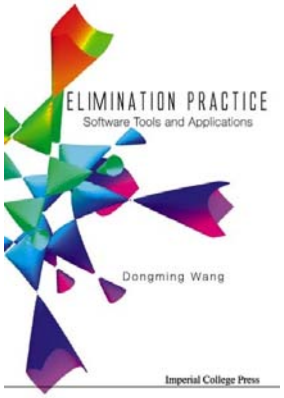
Textbooks

About 50 texts

- Undergraduate
- Graduate







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Maplesoft™
command the brilliance

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- Sponsors various meetings: ISSAC/ACA.

Commericial Software Systems

Mathematica

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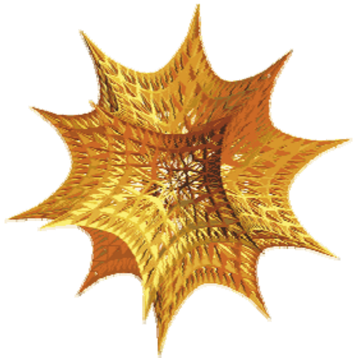

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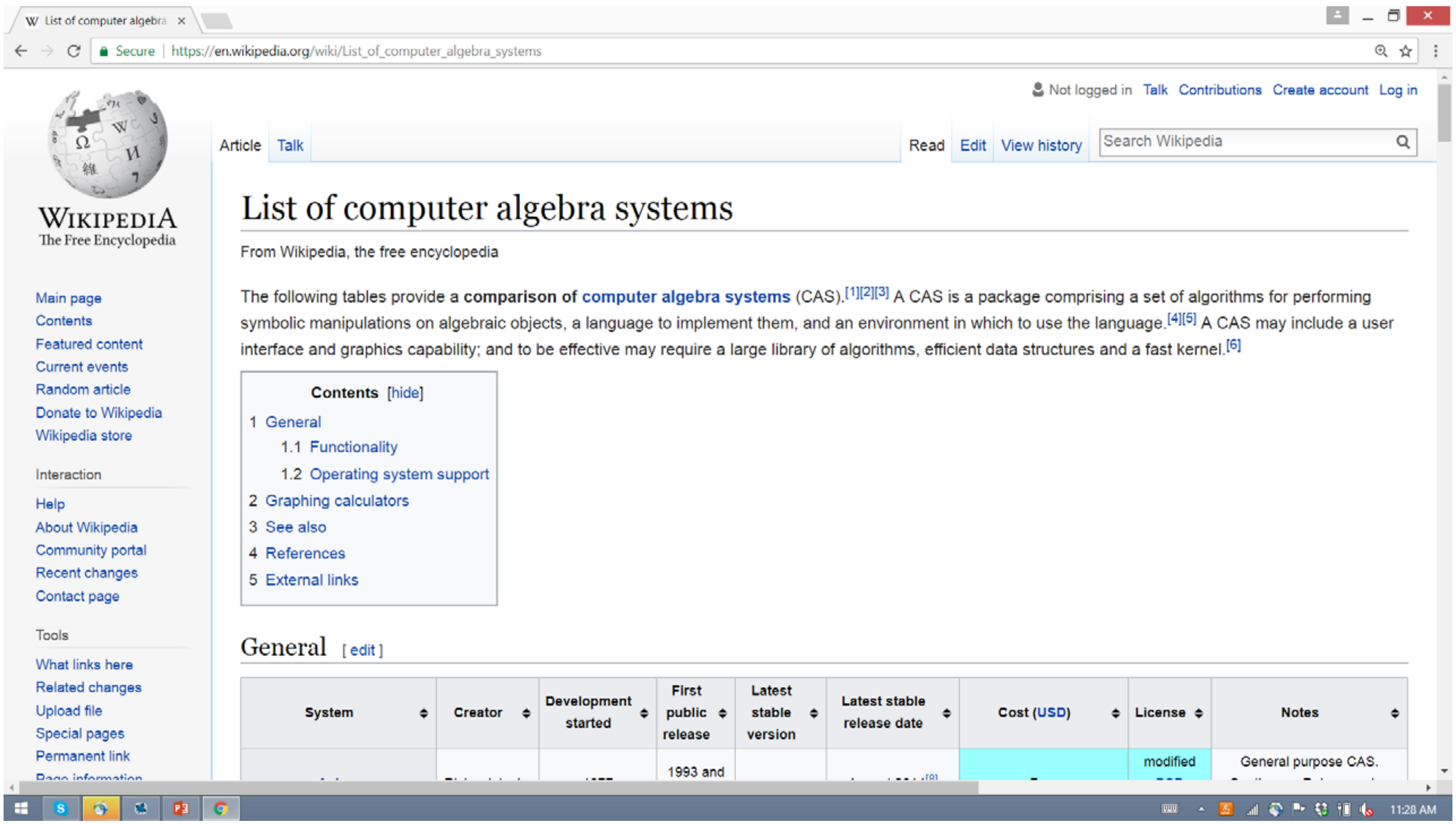
- Organizes major conferences for users

Academic Software Systems

https://en.wikipedia.org/wiki/List_of_computer_algebra_systems



List of computer algebra

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List of computer algebra systems

From Wikipedia, the free encyclopedia

The following tables provide a **comparison of computer algebra systems** (CAS).[1][2][3] A CAS is a package comprising a set of algorithms for performing symbolic manipulations on algebraic objects, a language to implement them, and an environment in which to use the language.[4][5] A CAS may include a user interface and graphics capability; and to be effective may require a large library of algorithms, efficient data structures and a fast kernel.[6]

Contents [hide]

General [edit]

System	Creator	Development started	First public release	Latest stable version	Latest stable release date	Cost (USD)	License	Notes
			1993 and				modified	General purpose CAS.

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Academic Software Systems

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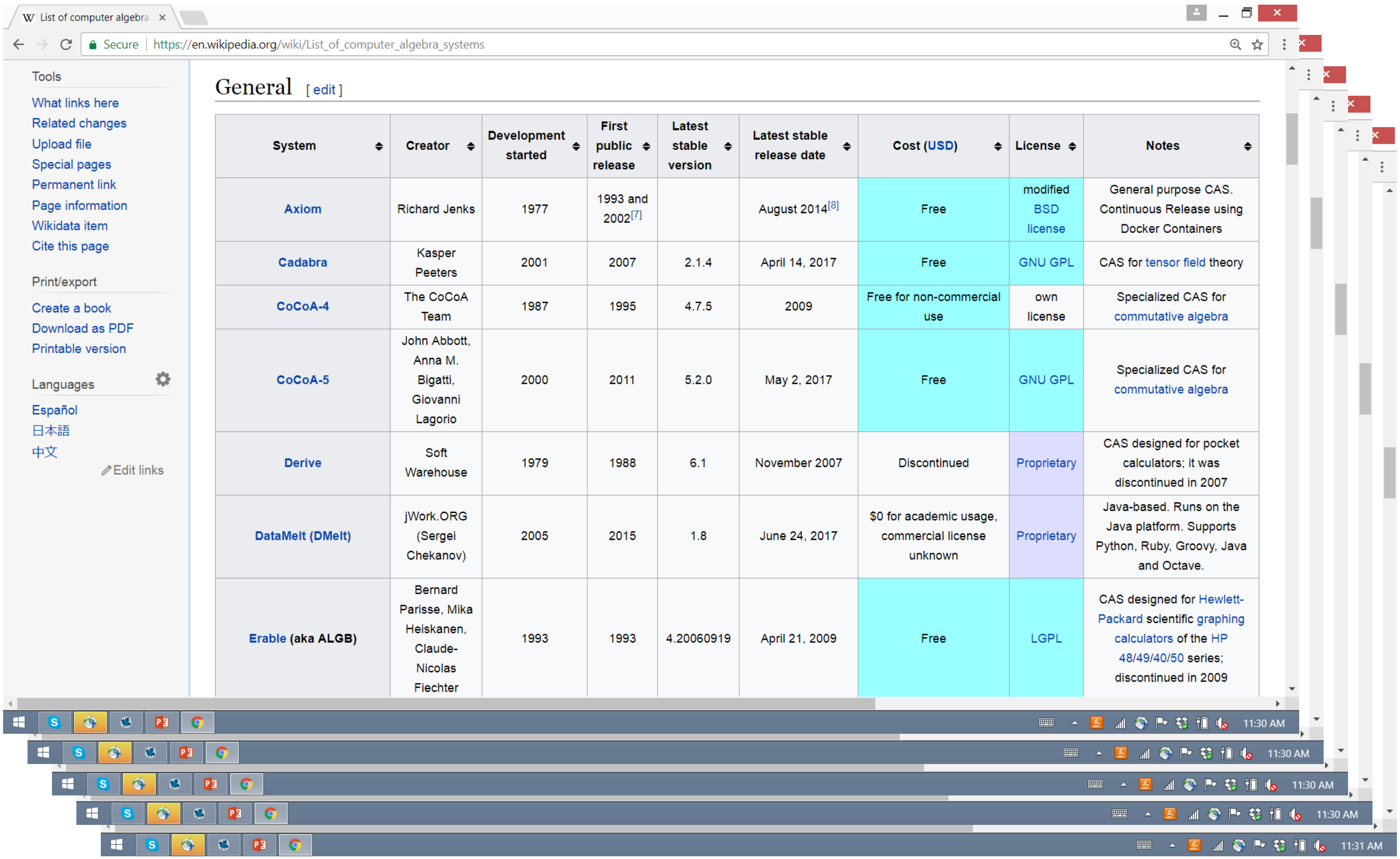
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- 日本語
- 中文

Edit links

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Axiom	Richard Jenks	1977	1993 and 2002[7]		August 2014[8]	Free	modified BSD license	General purpose CAS. Continuous Release using Docker Containers
Cadabra	Kasper Peeters	2001	2007	2.1.4	April 14, 2017	Free	GNU GPL	CAS for tensor field theory
CoCoA-4	The CoCoA Team	1987	1995	4.7.5	2009	Free for non-commercial use	own license	Specialized CAS for commutative algebra
CoCoA-5	John Abbott, Anna M. Bigatti, Giovanni Lagorio	2000	2011	5.2.0	May 2, 2017	Free	GNU GPL	Specialized CAS for commutative algebra
Derive	Soft Warehouse	1979	1988	6.1	November 2007	Discontinued	Proprietary	CAS designed for pocket calculators; it was discontinued in 2007
DataMelt (DMelt)	jWork.ORG (Sergei Chekanov)	2005	2015	1.8	June 24, 2017	$0 for academic usage, commercial license unknown	Proprietary	Java-based. Runs on the Java platform. Supports Python, Ruby, Groovy, Java and Octave.
Erable (aka ALGB)	Bernard Parisse, Mika Heiskanen, Claude-Nicolas Fiechter	1993	1993	4.20060919	April 21, 2009	Free	LGPL	CAS designed for Hewlett-Packard scientific graphing calculators of the HP 48/49/40/50 series; discontinued in 2009

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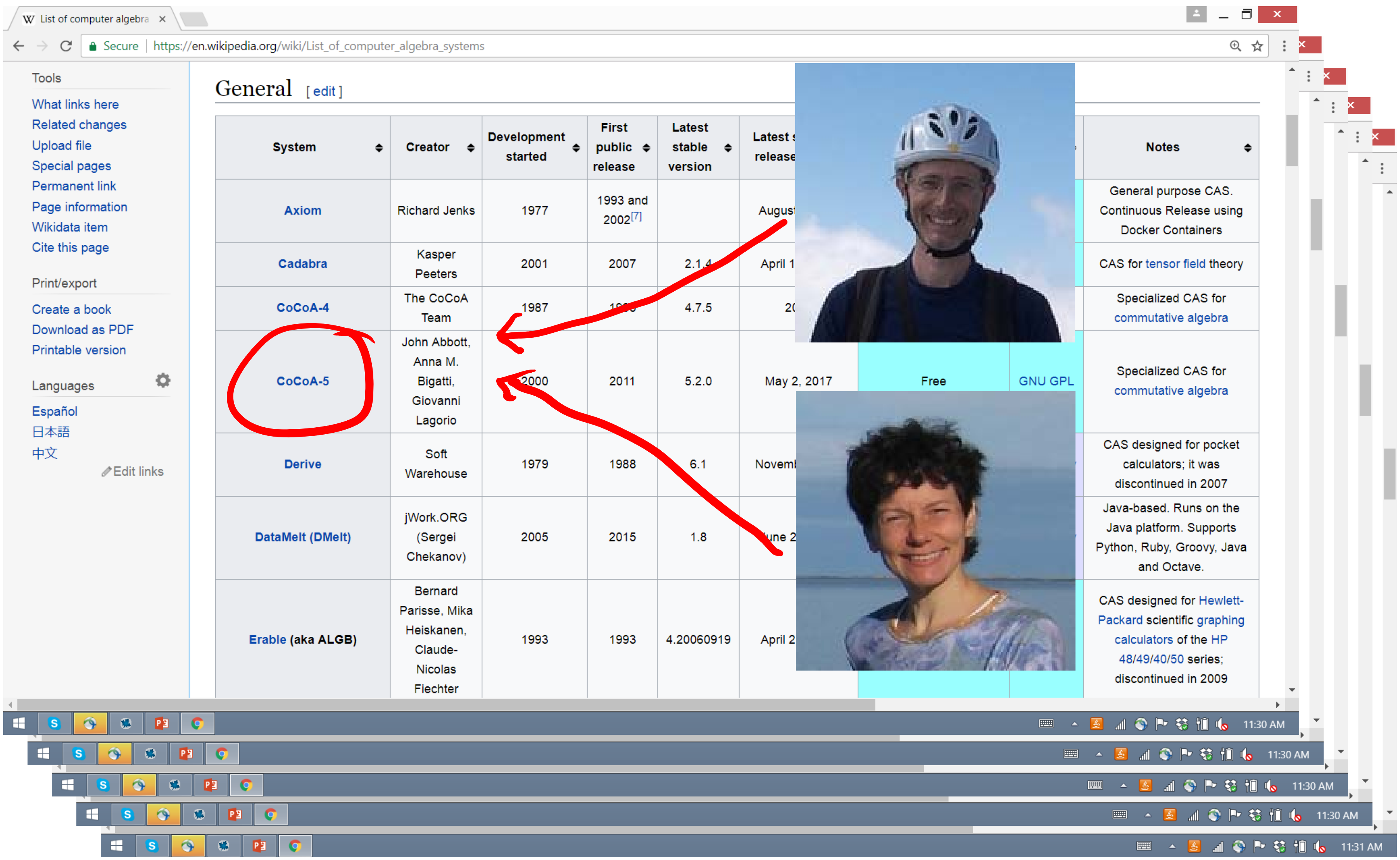

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Quantifier Elimination

"Sociological Overview"

Problem : Quantifier Elimination

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Input:

Output:

Problem : Quantifier Elimination

Input: $\forall x \quad x^2 + 1 > 0$

Output:

Problem : Quantifier Elimination

Input: $\forall x \quad x^2 + 1 > 0$

Output: $True$

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Input: $\forall x \quad x^2 + 1 > 0$

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Output:

Problem : Quantifier Elimination

Input: $\forall x \quad x^2 + 1 > 0$

Output: $True$

Input: $\forall x \quad x^2 + 3x + 1 > 0$

Output:

Problem : Quantifier Elimination

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Output: $True$

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Output: $False$

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Output: $True$

Input: $\forall x \quad x^2 + 3x + 1 > 0$

Output: $False$

Input:

Output:

Problem : Quantifier Elimination

Input: $\forall x \quad x^2+1>0$

Output: *True*

Input: $\forall x \quad x^2+3x+1>0$

Output: *False*

Input: $\forall x \quad x^2+bx+1>0$

Output:

Problem : Quantifier Elimination

Input: $\forall x \quad x^2 + 1 > 0$

Output: $True$

Input: $\forall x \quad x^2 + 3x + 1 > 0$

Output: $False$

Input: $\forall x \quad x^2 + bx + 1 > 0$

Output: $-2 < b < 2$

Problem : Quantifier Elimination

Problem : Quantifier Elimination

Input:

Output:

Problem : Quantifier Elimination

Input: $\forall x \exists y \quad x^2 + xy + b > 0$

$$\wedge$$

$$x + ay^2 + b \leq 0$$

Output:

Problem : Quantifier Elimination

Input: $\forall x \exists y \quad x^2 + xy + b > 0$

$$\wedge$$

$$x + ay^2 + b \leq 0$$

Output: $a < 0 \wedge b > 0$

Problem : Quantifier Elimination

Problem : Quantifier Elimination

Input:

Output:

Problem : Quantifier Elimination

Input: $\exists Y \quad F(X,Y)=0 \quad \wedge \quad G(X,Y)>0$

$$X=\{a,b\}$$
$$Y=\{c_1,s_1,c_2,s_2\}$$
$$F=\{c_1^2+s_1^2-1,\ c_2^2+s_2^2-1\}$$
$$G=\{g\}$$

$$\begin{aligned} g = {} & 4a^6b^2{c_1}^4{c_2}^2 - 8a^5b^3s_1s_2{c_1}^3c_2 - 8a^5b^3s_1s_2{c_1}^2{c_2}^2 + 4a^4b^4{c_1}^4{c_2}^2 + \\ & 16a^4b^4{c_1}^3{c_2}^3 + 4a^4b^4{c_1}^2{c_2}^4 - 8a^3b^5s_1s_2{c_1}^2{c_2}^2 - 8a^3b^5s_1s_2c_1{c_2}^3 + \\ & 4a^2b^6{c_1}^2{c_2}^4 - 4a^7bs_1s_2{c_1}^3 + 4a^6b^2{c_1}^4c_2 - 4a^6b^2{c_1}^3{c_2}^2 + 8a^5b^3s_1s_2{c_1}^3 + \\ & 12a^5b^3s_1s_2{c_1}^2c_2 + 16a^5b^3s_1s_2c_1{c_2}^2 - 8a^4b^4{c_1}^4c_2 - 24a^4b^4{c_1}^3{c_2}^2 - \\ & 24a^4b^4{c_1}^2{c_2}^3 - 8a^4b^4c_1{c_2}^4 + 16a^3b^5s_1s_2{c_1}^2c_2 + 12a^3b^5s_1s_2c_1{c_2}^2 + \\ & 8a^3b^5s_1s_2{c_2}^3 - 4a^2b^6{c_1}^2{c_2}^3 + 4a^2b^6c_1{c_2}^4 - 4ab^7s_1s_2{c_2}^3 + a^8{c_1}^4 + \\ & 12a^7bs_1s_2{c_1}^2 - 8a^6b^2{c_1}^4 - 12a^6b^2{c_1}^3c_2 - 12a^6b^2{c_1}^2{c_2}^2 - 4a^5b^3s_1s_2{c_1}^2 - \\ & 8a^5b^3s_1s_2{c_2}^2 + 4a^4b^4{c_1}^4 + 22a^4b^4{c_1}^2{c_2}^2 + 4a^4b^4{c_2}^4 - 4a^4b^2{c_1}^4{c_2}^2 - \\ & 8a^3b^5s_1s_2{c_1}^2 - 4a^3b^5s_1s_2{c_2}^2 + 8a^3b^3s_1s_2{c_1}^2{c_2}^2 - 12a^2b^6{c_1}^2{c_2}^2 - \\ & 12a^2b^6c_1{c_2}^3 - 8a^2b^6{c_2}^4 - 4a^2b^4{c_1}^2{c_2}^4 + 12ab^7s_1s_2{c_2}^2 + b^8{c_2}^4 - 4a^8{c_1}^3 - \\ & 12a^7bs_1s_2c_1 + 16a^6b^2{c_1}^3 + 12a^6b^2{c_1}^2c_2 + 20a^6b^2c_1{c_2}^2 - 16a^5b^3s_1s_2c_1 - \\ & 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2{c_1}^3 + 8a^4b^4{c_1}^3 + 12a^4b^4{c_1}^2c_2 + 12a^4b^4c_1{c_2}^2 + \\ & 8a^4b^4{c_2}^3 + 4a^4b^2{c_1}^4c_2 + 4a^4b^2{c_1}^3{c_2}^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\ & 12a^3b^3s_1s_2{c_1}^2c_2 - 12a^3b^3s_1s_2c_1{c_2}^2 + 20a^2b^6{c_1}^2c_2 + 12a^2b^6c_1{c_2}^2 + \\ & 16a^2b^6{c_2}^3 + 4a^2b^4{c_1}^2{c_2}^3 + 4a^2b^4c_1{c_2}^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2{c_2}^3 - \\ & 4b^8{c_2}^3 + 6a^8{c_1}^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2{c_2}^2 - 2a^6{c_1}^4 + \\ & 12a^5b^3s_1s_2 - 12a^5bs_1s_2{c_1}^2 - 14a^4b^4{c_1}^2 + 8a^4b^4c_1c_2 - 14a^4b^4{c_2}^2 - \\ & 4a^4b^2{c_1}^3c_2 + 10a^4b^2{c_1}^2{c_2}^2 + 12a^3b^5s_1s_2 + 4a^3b^3s_1s_2{c_1}^2 + 16a^3b^3s_1s_2c_1c_2 + \\ & 4a^3b^3s_1s_2{c_2}^2 - 8a^2b^6{c_1}^2 - 4a^2b^6c_1c_2 + 10a^2b^4{c_1}^2{c_2}^2 - 4a^2b^4c_1{c_2}^3 + \\ & 4ab^7s_1s_2 - 12ab^5s_1s_2{c_2}^2 + 6b^8{c_2}^2 - 2b^6{c_2}^4 - 4a^8c_1 - 16a^6b^2c_1 + \\ & 8a^6{c_1}^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2{c_1}^2c_2 - 16a^4b^2c_1{c_2}^2 - \\ & 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4{c_1}^2c_2 - 8a^2b^4c_1{c_2}^2 + \\ & 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6{c_2}^3 + a^8 + 8a^6b^2 - 12a^6{c_1}^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\ & 2a^4b^2{c_1}^2 + 12a^4b^2c_1c_2 + 6a^4b^2{c_2}^2 + a^4{c_1}^4 + 8a^2b^6 + 6a^2b^4{c_1}^2 + 12a^2b^4c_1c_2 - \\ & 2a^2b^4{c_2}^2 + 2a^2b^2{c_1}^2{c_2}^2 - 4ab^5s_1s_2 + b^8 - 12b^6{c_2}^2 + b^4{c_2}^4 + 8a^6c_1 + 4a^4b^2c_1 - \\ & 4a^4b^2c_2 - 4a^4{c_1}^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\ & 4b^4{c_2}^3 - 2a^6 - 2a^4b^2 + 8a^4{c_1}^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2{c_1}^2 + 4a^2b^2c_1c_2 - \\ & 2a^2b^2{c_2}^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4{c_2}^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\ & 3a^4 + 6a^2b^2 - 2a^2{c_1}^2 + 3b^4 - 2b^2{c_2}^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2 \end{aligned}$$

Output:

Problem : Quantifier Elimination

Input: $\exists Y \quad F(X,Y)=0 \quad \wedge \quad G(X,Y)>0$

$$X=\{a,b\}$$
$$Y=\{c_1,s_1,c_2,s_2\}$$
$$F=\{c_1^2+s_1^2-1,\ c_2^2+s_2^2-1\}$$
$$G=\{g\}$$

$$\begin{aligned}
g = {} & 4a^6b^2c_1^4c_2^2 - 8a^5b^3s_1s_2c_1^3c_2 - 8a^5b^3s_1s_2c_1^2c_2^2 + 4a^4b^4c_1^4c_2^2 + \\
& 16a^4b^4c_1^3c_2^3 + 4a^4b^4c_1^2c_2^4 - 8a^3b^5s_1s_2c_1^2c_2^2 - 8a^3b^5s_1s_2c_1c_2^3 + \\
& 4a^2b^6c_1^2c_2^4 - 4a^7bs_1s_2c_1^3 + 4a^6b^2c_1^4c_2 - 4a^6b^2c_1^3c_2^2 + 8a^5b^3s_1s_2c_1^3 + \\
& 12a^5b^3s_1s_2c_1^2c_2 + 16a^5b^3s_1s_2c_1c_2^2 - 8a^4b^4c_1^4c_2 - 24a^4b^4c_1^3c_2^2 - \\
& 24a^4b^4c_1^2c_2^3 - 8a^4b^4c_1c_2^4 + 16a^3b^5s_1s_2c_1^2c_2 + 12a^3b^5s_1s_2c_1c_2^2 + \\
& 8a^3b^5s_1s_2c_2^3 - 4a^2b^6c_1^2c_2^3 + 4a^2b^6c_1c_2^4 - 4ab^7s_1s_2c_2^3 + a^8c_1^4 + \\
& 12a^7bs_1s_2c_1^2 - 8a^6b^2c_1^4 - 12a^6b^2c_1^3c_2 - 12a^6b^2c_1^2c_2^2 - 4a^5b^3s_1s_2c_1^2 - \\
& 8a^5b^3s_1s_2c_2^2 + 4a^4b^4c_1^4 + 22a^4b^4c_1^2c_2^2 + 4a^4b^4c_2^4 - 4a^4b^2c_1^4c_2^2 - \\
& 8a^3b^5s_1s_2c_1^2 - 4a^3b^5s_1s_2c_2^2 + 8a^3b^3s_1s_2c_1^2c_2^2 - 12a^2b^6c_1^2c_2^2 - \\
& 12a^2b^6c_1c_2^3 - 8a^2b^6c_2^4 - 4a^2b^4c_1^2c_2^4 + 12ab^7s_1s_2c_2^2 + b^8c_2^4 - 4a^8c_1^3 - \\
& 12a^7bs_1s_2c_1 + 16a^6b^2c_1^3 + 12a^6b^2c_1^2c_2 + 20a^6b^2c_1c_2^2 - 16a^5b^3s_1s_2c_1 - \\
& 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1^3 + 8a^4b^4c_1^3 + 12a^4b^4c_1^2c_2 + 12a^4b^4c_1c_2^2 + \\
& 8a^4b^4c_2^3 + 4a^4b^2c_1^4c_2 + 4a^4b^2c_1^3c_2^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\
& 12a^3b^3s_1s_2c_1^2c_2 - 12a^3b^3s_1s_2c_1c_2^2 + 20a^2b^6c_1^2c_2 + 12a^2b^6c_1c_2^2 + \\
& 16a^2b^6c_2^3 + 4a^2b^4c_1^2c_2^3 + 4a^2b^4c_1c_2^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2^3 - \\
& 4b^8c_2^3 + 6a^8c_1^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2^2 - 2a^6c_1^4 + \\
& 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1^2 - 14a^4b^4c_1^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2^2 - \\
& 4a^4b^2c_1^3c_2 + 10a^4b^2c_1^2c_2^2 + 12a^3b^5s_1s_2 + 4a^3b^3s_1s_2c_1^2 + 16a^3b^3s_1s_2c_1c_2 + \\
& 4a^3b^3s_1s_2c_2^2 - 8a^2b^6c_1^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1^2c_2^2 - 4a^2b^4c_1c_2^3 + \\
& 4ab^7s_1s_2 - 12ab^5s_1s_2c_2^2 + 6b^8c_2^2 - 2b^6c_2^4 - 4a^8c_1 - 16a^6b^2c_1 + \\
& 8a^6c_1^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1^2c_2 - 16a^4b^2c_1c_2^2 - \\
& 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1^2c_2 - 8a^2b^4c_1c_2^2 + \\
& 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2^3 + a^8 + 8a^6b^2 - 12a^6c_1^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\
& 2a^4b^2c_1^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2^2 + a^4c_1^4 + 8a^2b^6 + 6a^2b^4c_1^2 + 12a^2b^4c_1c_2 - \\
& 2a^2b^4c_2^2 + 2a^2b^2c_1^2c_2^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2^2 + b^4c_2^4 + 8a^6c_1 + 4a^4b^2c_1 - \\
& 4a^4b^2c_2 - 4a^4c_1^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\
& 4b^4c_2^3 - 2a^6 - 2a^4b^2 + 8a^4c_1^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1^2 + 4a^2b^2c_1c_2 - \\
& 2a^2b^2c_2^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\
& 3a^4 + 6a^2b^2 - 2a^2c_1^2 + 3b^4 - 2b^2c_2^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2
\end{aligned}$$

Output: $a^6+3a^4b^2+3a^2b^4+b^6-3a^4+21a^2b^2-3b^4+3a^2+3b^2>1$

Problem : Quantifier Elimination

Problem : Quantifier Elimination

Input:

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

$\forall \quad \exists$

Output:

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

$\forall \quad \exists$

Output: *Expression*

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

$\forall \quad \exists$

Output: *Expression*

equivalent

Problem : Quantifier Elimination

Input: *Expression* made of

Algebraic expressions

$= \quad \neq \quad > \quad < \quad \geq \quad \leq$

$\wedge \quad \vee \quad \neg \quad \Rightarrow \quad \Leftarrow \quad \Leftrightarrow$

$\forall \quad \exists$

Output: *Expression*

equivalent

without $\forall \quad \exists$

Motivation

Motivation

Foundation of Mathematics

- Hilbert's Program (1900)

Motivation

Foundation of Mathematics

- Hilbert's Program (1900)

Applications in Science and Engineering

- Stability analysis of PDE and Finite differences
- Robust control system design
- Reachability analysis
- Parametric optimization
- Hybrid system analysis
- Parameter estimation
- Robot motion planning
- Computer vision
- Dynamic geometric constraint solving
- Education software for real analysis
- ….

Geometric Reasoning

Geometric Reasoning

A. Dolzman, ADG *(2000), (Edited by X. Gao, D.Wang and L.Yang)*

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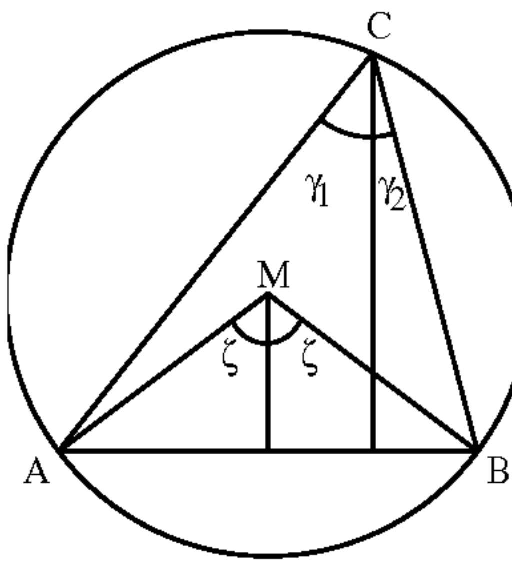
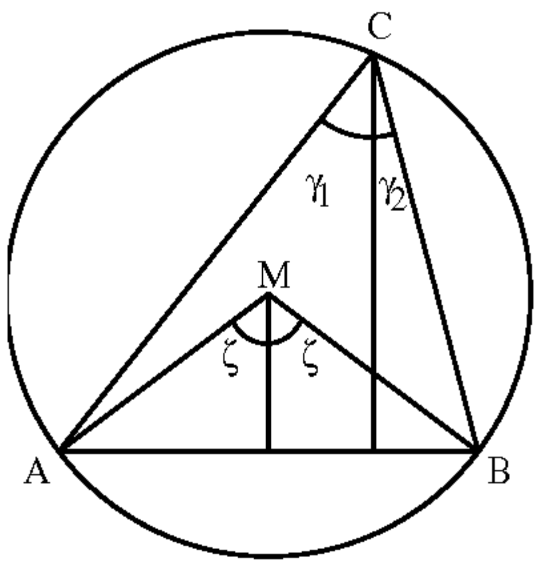


Geometric Reasoning

A. Dolzman, ADG *(2000), (Edited by X. Gao, D.Wang and L.Yang)*



$$\forall x \forall t_1 \forall t_2 \forall t \forall b \big(c^2 = a^2 + b^2 \wedge c^2 = x_0^2 + (y_0 - b)^2 \wedge$$
$$y_0 t_1 = a + x_0 \wedge y_0 t_2 = a - x_0 \wedge (1 - t_1 t_2) t = t_1 + t_2 \longrightarrow bt = a \big)$$

Nonlinear Control of Aircraft

Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)

Nonlinear Control of Aircraft

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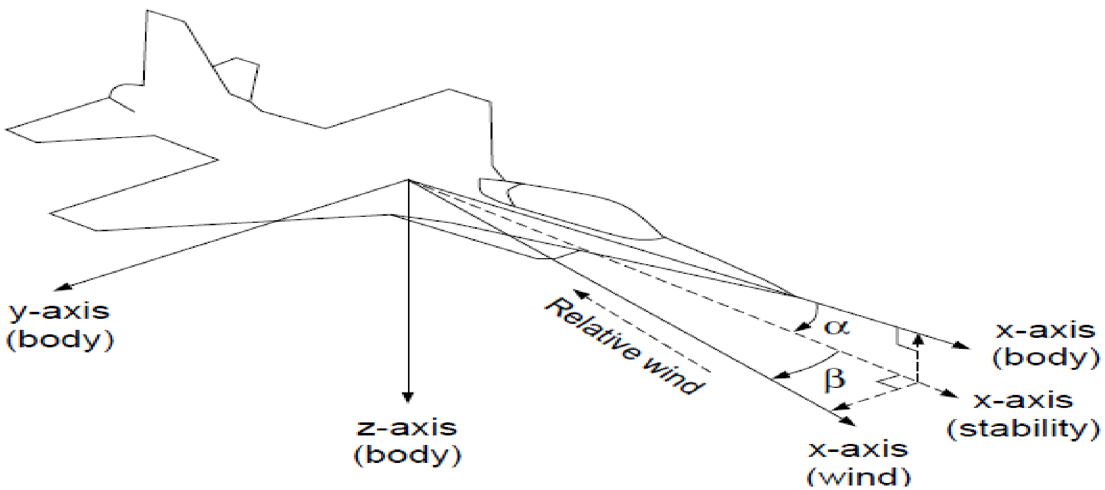
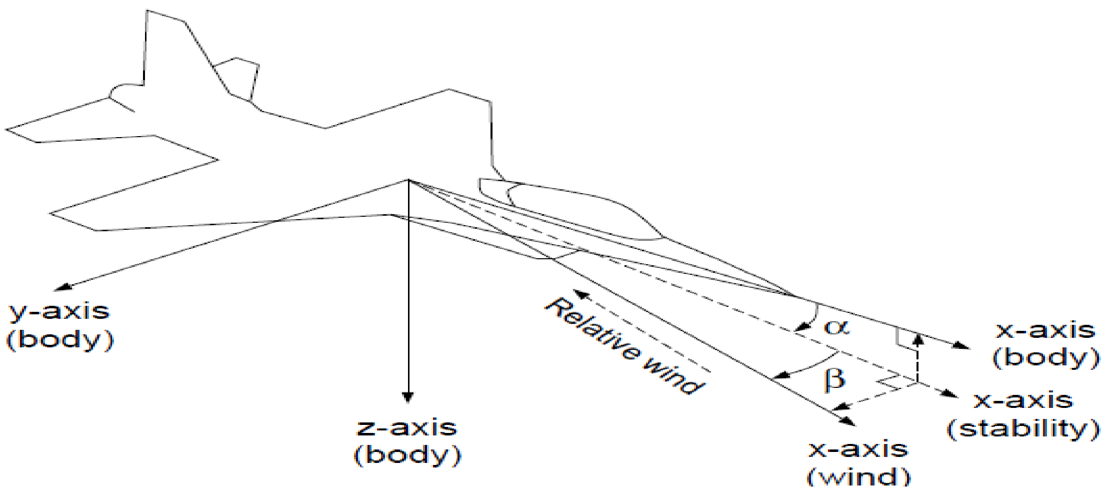


Nonlinear Control of Aircraft

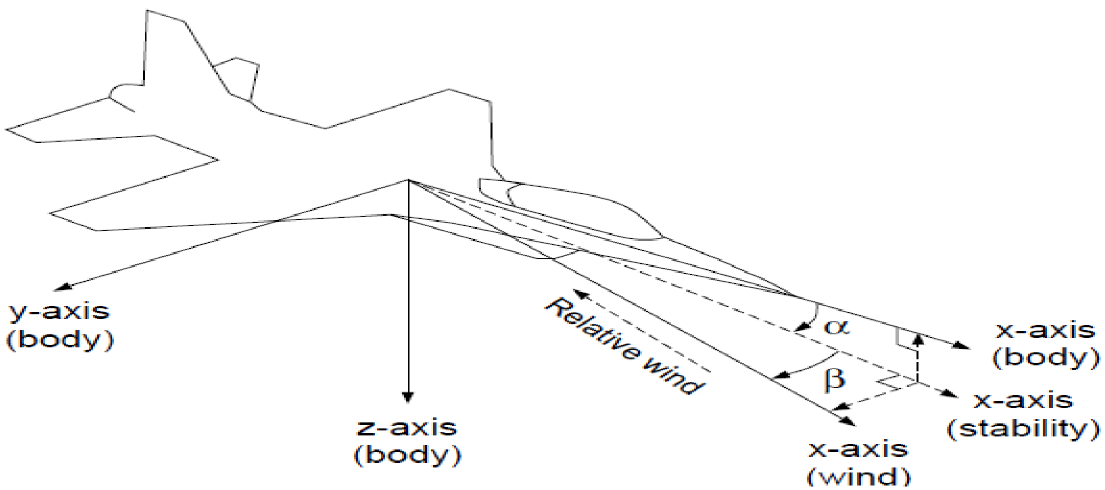
M. Jirstrand *J. Symbolic Computation* (1997)



$$\exists u_1 \, \exists u_2 \, \exists u_3 \, [C_L = 0 \, \wedge \, C_M = 0 \, \wedge \, C_N = 0 \, \wedge \, u_i^2 \leq 1, \, i = 1,2,3]$$

Nonlinear Control of Aircraft

M. Jirstrand *J. Symbolic Computation* (1997)

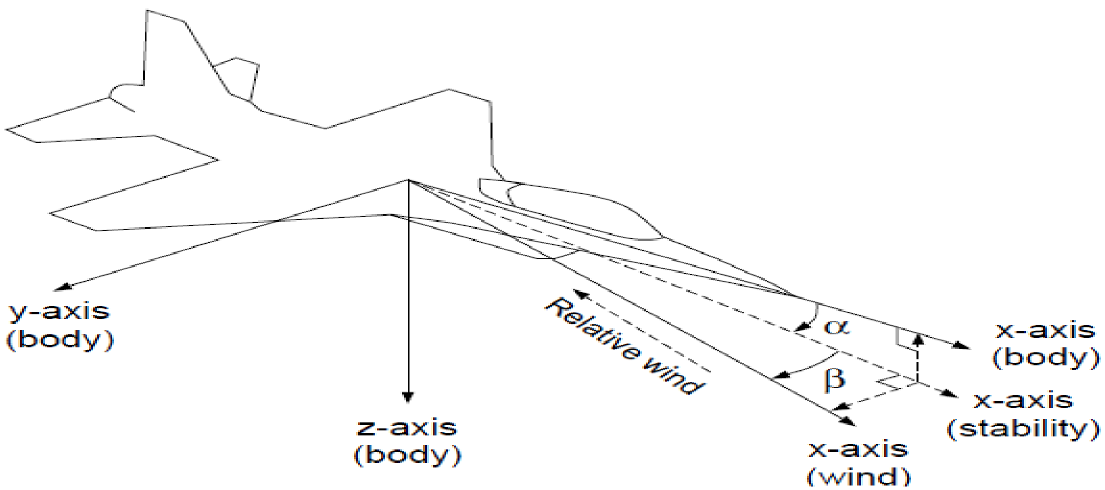


$$\exists u_1\, \exists u_2\, \exists u_3\, \left[C_L = 0 \;\wedge\; C_M = 0 \;\wedge\; C_N = 0 \;\wedge\; u_i^2 \le 1,\, i = 1,2,3 \right]$$

$$\begin{aligned}
C_L(x_1, x_2, u1, u3) =& -38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3\\
&+ u_1(-52 - 2x_1 + 114x_1^2 - 79x_1^3 + 7x_2^2 + 14x_1x_2^2)\\
&+ u_3(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 - 13x_1x_2^2\\
&+ 20x_1^2x_2^2 + 11x_2^4)\\
C_M(x_1, u_2) =& -12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 + 17u_2^2x_1\\
&- 202x_1^2 + 81u_2x_1^2 + 139x_1^3\\
C_N(x_1, x_2, u1, u3) =& 139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 - 38x_2^3 + 185x_1x_2^3\\
&+ u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2)\\
&+ u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 + 63x_1x_2^2 - 54x_1^4\\
&- 69x_1^2x_2^2 - 26x_2^4)
\end{aligned}$$

Nonlinear Control of Aircraft

M. Jirstrand   *J. Symbolic Computation* (1997)



$$\exists u_1\, \exists u_2\, \exists u_3\, \left[C_L = 0 \;\wedge\; C_M = 0 \;\wedge\; C_N = 0 \;\wedge\; u_i^2 \le 1,\, i = 1,2,3\right]$$

$$\begin{aligned}
C_L(x_1, x_2, u1, u3) = & -38x_2 - 170x_1x_2 + 148x_1^2x_2 + 4x_2^3 \\
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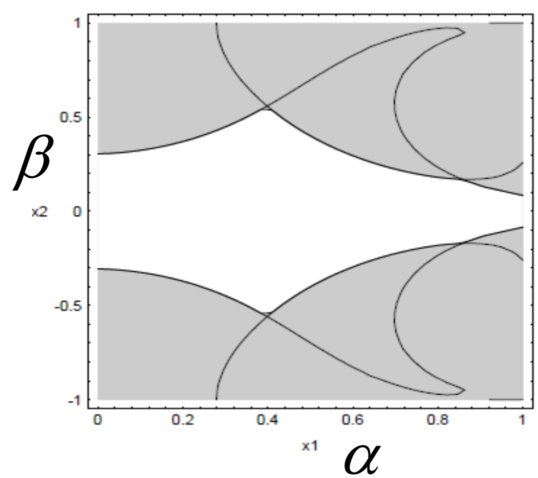
& + u_3(14 - 10x_1 + 37x_1^2 - 48x_1^3 + 8x_1^4 - 13x_2^2 - 13x_1x_2^2 \\
& + 20x_1^2x_2^2 + 11x_2^4) \\
C_M(x_1, u_2) = & -12 - 125u_2 + u_2^2 + 6u_2^3 + 95x_1 - 21u_2x_1 + 17u_2^2x_1 \\
& - 202x_1^2 + 81u_2x_1^2 + 139x_1^3 \\
C_N(x_1, x_2, u1, u3) = & 139x_2 - 112x_1x_2 - 388x_1^2x_2 + 215x_1^3x_2 - 38x_2^3 + 185x_1x_2^3 \\
& + u_1(-11 + 35x_1 - 22x_1^2 + 5x_2^2 + 10x_1^3 - 17x_1x_2^2) \\
& + u_3(-44 + 3x_1 - 63x_1^2 + 34x_2^2 + 142x_1^3 + 63x_1x_2^2 - 54x_1^4 \\
& - 69x_1^2x_2^2 - 26x_2^4)
\end{aligned}$$



Design of HDD Swing-arm

Design of HDD Swing-arm

H. Anai, S. Hara (2000) Fujitsu

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Design of HDD Swing-arm

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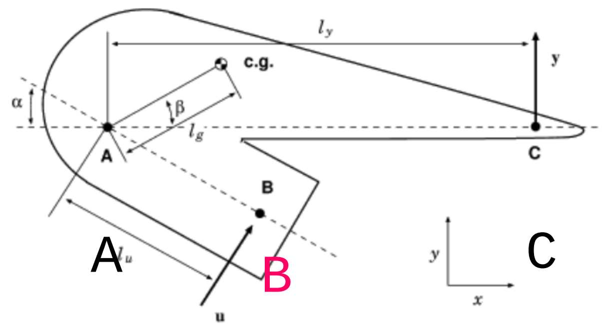

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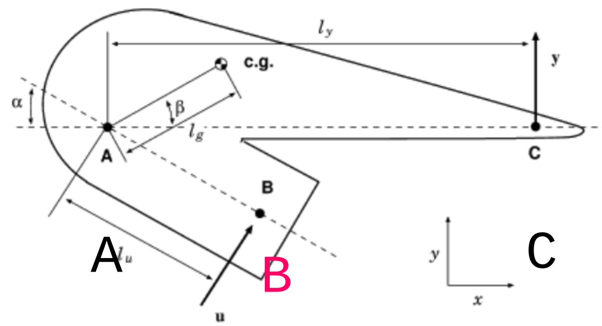

Where should the actuator **B** be located so that the arm satisfies "stability" and "positive realness (PR)" ?

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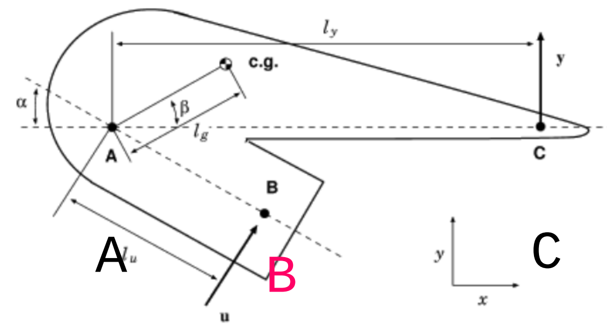

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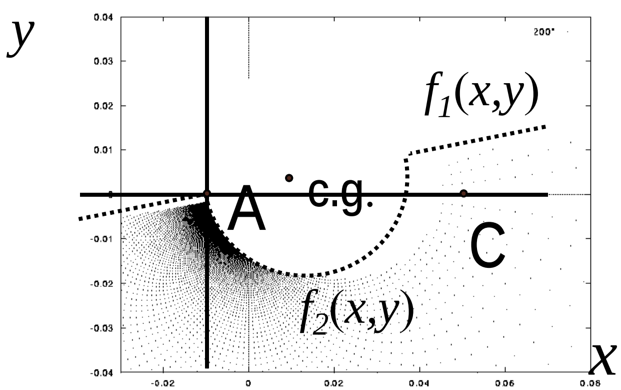

Optimal Numerical Algorithm

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M. Erascu, H. Hong, Reliable Computing (2013), JSC (2016)

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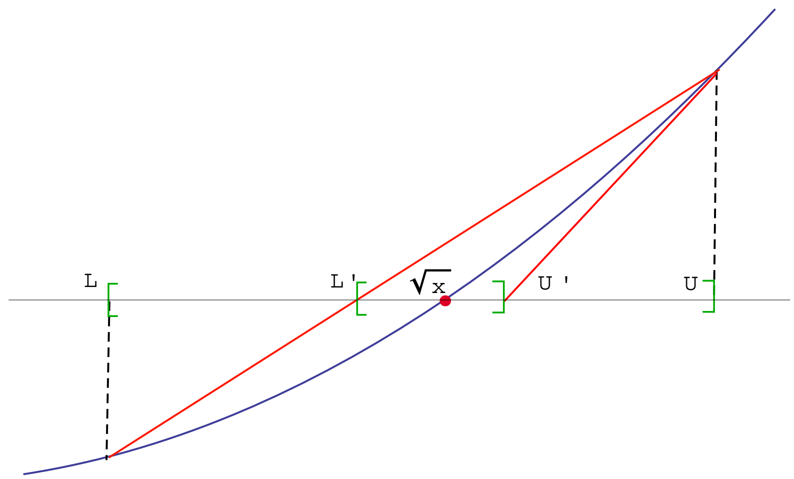

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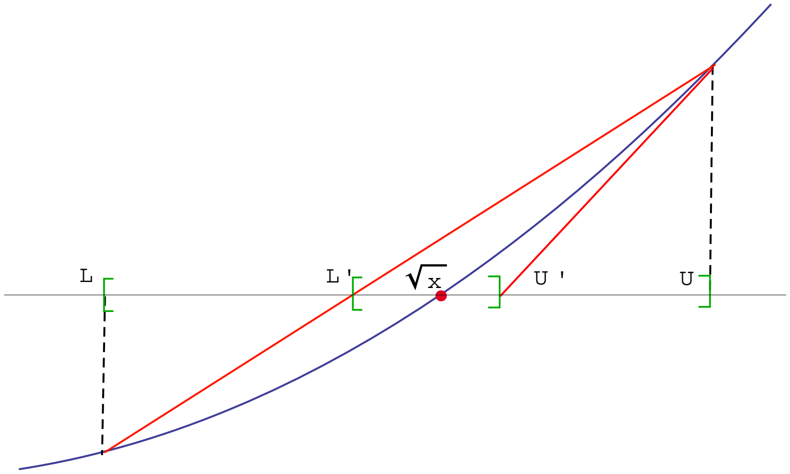

$$L' = L + \frac{x + p_0 L^2 + p_1 L U + p_2 U^2}{p_3 L + p_4 U} \qquad U' = U + \frac{x + q_0 U^2 + q_1 U L + q_2 L^2}{q_3 U + q_4 L}$$

$$Correctness(p, q) :\iff \underset{L,U,x}{\forall}\ 0 < L \le \sqrt{x} \le U \implies 0 < L' \le \sqrt{x} \le U'$$

$$Termination(p, q) :\iff \underset{c \in (0,1)}{\exists}\ \underset{L,U,x}{\forall}\ 0 < L \le \sqrt{x} \le U \implies 0 \le U' - L' \le c(U - L)$$

Stability of Numerical PDE

McCormack Scheme

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H. Hong and M. Safey-Eldin, *J. of Symbolic Computation* (2012)

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$$u(x, y, t + 2\Delta_t) \approx M u(x, y, t).$$

$$\begin{aligned}
G_{++} &= I + a(T_x - I) + b(T_y - I), \\
G_{--} &= I + a(I - T_x^{-1}) + b(I - T_y^{-1}), \\
G_{-+} &= I + a(I - T_x^{-1}) + b(T_y - I), \\
G_{+-} &= I + a(T_x - I) + b(I - T_y^{-1}), \\
M_1 &= \frac{1}{2}(I + G_{++}G_{--}), \\
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$$\exists Y \quad F(X, Y) = 0 \quad \wedge \quad G(X, Y) > 0$$

$$\begin{aligned}
X &= \{a, b\} \\
Y &= \{c_1, s_1, c_2, s_2\} \\
F &= \{c_1^2 + s_1^2 - 1,\ c_2^2 + s_2^2 - 1\} \\
G &= \{g\}
\end{aligned}$$

$$\begin{aligned}
g = {} & 4a^6b^2c_1{}^4c_2{}^2 - 8a^5b^3s_1s_2c_1{}^3c_2 - 8a^5b^3s_1s_2c_1{}^2c_2{}^2 + 4a^4b^4c_1{}^4c_2{}^2 + \\
& 16a^4b^4c_1{}^3c_2{}^3 + 4a^4b^4c_1{}^2c_2{}^4 - 8a^3b^5s_1s_2c_1{}^2c_2{}^2 - 8a^3b^5s_1s_2c_1c_2{}^3 + \\
& 4a^2b^6c_1{}^2c_2{}^4 - 4a^7bs_1s_2c_1{}^3 + 4a^6b^2c_1{}^4c_2 - 4a^6b^2c_1{}^3c_2{}^2 + 8a^5b^3s_1s_2c_1{}^3 + \\
& 12a^5b^3s_1s_2c_1{}^2c_2 + 16a^5b^3s_1s_2c_1c_2{}^2 - 8a^4b^4c_1{}^4c_2 - 24a^4b^4c_1{}^3c_2{}^2 - \\
& 24a^4b^4c_1{}^2c_2{}^3 - 8a^4b^4c_1c_2{}^4 + 16a^3b^5s_1s_2c_1{}^2c_2 + 12a^3b^5s_1s_2c_1c_2{}^2 + \\
& 8a^3b^5s_1s_2c_2{}^3 - 4a^2b^6c_1{}^2c_2{}^3 + 4a^2b^6c_1c_2{}^4 - 4ab^7s_1s_2c_2{}^3 + a^8c_1{}^4 + \\
& 12a^7bs_1s_2c_1{}^2 - 8a^6b^2c_1{}^4 - 12a^6b^2c_1{}^3c_2 - 12a^6b^2c_1{}^2c_2{}^2 - 4a^5b^3s_1s_2c_1{}^2 - \\
& 8a^5b^3s_1s_2c_2{}^2 + 4a^4b^4c_1{}^4 + 22a^4b^4c_1{}^2c_2{}^2 + 4a^4b^4c_2{}^4 - 4a^4b^2c_1{}^4c_2{}^2 - \\
& 8a^3b^5s_1s_2c_1{}^2 - 4a^3b^5s_1s_2c_2{}^2 + 8a^3b^3s_1s_2c_1{}^2c_2{}^2 - 12a^2b^6c_1{}^2c_2{}^2 - \\
& 12a^2b^6c_1c_2{}^3 - 8a^2b^6c_2{}^4 - 4a^2b^4c_1{}^2c_2{}^4 + 12ab^7s_1s_2c_2{}^2 + b^8c_2{}^4 - 4a^8c_1{}^3 - \\
& 12a^7bs_1s_2c_1 + 16a^6b^2c_1{}^3 + 12a^6b^2c_1{}^2c_2 + 20a^6b^2c_1c_2{}^2 - 16a^5b^3s_1s_2c_1 - \\
& 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1{}^3 + 8a^4b^4c_1{}^3 + 12a^4b^4c_1{}^2c_2 + 12a^4b^4c_1c_2{}^2 + \\
& 8a^4b^4c_2{}^3 + 4a^4b^2c_1{}^4c_2 + 4a^4b^2c_1{}^3c_2{}^2 - 4a^3b^5s_1s_2c_1 - 16a^3b^5s_1s_2c_2 - \\
& 12a^3b^3s_1s_2c_1{}^2c_2 - 12a^3b^3s_1s_2c_1c_2{}^2 + 20a^2b^6c_1{}^2c_2 + 12a^2b^6c_1c_2{}^2 + \\
& 16a^2b^6c_2{}^3 + 4a^2b^4c_1{}^2c_2{}^3 + 4a^2b^4c_1c_2{}^4 - 12ab^7s_1s_2c_2 + 4ab^5s_1s_2c_2{}^3 - \\
& 4b^8c_2{}^3 + 6a^8c_1{}^2 + 4a^7bs_1s_2 - 4a^6b^2c_1c_2 - 8a^6b^2c_2{}^2 - 2a^6c_1{}^4 + \\
& 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1{}^2 - 14a^4b^4c_1{}^2 + 8a^4b^4c_1c_2 - 14a^4b^4c_2{}^2 - \\
& 4a^4b^2c_1{}^3c_2 + 10a^4b^2c_1{}^2c_2{}^2 + 12a^3b^5s_1s_2 + 4a^3b^3s_1s_2c_1{}^2 + 16a^3b^3s_1s_2c_1c_2 + \\
& 4a^3b^3s_1s_2c_2{}^2 - 8a^2b^6c_1{}^2 - 4a^2b^6c_1c_2 + 10a^2b^4c_1{}^2c_2{}^2 - 4a^2b^4c_1c_2{}^3 + \\
& 4ab^7s_1s_2 - 12ab^5s_1s_2c_2{}^2 + 6b^8c_2{}^2 - 2b^6c_2{}^4 - 4a^8c_1 - 16a^6b^2c_1 + \\
& 8a^6c_1{}^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 - 8a^4b^2c_1{}^2c_2 - 16a^4b^2c_1c_2{}^2 - \\
& 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - 16a^2b^4c_1{}^2c_2 - 8a^2b^4c_1c_2{}^2 + \\
& 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2{}^3 + a^8 + 8a^6b^2 - 12a^6c_1{}^2 - 4a^5bs_1s_2 + 14a^4b^4 - \\
& 2a^4b^2c_1{}^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2{}^2 + a^4c_1{}^4 + 8a^2b^6 + 6a^2b^4c_1{}^2 + 12a^2b^4c_1c_2 - \\
& 2a^2b^4c_2{}^2 + 2a^2b^2c_1{}^2c_2{}^2 - 4ab^5s_1s_2 + b^8 - 12b^6c_2{}^2 + b^4c_2{}^4 + 8a^6c_1 + 4a^4b^2c_1 - \\
& 4a^4b^2c_2 - 4a^4c_1{}^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4a^2b^4c_2 - 4ab^3s_1s_2c_2 + 8b^6c_2 - \\
& 4b^4c_2{}^3 - 2a^6 - 2a^4b^2 + 8a^4c_1{}^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1{}^2 + 4a^2b^2c_1c_2 - \\
& 2a^2b^2c_2{}^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2{}^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 + \\
& 3a^4 + 6a^2b^2 - 2a^2c_1{}^2 + 3b^4 - 2b^2c_2{}^2 + 4a^2c_1 + 4b^2c_2 - 2a^2 - 2b^2
\end{aligned}$$

$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$$

Stability of Numerical PDE

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& 16a^4b^4c_1{}^3c_2{}^3 + 4a^4b^4c_1{}^2c_2{}^4 - 8a^3b^5s_1s_2c_1{}^2c_2{}^2 - 8a^3b^5s_1s_2c_1c_2{}^3 +\\
& 4a^2b^6c_1{}^2c_2{}^4 - 4a^7bs_1s_2c_1{}^3 + 4a^6b^2c_1{}^4c_2 - 4 \cdots\\
& 12a^5b^3s_1s_2c_1{}^2c_2 + 16a^5b^3s_1s_2c_1c_2{}^2 - 8a \cdots\\
& 24a^4b^4c_1{}^2c_2{}^3 - 8a^4b^4c_1c_2{}^4 + 16a^3b^5s_1s_2 \cdots\\
& 8a^3b^5s_1s_2c_2{}^3 - 4a^2b^6c_1{}^2c_2{}^3 + 4a^2b^6c_1c_2 \cdots\\
& 12a^7bs_1s_2c_1{}^2 - 8a^6b^2c_1{}^4 - 12a^6b^2c_1{}^3c_2 - 12 \cdots\\
& 8a^5b^3s_1s_2c_2{}^2 + 4a^4b^4c_1{}^4 + 22a^4b^4c_1{}^2c_2{}^2 \cdots\\
& 8a^3b^5s_1s_2c_1{}^2 - 4a^3b^5s_1s_2c_2{}^2 + 8a^3b^3s_1 \cdots\\
& 12a^2b^6c_1c_2{}^3 - 8a^2b^6c_2{}^4 - 4a^2b^4c_1{}^2c_2{}^4 + 12a \cdots\\
& 12a^7bs_1s_2c_1 + 16a^6b^2c_1{}^3 + 12a^6b^2c_1{}^2c_2 + 2 \cdots\\
& 4a^5b^3s_1s_2c_2 + 4a^5bs_1s_2c_1{}^3 + 8a^4b^4c_1{}^3 + 1 \cdots\\
& 8a^4b^4c_2{}^3 + 4a^4b^2c_1{}^4c_2 + 4a^4b^2c_1{}^3c_2{}^2 - 4 \cdots\\
& 12a^3b^3s_1s_2c_1{}^2c_2 - 12a^3b^3s_1s_2c_1c_2{}^2 + 20 \cdots\\
& 16a^2b^6c_2{}^3 + 4a^2b^4c_1{}^2c_2{}^3 + 4a^2b^4c_1c_2{}^4 - 1 \cdots\\
& 4b^8c_2{}^3 + 6a^8c_1{}^2 + 4a^7bs_1s_2 - 4a^6b^2c_1 \cdots\\
& 12a^5b^3s_1s_2 - 12a^5bs_1s_2c_1{}^2 - 14a^4b^4c_1{}^2 \cdots\\
& 4a^4b^2c_1{}^3c_2 + 10a^4b^2c_1{}^2c_2{}^2 + 12a^3b^5s_1s_2 + 4a \cdots\\
& 4a^3b^3s_1s_2c_2{}^2 - 8a^2b^6c_1{}^2 - 4a^2b^6c_1c_2 + 1 \cdots\\
& 4ab^7s_1s_2 - 12ab^5s_1s_2c_2{}^2 + 6b^8c_2{}^2 - 2b^6 \cdots\\
& 8a^6c_1{}^3 + 12a^5bs_1s_2c_1 - 12a^4b^4c_1 - 12a^4b^4c_2 \cdots\\
& 4a^3b^3s_1s_2c_1 - 4a^3b^3s_1s_2c_2 - 16a^2b^6c_2 - \cdots\\
& 12ab^5s_1s_2c_2 - 4b^8c_2 + 8b^6c_2{}^3 + a^8 + 8a^6b^2 - 1 \cdots\\
& 2a^4b^2c_1{}^2 + 12a^4b^2c_1c_2 + 6a^4b^2c_2{}^2 + a^4c_1{}^4 + 8 \cdots\\
& 2a^2b^4c_2{}^2 + 2a^2b^2c_1{}^2c_2{}^2 - 4ab^5s_1s_2 + b^8 - 12b^6 \cdots\\
& 4a^4b^2c_2 - 4a^4c_1{}^3 - 4a^3bs_1s_2c_1 - 4a^2b^4c_1 + 4 \cdots\\
& 4b^4c_2{}^3 - 2a^6 - 2a^4b^2 + 8a^4c_1{}^2 + 4a^3bs_1s_2 - 2a^2b^4 - 2a^2b^2c_1{}^2 + 4a^2b^2c_1c_2 -\\
& 2a^2b^2c_2{}^2 + 4ab^3s_1s_2 - 2b^6 + 8b^4c_2{}^2 - 8a^4c_1 - 4a^2b^2c_1 - 4a^2b^2c_2 - 8b^4c_2 +\\
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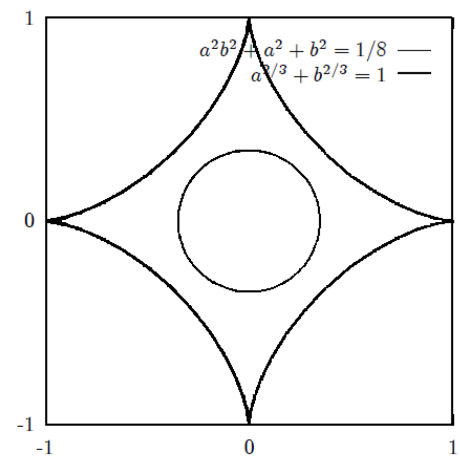
\end{aligned}$$



$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 > 1$$

Stability Analysis

Multi-stable model of Cell Differentiation

Stability Analysis

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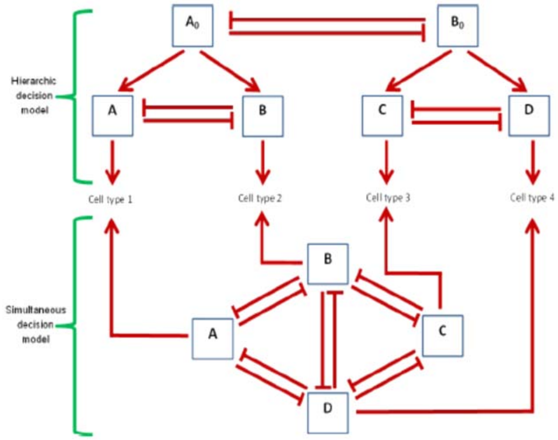
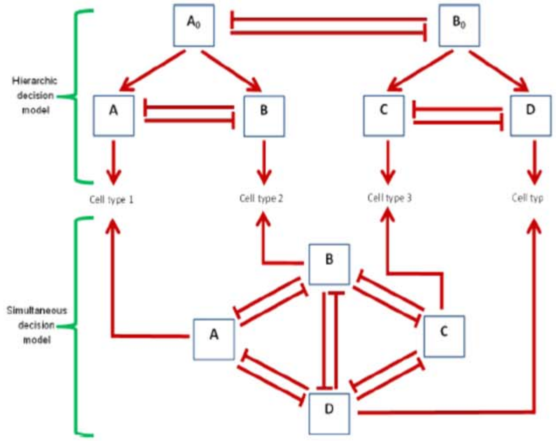


Stability Analysis

Multi-stable model of Cell Differentiation

H, Hong, X. Tang, B. Xia (2013)



$$\frac{dx_k}{dt} = -x_k + \sigma \frac{1}{1 + \Sigma_{m=1}^{n} x_m^c - x_k^c}$$

Stability Analysis

Multi-stable model of Cell Differentiation

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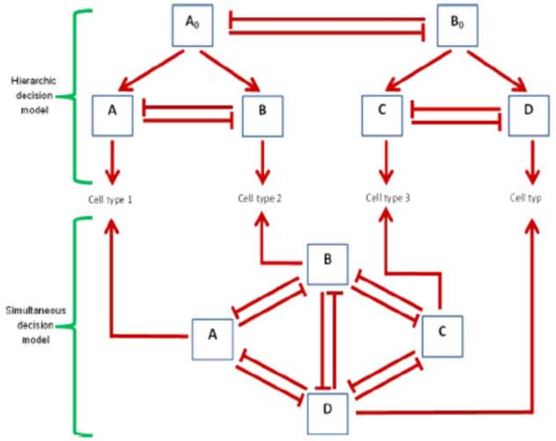

$$\frac{dx_k}{dt} = -x_k + \sigma \frac{1}{1 + \Sigma_{m=1}^{n} x_m^c - x_k^c}$$

$$\exists \mathbf{x} \left(\mathbf{f}(\sigma, \mathbf{x}) = 0 \ \wedge \ \det\left(J_{\mathbf{f}}\left(\sigma, \mathbf{x}\right)\right) \cdot \prod_{k=1}^{n} \Delta_k(\sigma, \mathbf{x}) = 0 \right) \right)$$

Hopf Bifurcation

Gene regulatory network

Circadian clock of green alga

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T. Sturm, et al (2013)

Hopf Bifurcation

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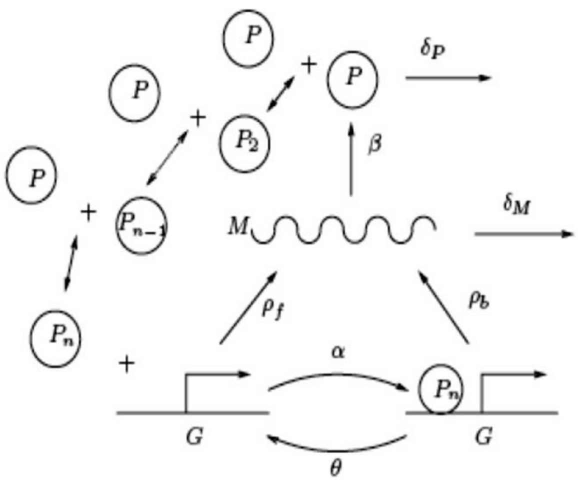

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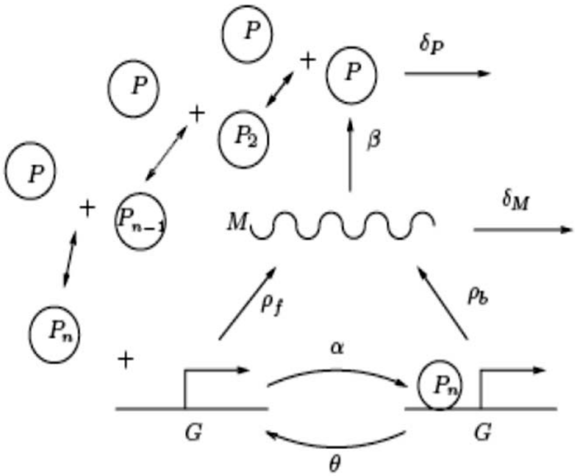

$$
\begin{aligned}
&\exists v_2 \exists v_1 \exists v_3 (0 < v_1 \wedge 0 < v_3 \wedge 0 < v_2 \wedge \vartheta\gamma_0 - v_1 - v_1 v_3{}^9 = 0 \wedge \\
&\lambda_1 v_1 + \gamma_0 \mu - v_2 = 0 \wedge 9\alpha\gamma_0 - v_1 - v_1 v_3{}^9 + \delta v_2 - v_3 = 0 \wedge \\
&0 < \vartheta\delta + \vartheta v_3{}^9 \delta + 9\lambda_1 \vartheta v_1 v_3{}^8 \delta \wedge \\
&162\vartheta v_3{}^{17} \alpha v_1 + 162\vartheta\alpha v_1 v_3{}^8 + 162\alpha v_1 v_3{}^8 \delta + \vartheta + 2\vartheta v_3{}^9 \delta + \vartheta^2 v_3{}^{18} \delta \\
&\quad + \vartheta v_3{}^9 \vartheta\delta + 81\alpha v_1 v_3{}^8 \vartheta\delta + 81\alpha v_1 v_3{}^{17} \vartheta\delta + \delta^2 + \vartheta\delta^2 + \vartheta^2\delta + \vartheta^2 \\
&\quad + 2\vartheta^2 v_3{}^9 + \vartheta^2 v_3{}^{18} + 6561\alpha^2 v_1{}^2 v_3{}^{16} + 2\vartheta^2 v_3{}^9 \delta + \delta + 81\alpha v_1 v_3{}^8 \\
&\quad + \vartheta v_3{}^9 \delta^2 - 9\lambda_1 \vartheta v_1 v_3{}^8 \delta = 0 \wedge \\
&0 < \vartheta \wedge 0 < \gamma_0 \wedge 0 < \mu \wedge 0 < \delta \wedge 0 < \alpha)
\end{aligned}
$$

Motivation (Recap)

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- *Arise as a fundamental question in the logical foundation of mathematics*

Motivation (Recap)

- *Arise as a fundamental question in the logical foundation of mathematics*

- *Numerous problems from science and engineering can be reduced to QE problems.*

History/State of the Art

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- ~1930: Tarski

$$2^{2^{2^{\cdot^{\cdot^{\cdot^{n}}}}}}$$

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  2^{n} for no alternation of quantifiers

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- ------ : Arnon, McCallum, Hong, Brown, Strezbonski,

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  Efficient algorithms for moderate inputs

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- ------ : Arnon, McCallum, Hong, Brown, Strezbonski, Xia, Chen, Kosta, England, …

  Efficient algorithms for moderate inputs

- ------ : Weispfenning, Sturm, Dolzman, Gonzalez-Vega, Anai, Hong, Safey-Eldin, Moura, Jovanovic, Abraham,…..

  Efficient algorithms for special inputs

Software Packages

Software Packages

- General Inputs

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- General Inputs
  - QEPCAD (in C)

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- General Inputs
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Software Packages

- General Inputs
  - QEPCAD (in C)
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  - …
- Special Inputs
  - Redlog (in REDUCE)

Software Packages

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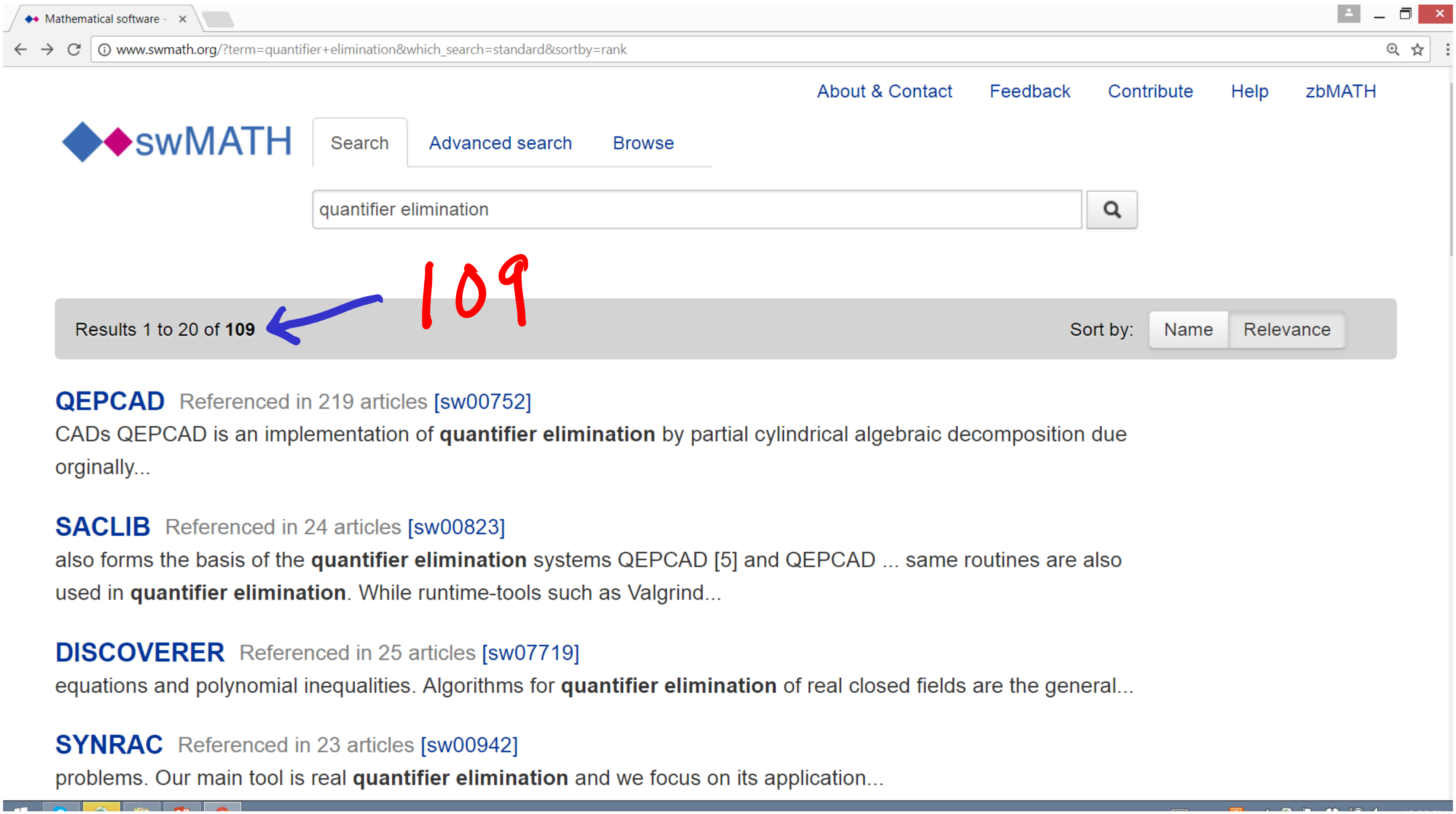
Software Packages



References

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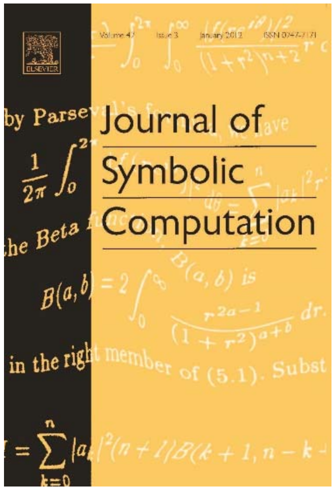

Quantifier Elimination

"Scientific Overview"

Mathematical Theories

- Algebraic geometry
- Real algebraic geometry
- Commutative algebra
- Complex analysis
- Real analysis
- Topology

Plan

Plan

- *Tarski's*

Plan

- *Tarski's*
  - *Original*

Plan

- *Tarski's*
  - *Original*
  - *Repackaging (Polynomial remaindering - Sign table)*

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- *New methods*

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Tarski

Key Idea

Key Idea

1. Reduce to

"Generalized Root Counting"

Key Idea

1. Reduce to

   "Generalized Root Counting"

2. Solve it using

   "Sturm - Sylvester Theorem"

Generalized Root Counting

Generalized Root Counting

In: $f(x)$, $g(x)$

Generalized Root Counting

In: $f(x)$, $g(x)$

Out: $N = N_+ - N_-$

where

$$N_+ = \#\{x : f(x) = 0 \wedge g(x) > 0\}$$

$$N_- = \#\{x : f(x) = 0 \wedge g(x) < 0\}$$

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Example

In: $f(x) = x^3 - 4x$, $g(x) = x - 1$

Out:

Generalized Root Counting

In: $f(x)$, $g(x)$

Out: $N = N_+ - N_-$

where

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Example

In: $f(x) = x^3 - 4x$, $g(x) = x - 1$

Out: $N =$

Generalized Root Counting

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where

$$N_+ = \#\{x : f(x) = 0 \wedge g(x) > 0\}$$

$$N_- = \#\{x : f(x) = 0 \wedge g(x) < 0\}$$

<u>Example</u>

In: $f(x) = x^3 - 4x$, $g(x) = x - 1$

Out: $N = -1$

Reduction

Reduction

$$\exists x \;\; f(x) = 0 \;\wedge\; g(x) > 0$$

Reduction

$$\exists x \quad f(x) = 0 \wedge g(x) > 0$$

$$\Updownarrow$$

Reduction

$$\exists x \quad f(x) = 0 \ \wedge \ g(x) > 0$$

$$\Updownarrow$$

$$\frac{N(f, g) + N(f, g^2)}{2} > 0$$

Reduction

Reduction

$$\exists x \; f(x) = 0 \wedge g(x) < 0$$

Reduction

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$$\Updownarrow$$

Reduction

$$\exists x \quad f(x) = 0 \ \wedge \ g(x) < 0$$

$$\Updownarrow$$

$$\frac{-N(f, g) + N(f, g^2)}{2} > 0$$

Reduction

Reduction

$$\exists x \ \ f(x) = 0 \ \wedge \ g(x) = 0$$

Reduction

$$\exists x \;\; f(x) = 0 \wedge g(x) = 0$$

$$\Updownarrow$$

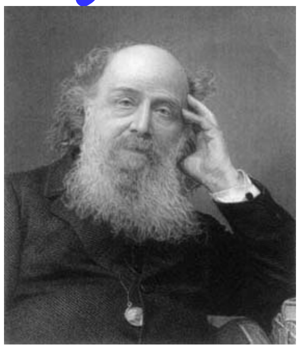
Reduction

$$\exists x \quad f(x) = 0 \ \wedge \ g(x) = 0$$

$$\Updownarrow$$

$$N(f, 1) - N(f, g^2) > 0$$

Sturm - Sylvester



Sturm - Sylvester

Let

$$p_1 = f$$

$$p_2 = f' \cdot g$$

$$p_3 = -\operatorname{rem}(p_1, p_2)$$

$$p_4 = -\operatorname{rem}(p_2, p_3)$$

$$\vdots$$

Sturm - Sylvester

Let

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$$\vdots$$

$P_1(-\infty)$

$P_2(-\infty)$

$P_3(-\infty)$

$P_4(-\infty)$

$\vdots$

Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\operatorname{rem}(P_1, P_2)$$

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$$\vdots$$

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
$\vdots$	$\vdots$

Sturm - Sylvester

Let

$p_1 = f$

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$\vdots$

$p_1(-\infty)$	$p_1(\infty)$
$p_2(-\infty)$	$p_2(\infty)$
$p_3(-\infty)$	$p_3(\infty)$
$p_4(-\infty)$	$p_4(\infty)$
$\vdots$	$\vdots$
V_-	

Sturm - Sylvester

Let

$$\begin{array}{l|ll}
p_1 = f & p_1(-\infty) & p_1(\infty) \\
p_2 = f' \cdot g & p_2(-\infty) & p_2(\infty) \\
p_3 = -\operatorname{rem}(p_1, p_2) & p_3(-\infty) & p_3(\infty) \\
p_4 = -\operatorname{rem}(p_2, p_3) & p_4(-\infty) & p_4(\infty) \\
\vdots & \vdots & \vdots \\
 & V_- & V_+
\end{array}$$

Sturm - Sylvester

Let

$$P_1 = f$$

$$P_2 = f' \cdot g$$

$$P_3 = -\operatorname{rem}(P_1, P_2)$$

$$P_4 = -\operatorname{rem}(P_2, P_3)$$

$$\vdots$$

$P_1(-\infty)$	$P_1(\infty)$
$P_2(-\infty)$	$P_2(\infty)$
$P_3(-\infty)$	$P_3(\infty)$
$P_4(-\infty)$	$P_4(\infty)$
$\vdots$	$\vdots$
V_-	V_+

Then

$$N = V_- - V_+$$

Challenges

- Remove redundancy
- Remove inconsistency
- Make incremental

Repackaging

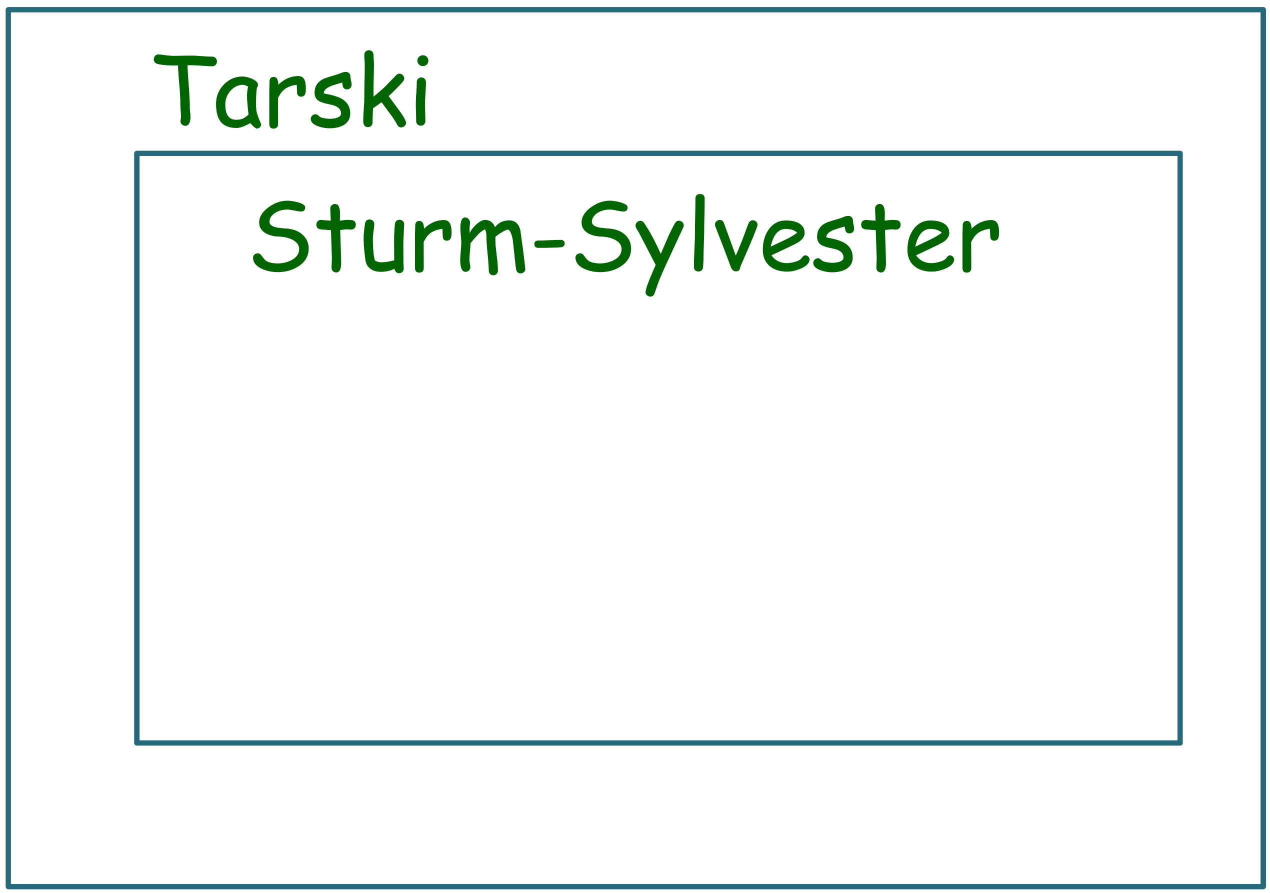
Polynomial remaindering - Sign table

Observe

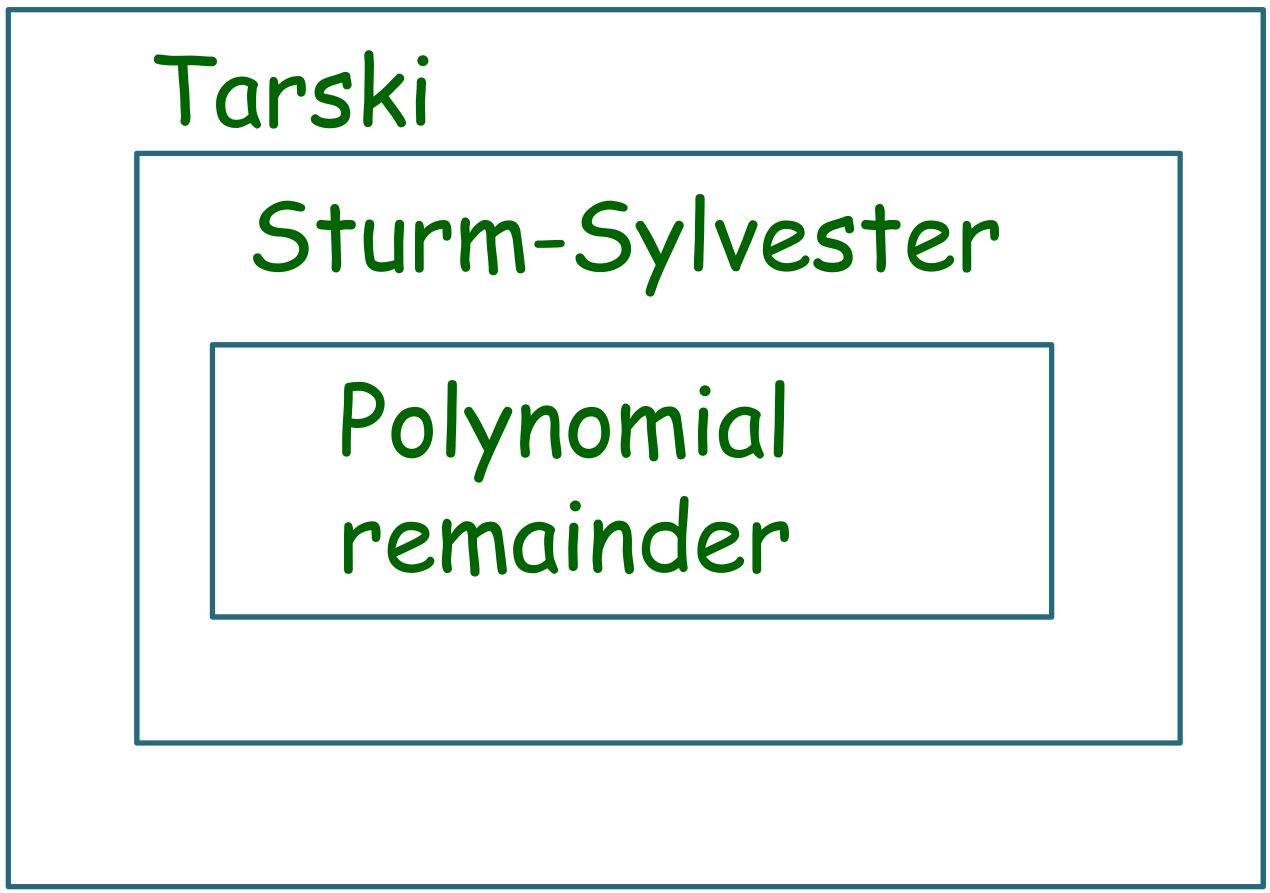
Observe

Tarski

Observe



Observe



Remaindering - Sign table

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$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \;\wedge\; \underbrace{x}_{g} > 0$$

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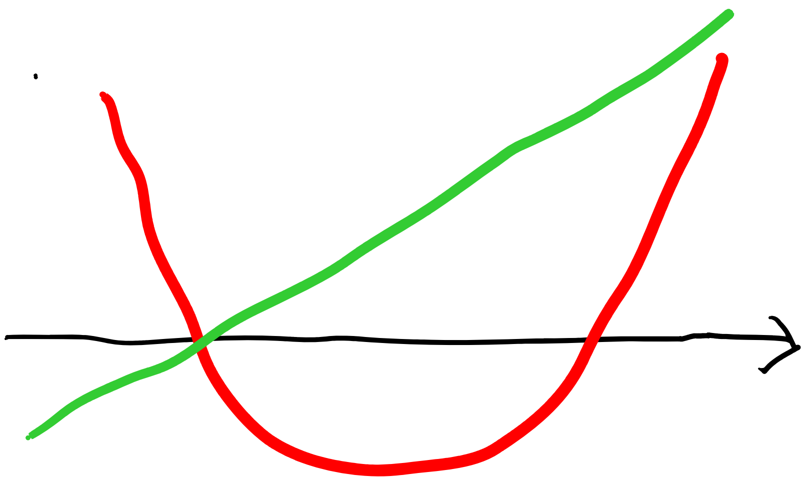

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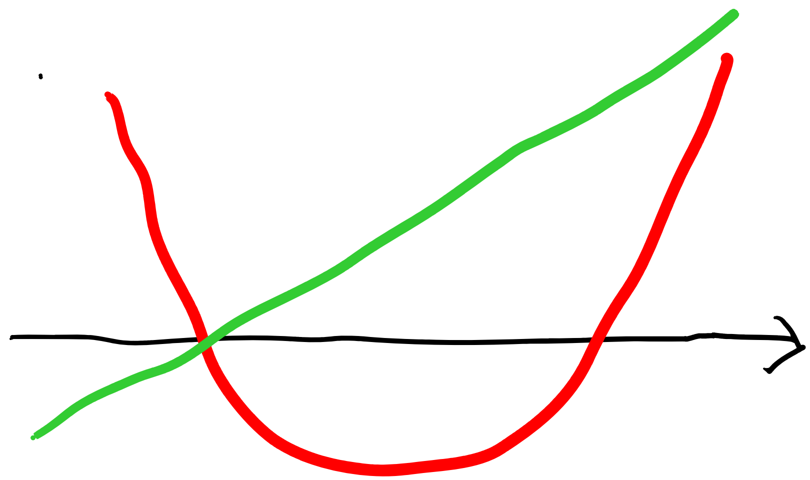
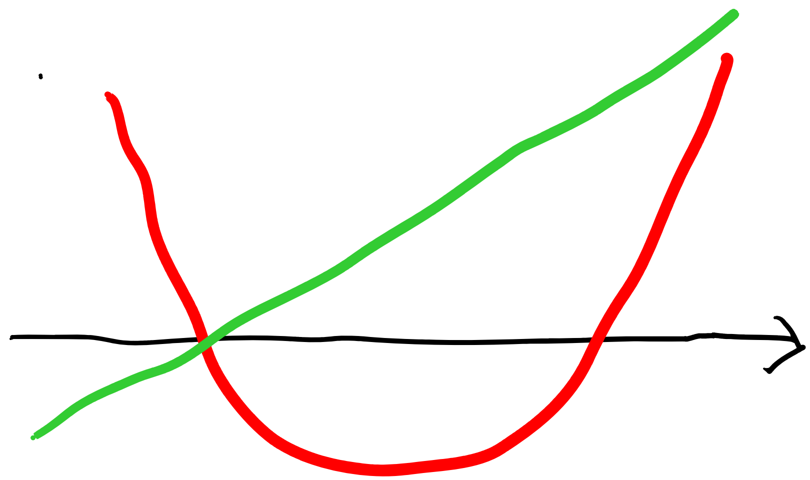


f	+
g	−

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \;\wedge\; \underbrace{x}_{g} > 0$$



f	$+$	0
g	$-$	0

Remaindering - Sign table

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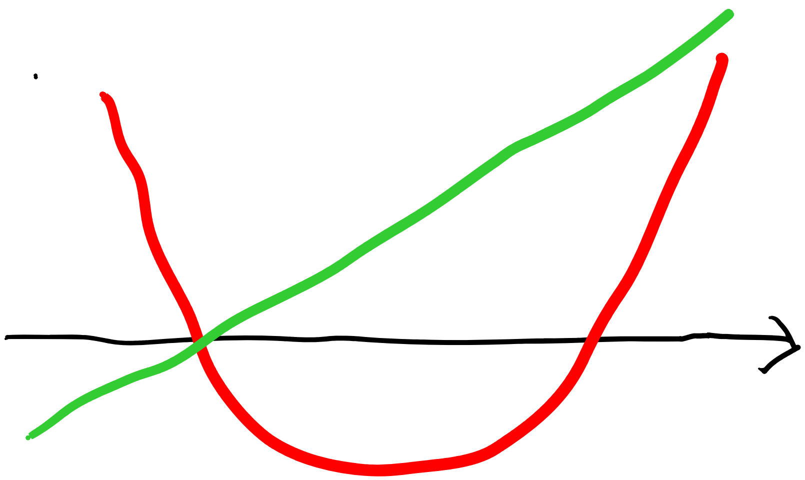

f	+	0	−
g	−	0	+

Remaindering - Sign table

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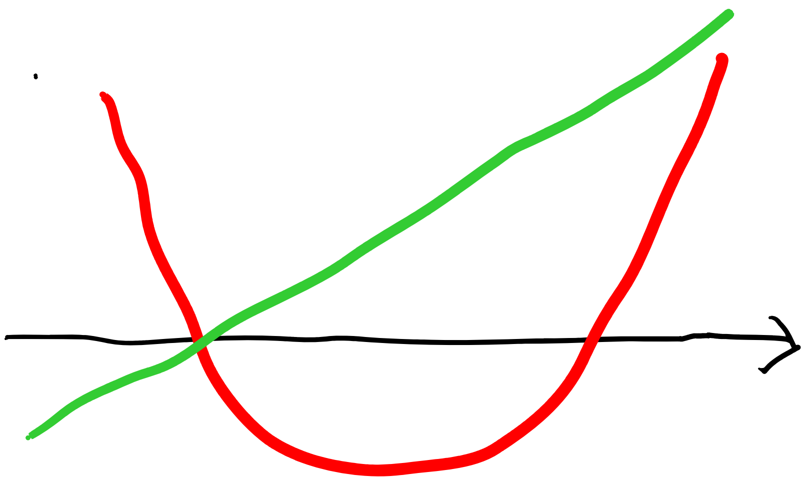
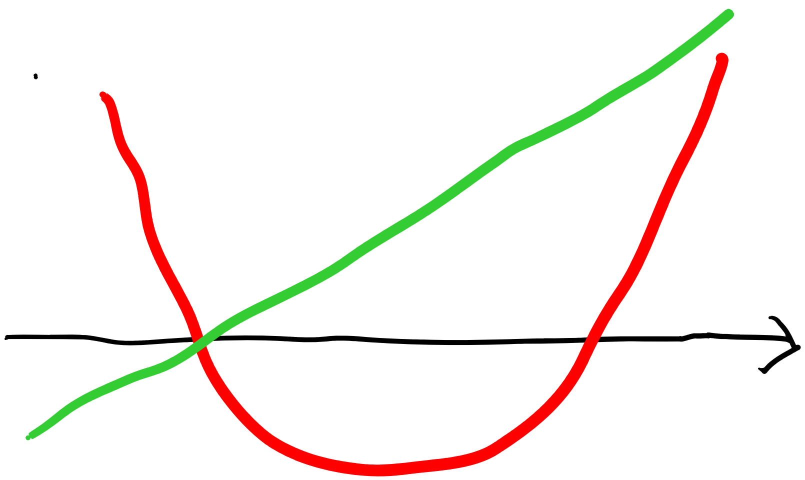


f	+	0	−	0
g	−	0	+	+

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \wedge \underbrace{x}_{g} > 0$$



f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

Remaindering - Sign table

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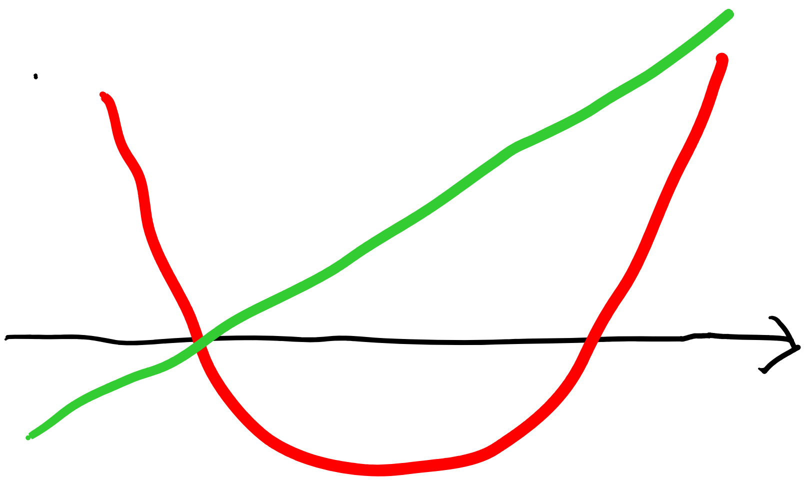
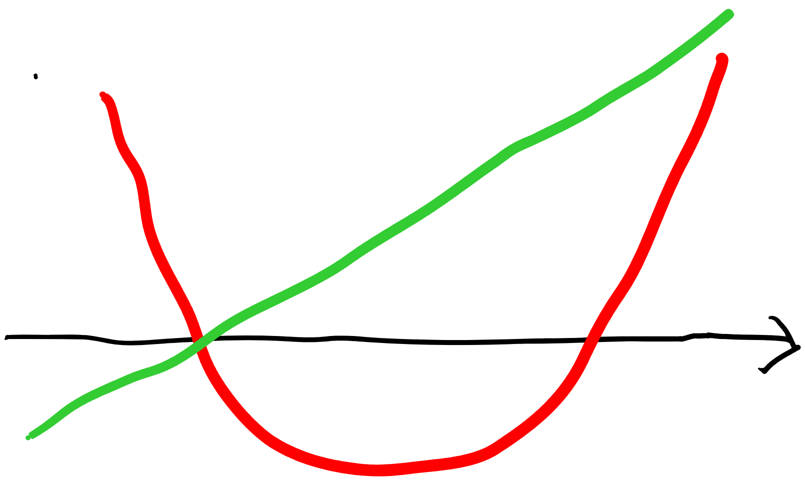


f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f, f')$	−			−	−
$r(f, g)$	0	0	0		
f					
g					

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \ \wedge \ \underbrace{x}_{g} > 0$$



f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f,f')$	−			−	−
$r(f,g)$	0	0	0		
f					
g	−	0	+	+	+

Remaindering - Sign table

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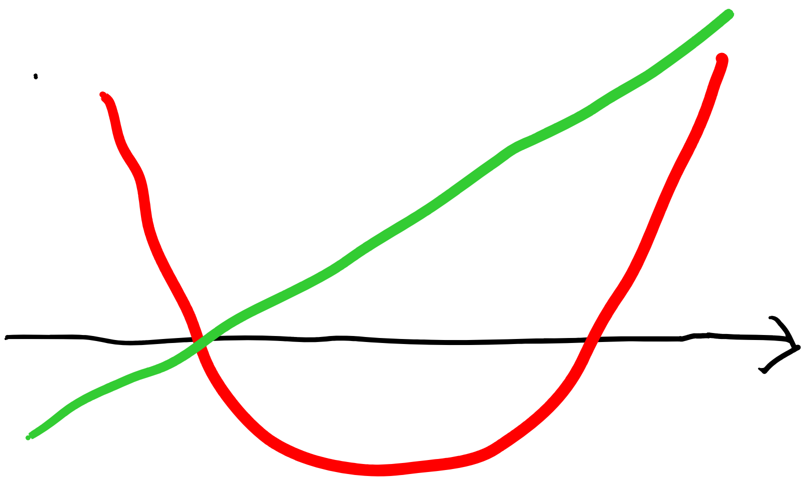

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f,f')$	−			−	−
$r(f,g)$	0	0	0		

f	+				+
g	−	0	+	+	+

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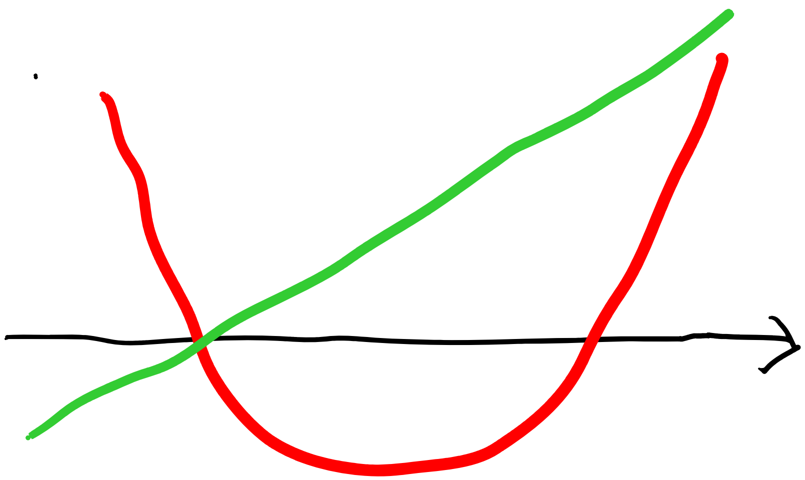

f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$
$r(f,f')$	$-$			$-$	$-$
$r(f,g)$	0	0	0		
f	$+$				$+$
g	$-$	0	$+$	$+$	$+$

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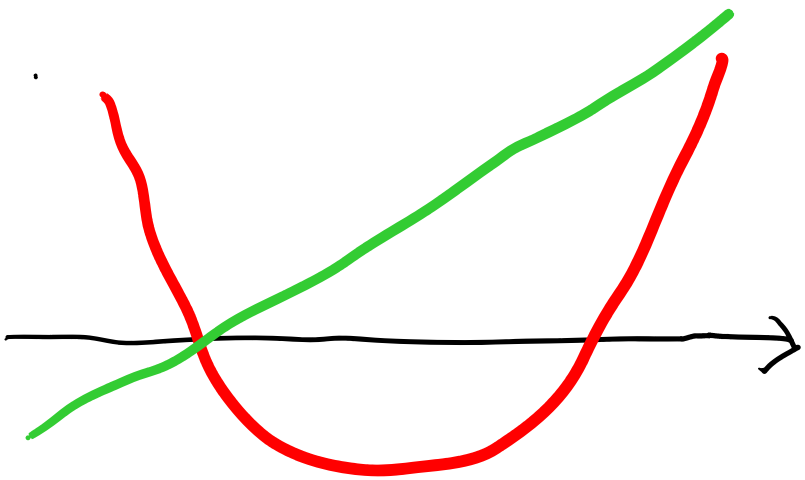

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f,f')$	−			−	−
$r(f,g)$	0	0	0		

f	+	0			+
g	−	0	+	+	+

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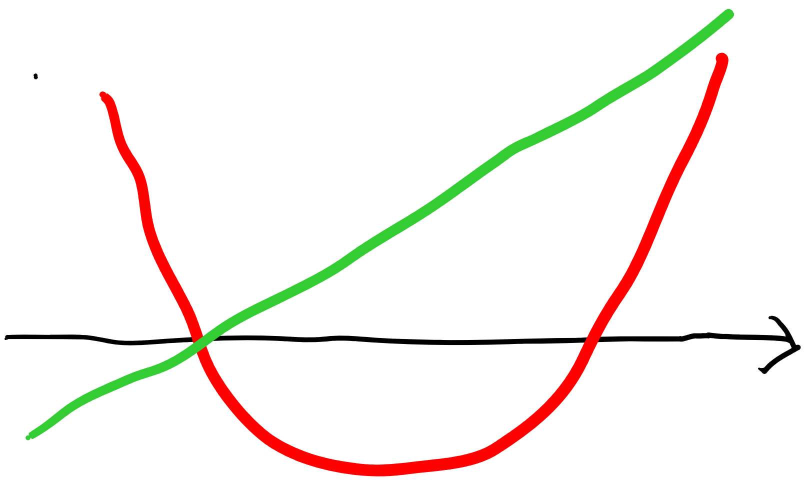


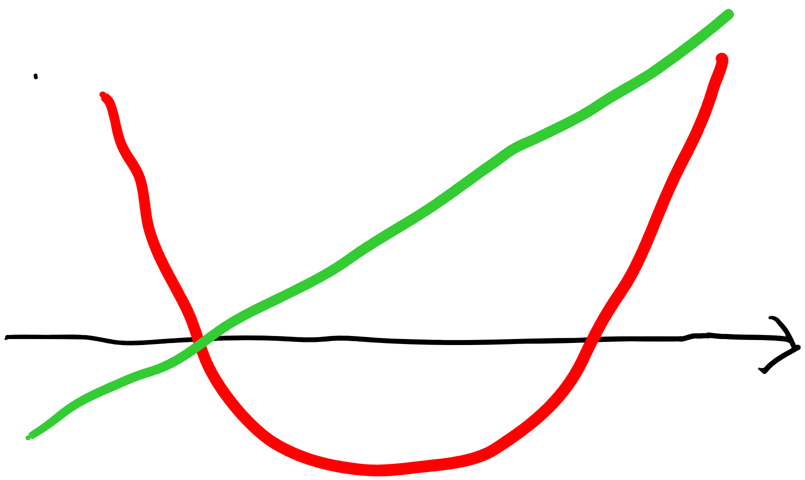
f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

0

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$
$r(f,f')$	$-$			$-$	$-$
$r(f,g)$	0	0	0		
f	$+$	0			$+$
g	$-$	0	$+$	$+$	$+$

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \wedge \underbrace{x}_{g} > 0$$



f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f, f')$	−			−	−
$r(f, g)$	0	0	0		
f	+	0			+
g	−	0	+	+	+

Remaindering - Sign table

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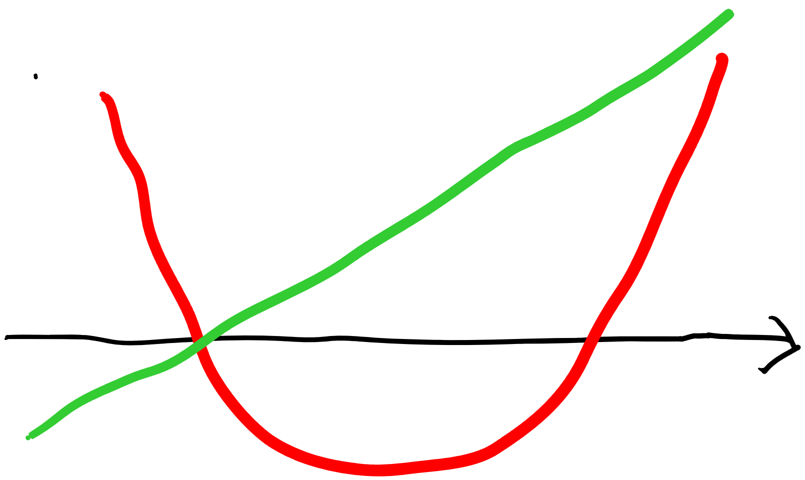

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f,f')$	−			−	−
$r(f,g)$	0	0	0		
f	+	0	−		+
g	−	0	+	+	+

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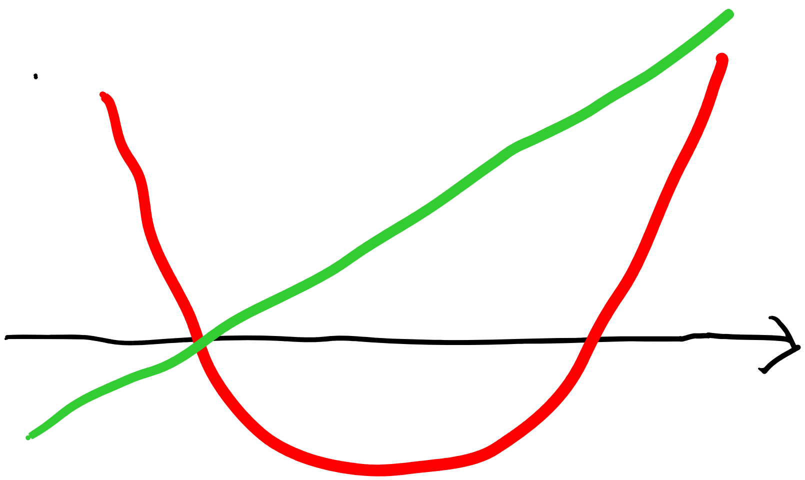

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
$r(f,f')$	−			−	−
$r(f,g)$	0	0	0		

f	+	0	−	0	+
g	−	0	+	+	+

Remaindering - Sign table

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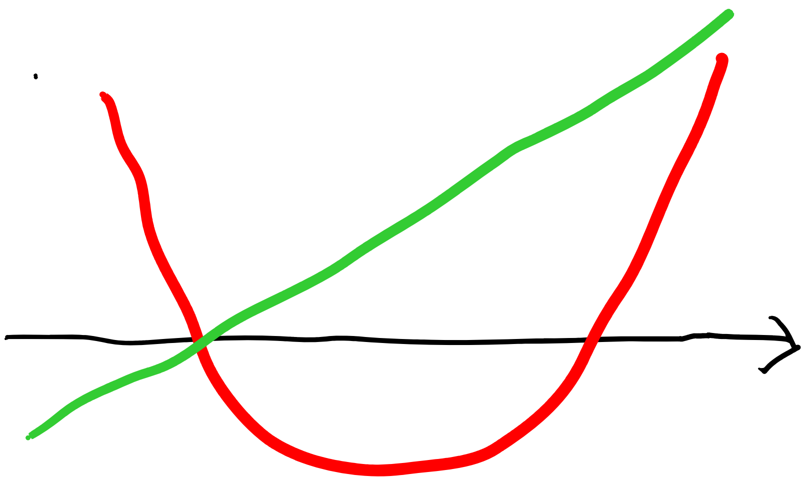

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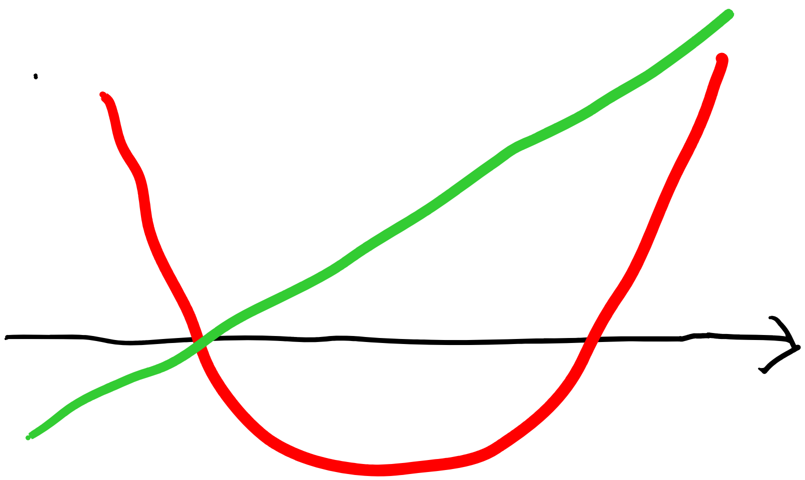
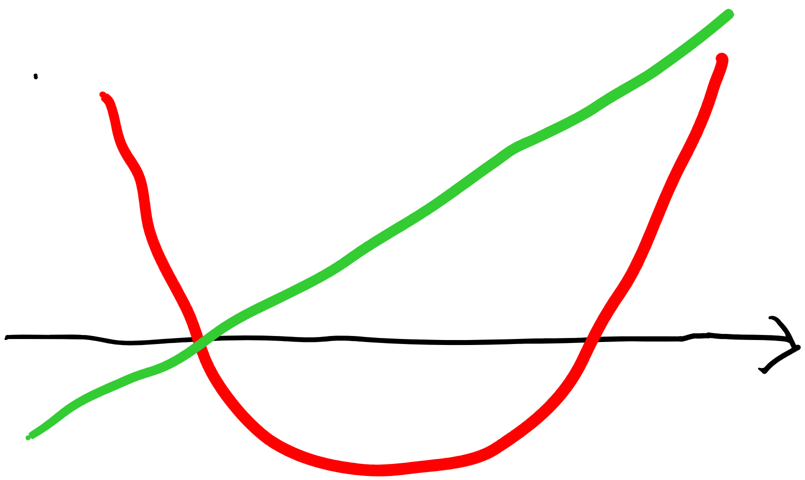


f	$+$
g	$-$

Remaindering - Sign table

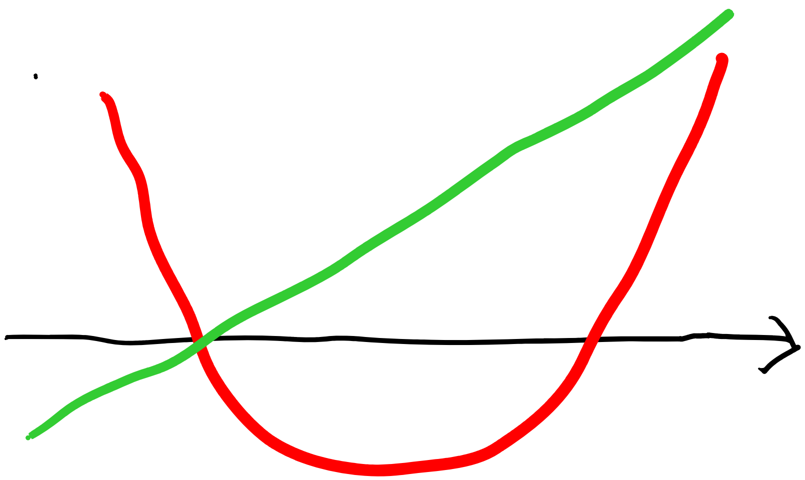
$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \wedge \underbrace{x}_{g} > 0$$



f	$+$	0
g	$-$	0

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \land \underbrace{x}_{g} > 0$$



f	$+$	0	$-$
g	$-$	0	$+$

Remaindering - Sign table

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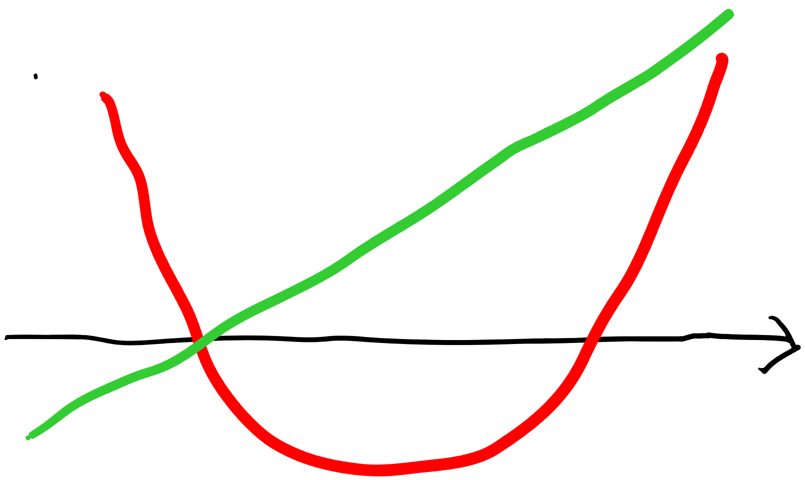
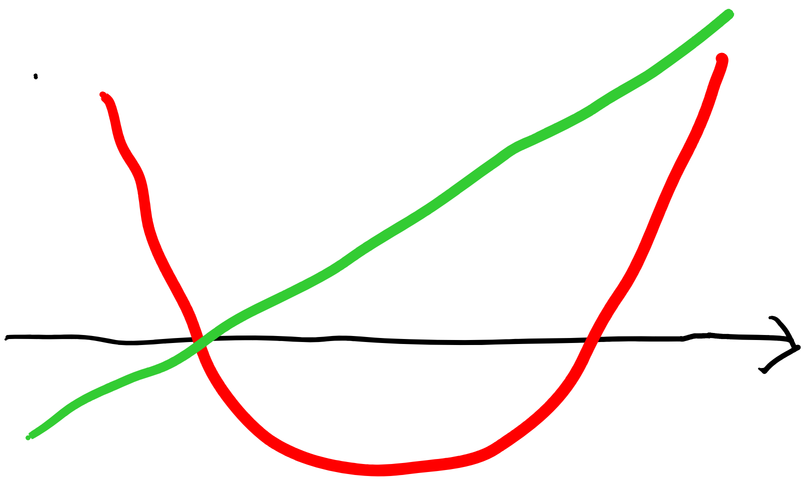


f	+	0	−	0
g	−	0	+	+

Remaindering - Sign table

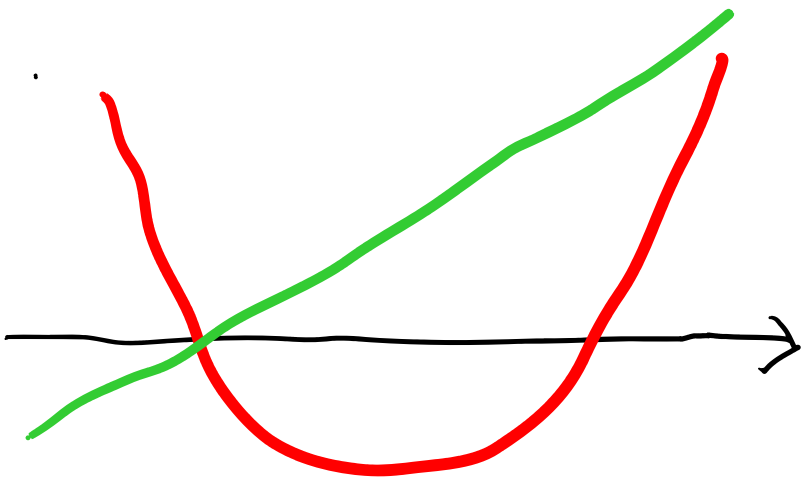
$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \;\wedge\; \underbrace{x}_{g} > 0$$



f	+	0	−	0	+
g	−	0	+	+	+

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \ \wedge\ \underbrace{x}_{g} > 0$$



f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$
f'	$-$			0	$+$
$r(f,f')$	$-$			$-$	$-$
g	$-$	0	$+$		
$r(f,g)$	0	0	0		
f					
g					

Remaindering - Sign table

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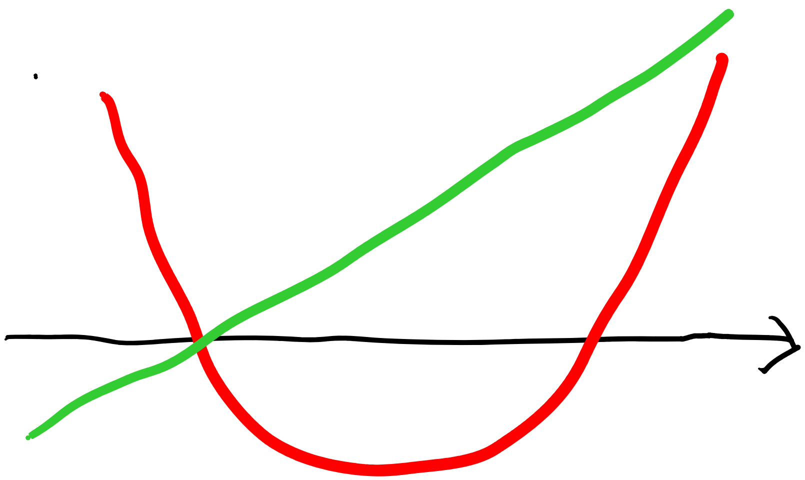


f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+

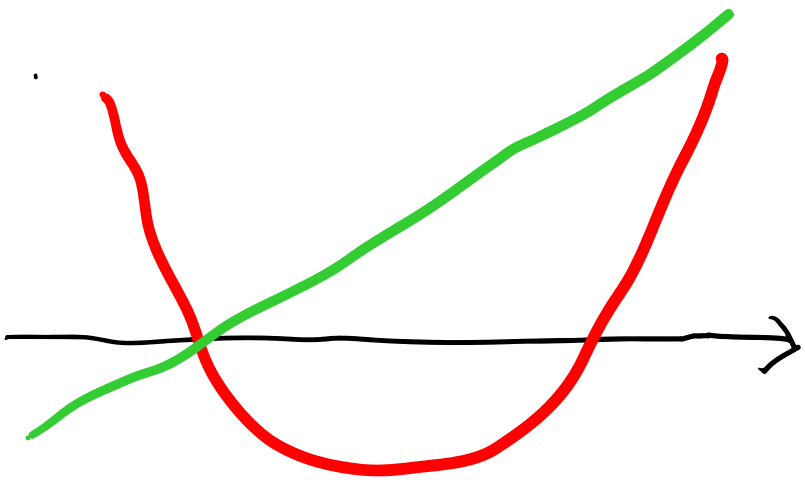
f'	−			0	+
$r(f,f')$	−			−	−

g	−	0	+		
$r(f,g)$	0	0	0		

f	+				+
g	−				+

Remaindering - Sign table

$$\exists x \quad \underbrace{x(x-1)}_{f} > 0 \;\wedge\; \underbrace{x}_{g} > 0$$



f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+

f'	−			0	+
$r(f,f')$	−			−	−

g	−	0	+		
$r(f,g)$	0	0	0		

f	+				+
g	−				+

Remaindering - Sign table

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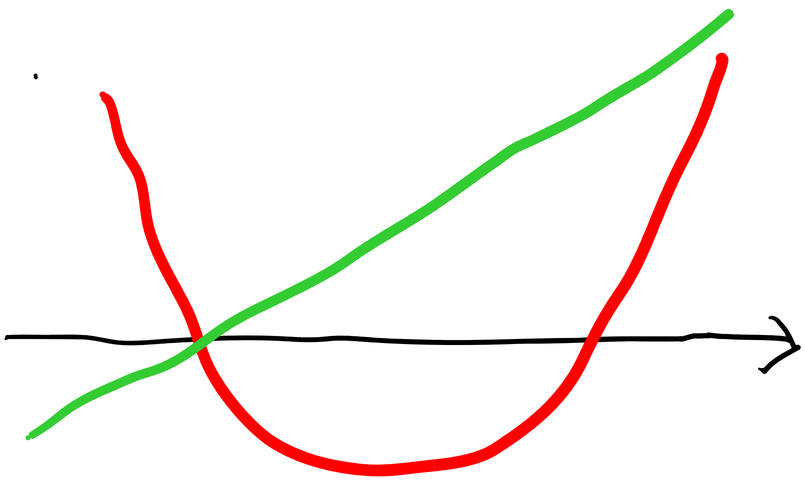

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
f'	−			0	+
$r(f, f')$	−			−	−
g	−	0	+		
$r(f, g)$	0	0	0		
f	+				+
g	−				+

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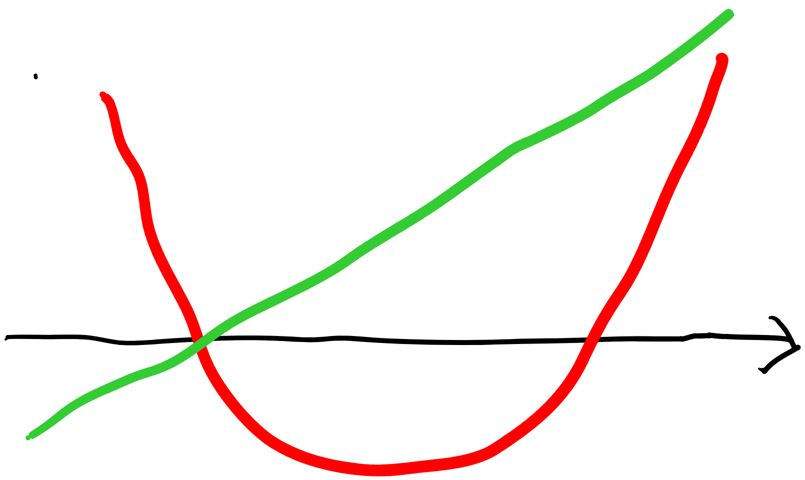

f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$
f'	$-$			0	$+$
$r(f,f')$	$-$			$-$	$-$
g	$-$	0	$+$		
$r(f,g)$	0	0	0		
f	$+$	0			$+$
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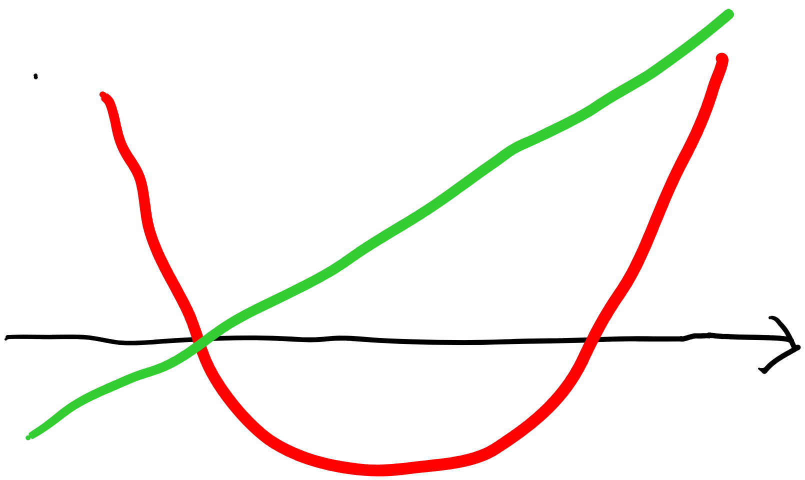

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+

f'	−			0	+
$r(f,f')$	−			−	−

g	−	0	+		
$r(f,g)$	0	0	0		

f	+	0			+
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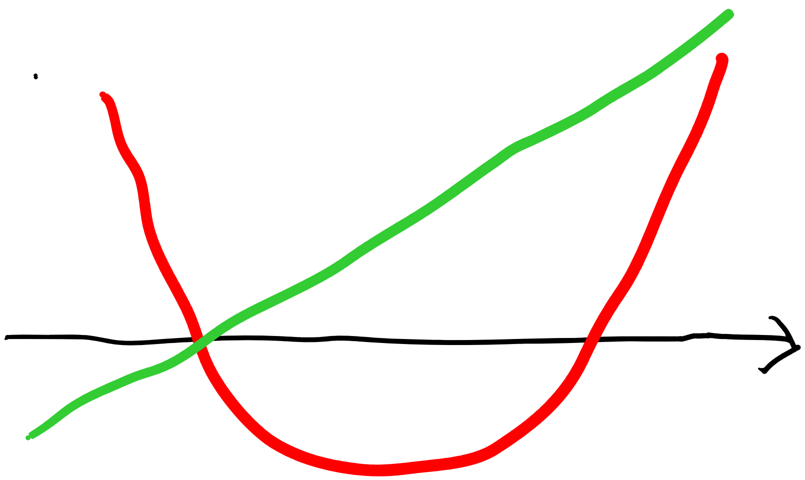

f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$
f'	$-$			0	$+$
$r(f,f')$	$-$			$-$	$-$
g	$-$	0	$+$		
$r(f,g)$	0	0	0		
f	$+$	0			$+$
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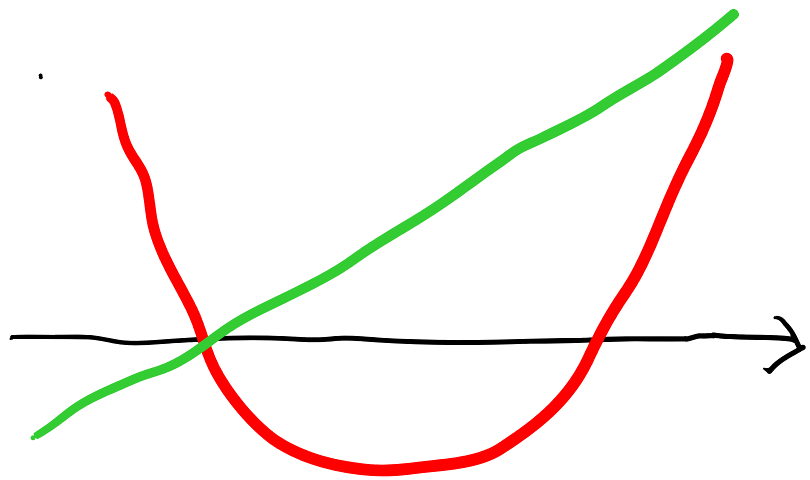

f	+	0	−	0	+
g	−	0	+	+	+

f'	−	−	−	0	+
g	−	0	+	+	+
f'	−			0	+
$r(f, f')$	−			−	−
g	−	0	+		
$r(f, g)$	0	0	0		
f	+	0	−		+
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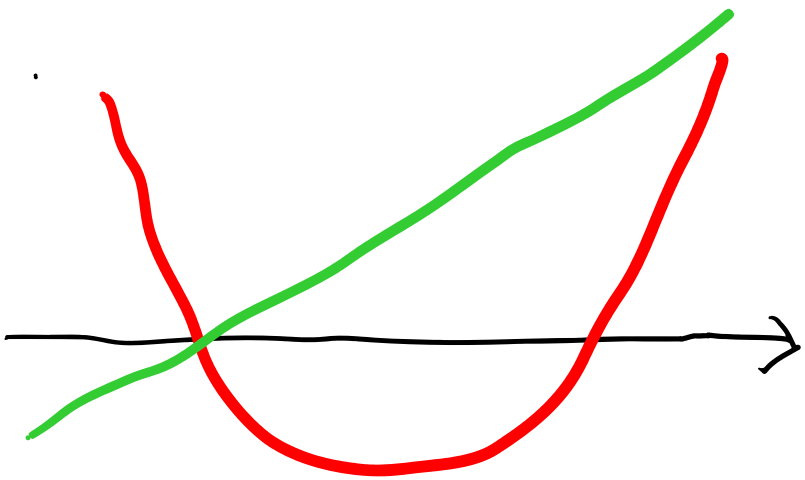

f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$	$-$	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

f'	$-$			0	$+$
$r(f, f')$	$-$			$-$	$-$

g	$-$	0	$+$		
$r(f, g)$	0	0	0		

f	$+$	0	$-$	0	$+$
g	$-$	0	$+$	$+$	$+$

Challenges

- Remove redundancy
- Remove inconsistency
- Incremental

Collins

Key Idea

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$\exists x \in \mathbb{R} \quad F(x)$

Key Idea

infinite set

$$\exists x \in \mathbb{R} \quad F(x)$$

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infinite set

$$\Updownarrow$$

$$\exists x \in S \quad F(x)$$

Finite set, "samples"

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Finite set, "samples"

$$\Updownarrow$$

$$\bigvee_{x \in S} F(x)$$

Key Idea

$$\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad F(x,y)$$

Key Idea

Infinite sets

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Infinite sets

$$\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad F(x,y)$$

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finite sets

$$\exists x \in S \ \forall y \in T_x \quad F(x,y)$$

Key Idea

Infinite sets

$$\exists x \in \mathbb{R} \ \forall y \in \mathbb{R} \quad F(x, y)$$

$$\Updownarrow$$

finite sets

$$\exists x \in S \ \forall y \in T_x \quad F(x, y)$$

$$\Updownarrow$$

$$\bigvee_{x \in S} \ \bigwedge_{y \in T_x} \quad F(x, y)$$

I_n: $\exists x \, \forall y \;\; x + y > 0 \;\wedge\; x - y > 0$

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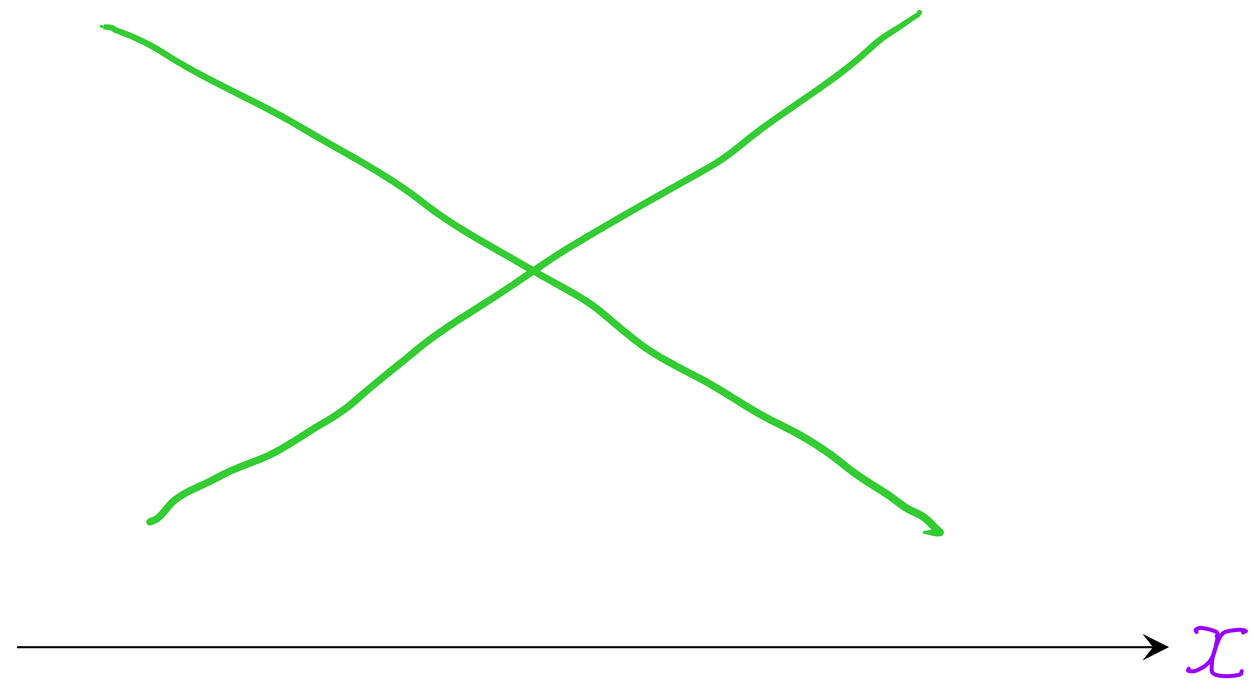

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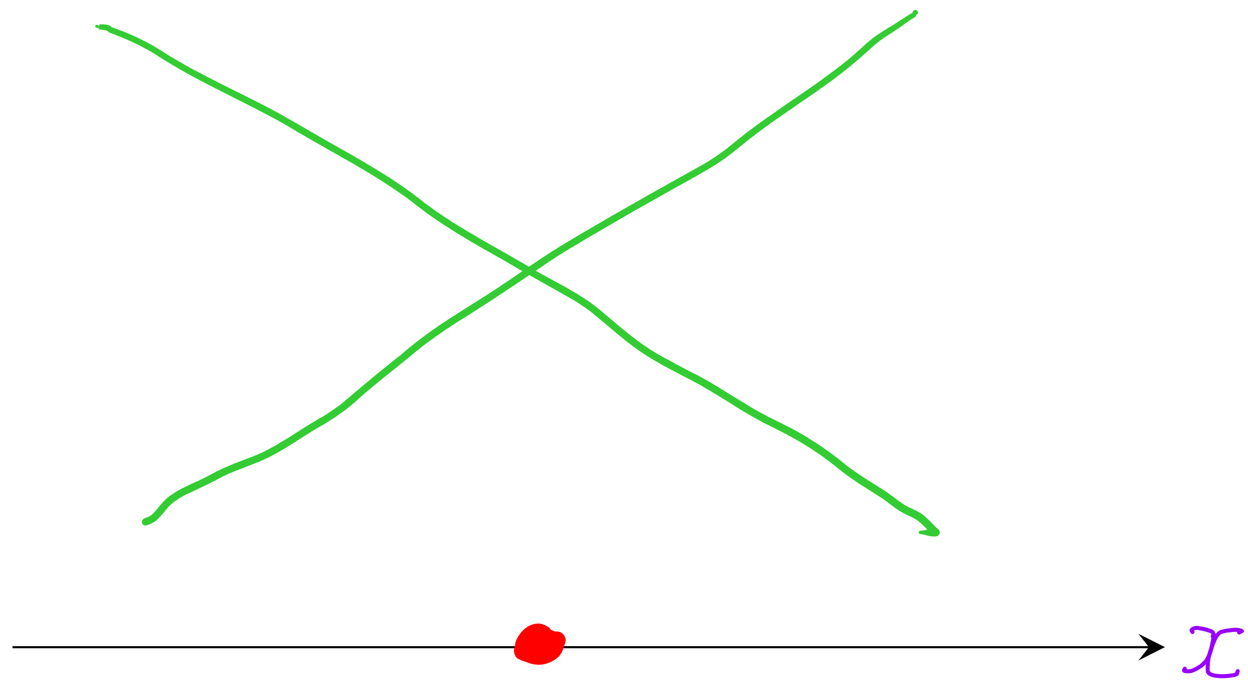
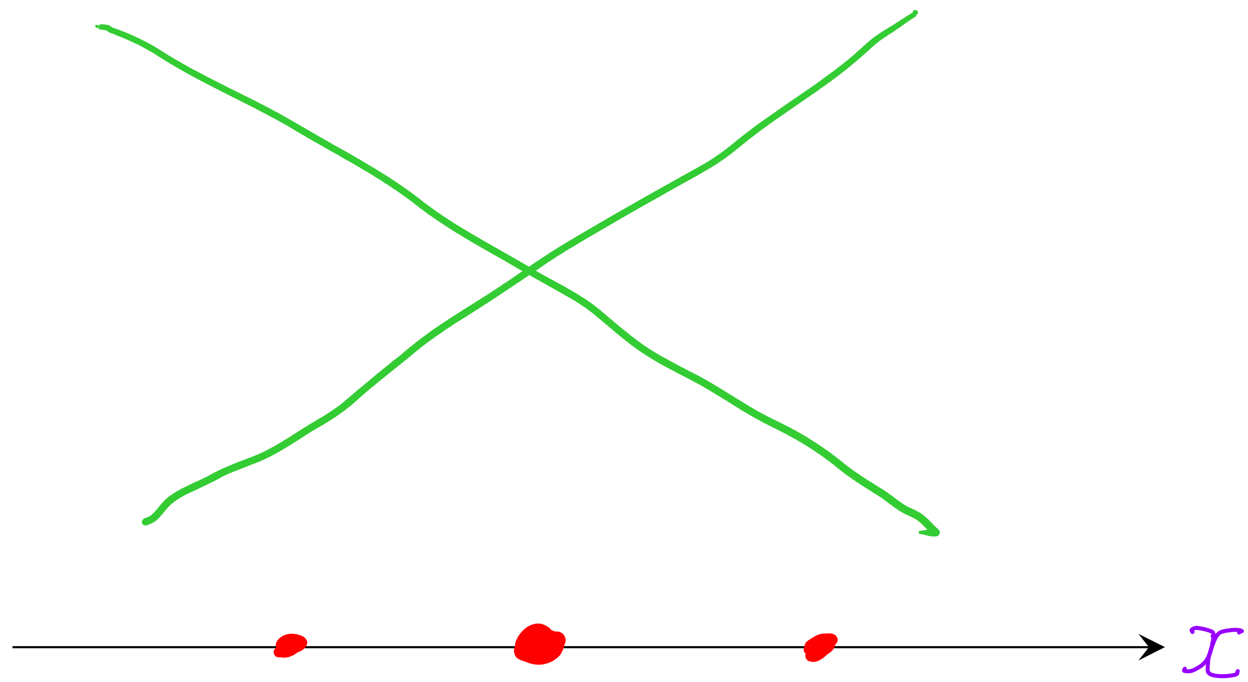
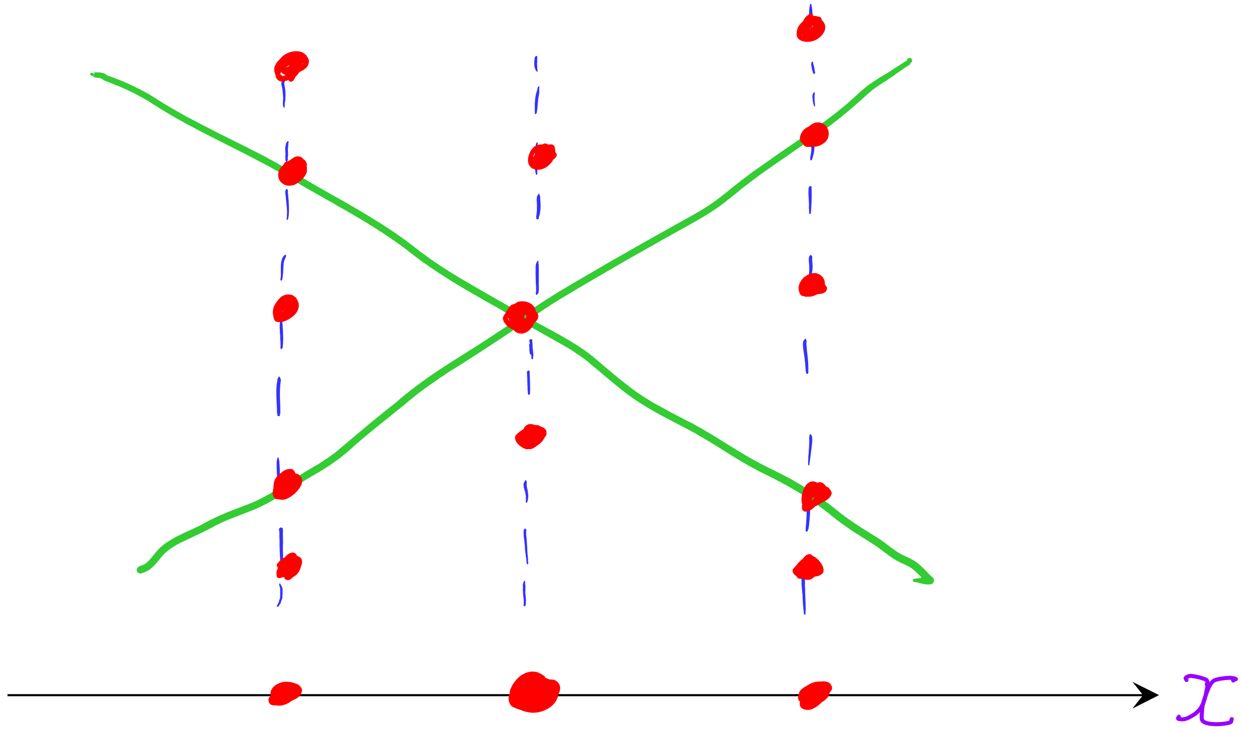


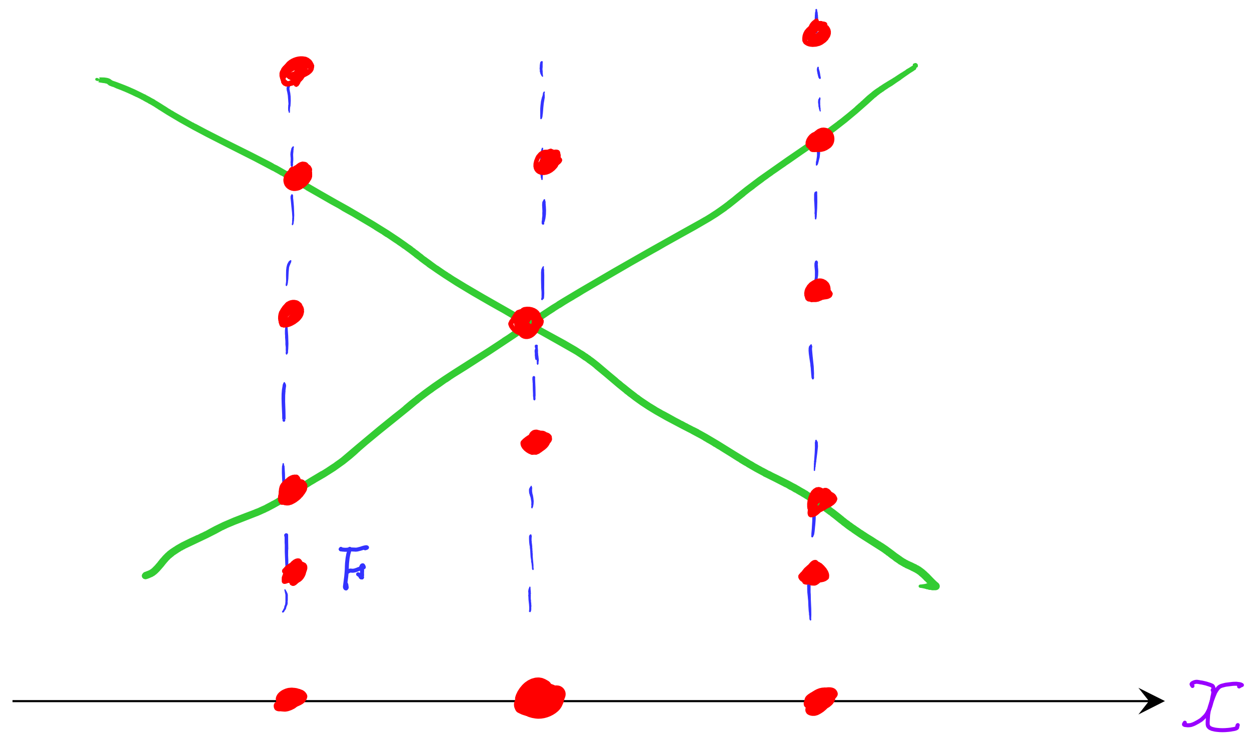
I_n: $\exists x\ \forall y\ \ x+y>0 \ \wedge\ x-y>0$



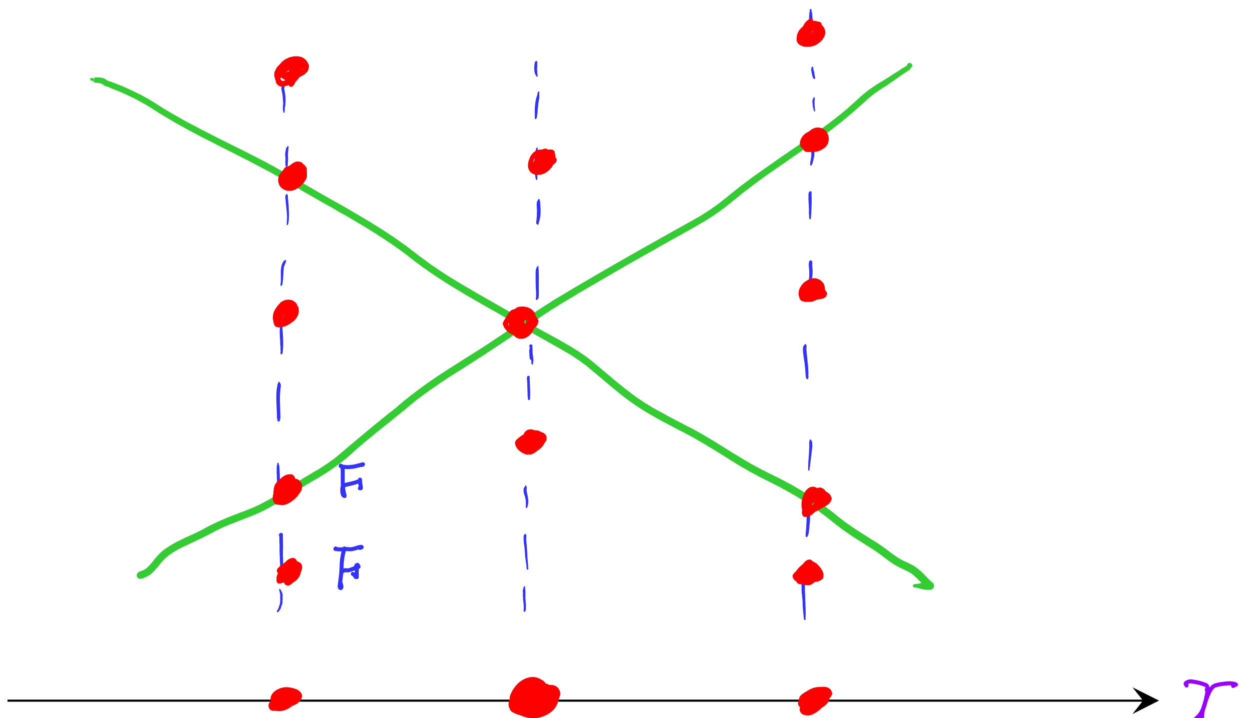
I_n: $\exists x\ \forall y\ \ x+y>0\ \wedge\ x-y>0$



$$I_n: \exists x \, \forall y \;\; x + y > 0 \;\wedge\; x - y > 0$$



$I_n: \exists x \forall y \quad x + y > 0 \wedge x - y > 0$



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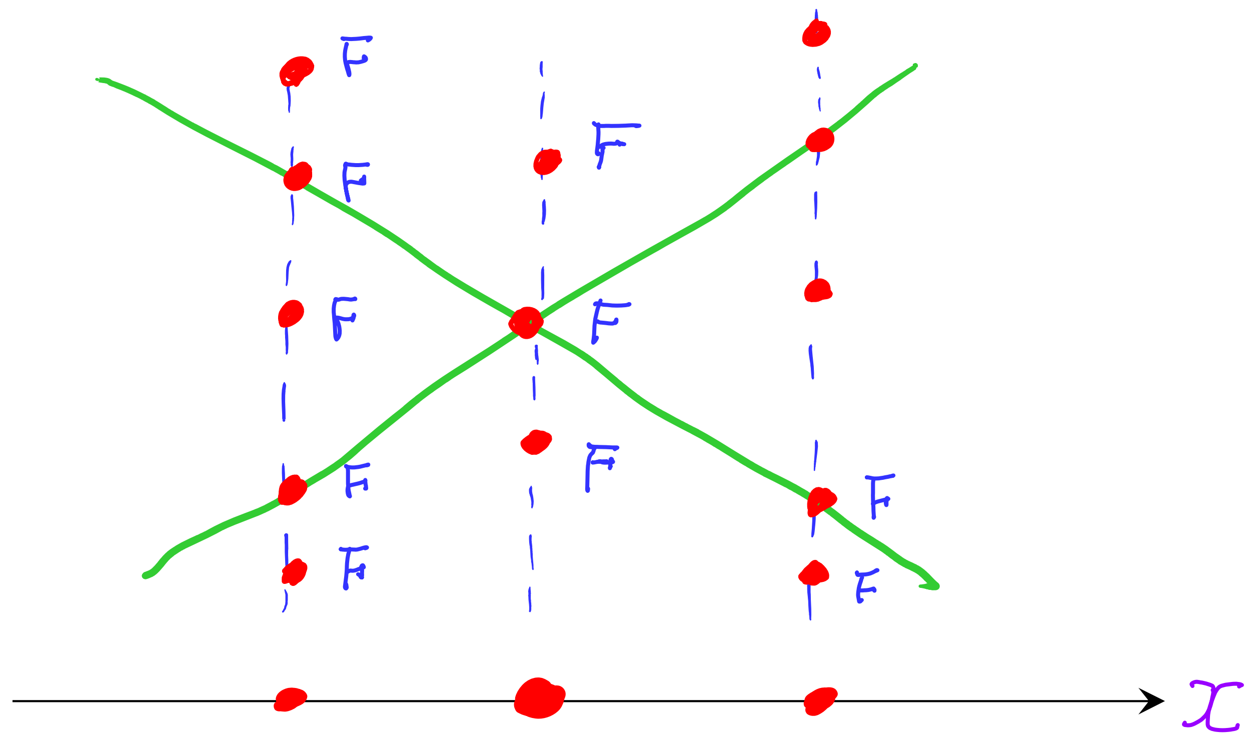

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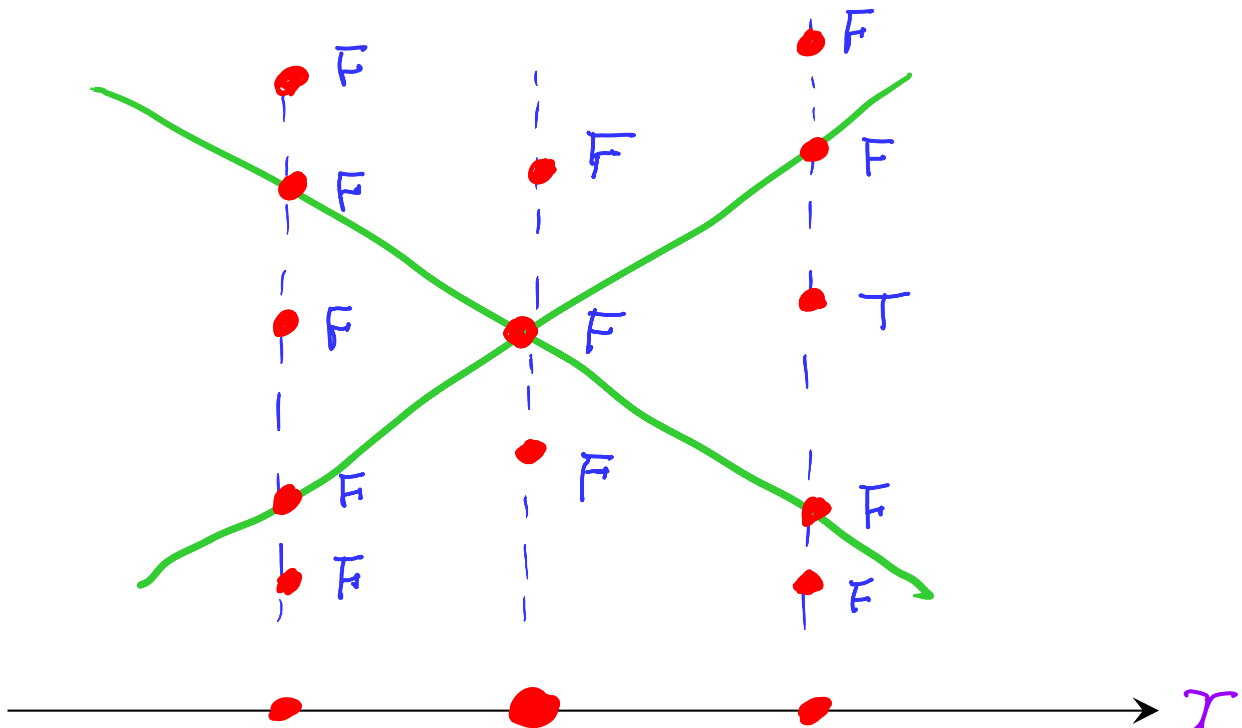

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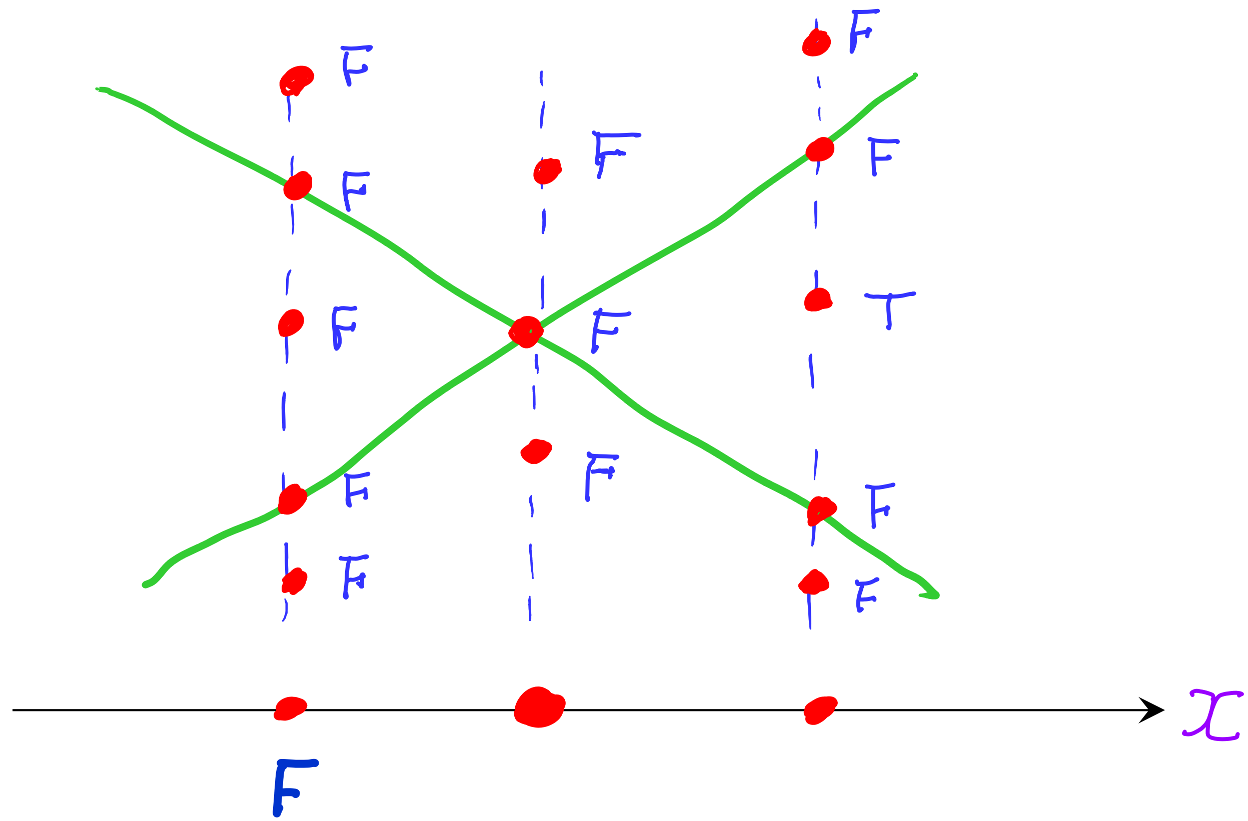

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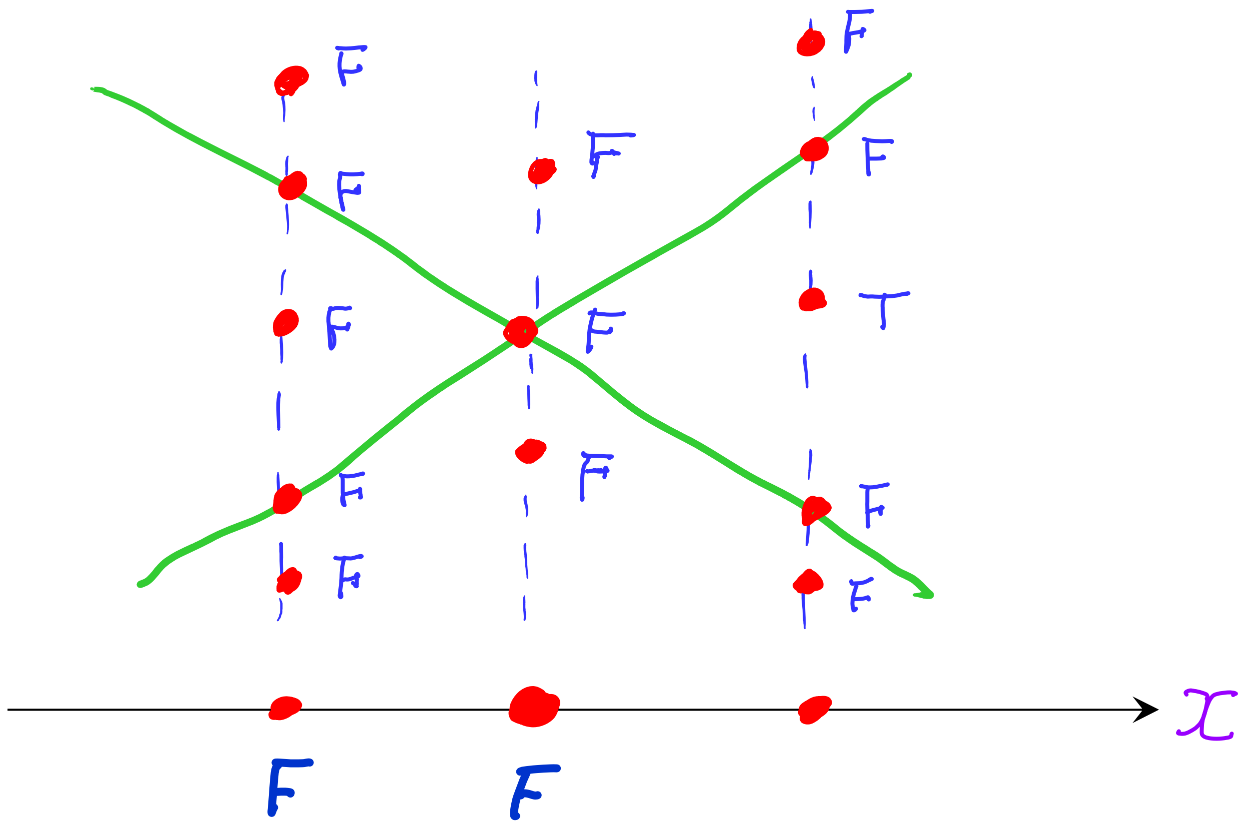

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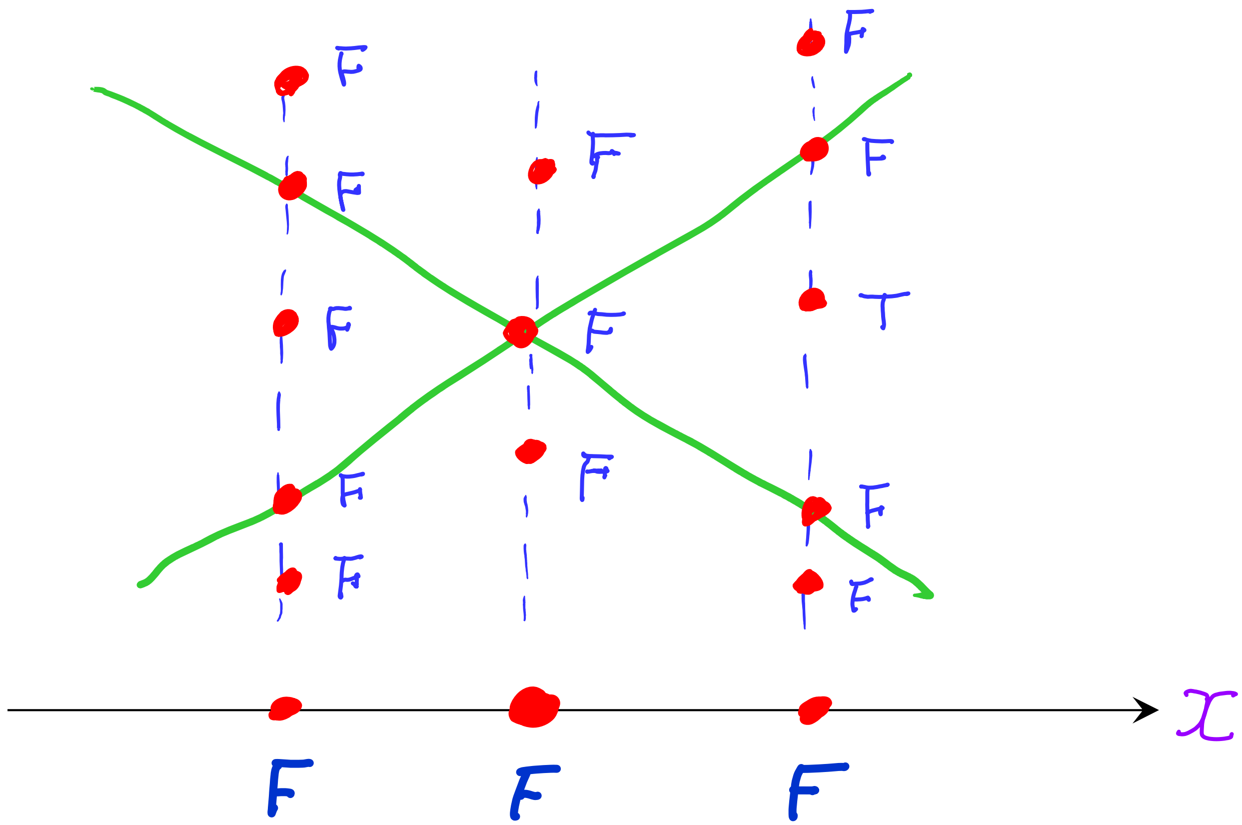
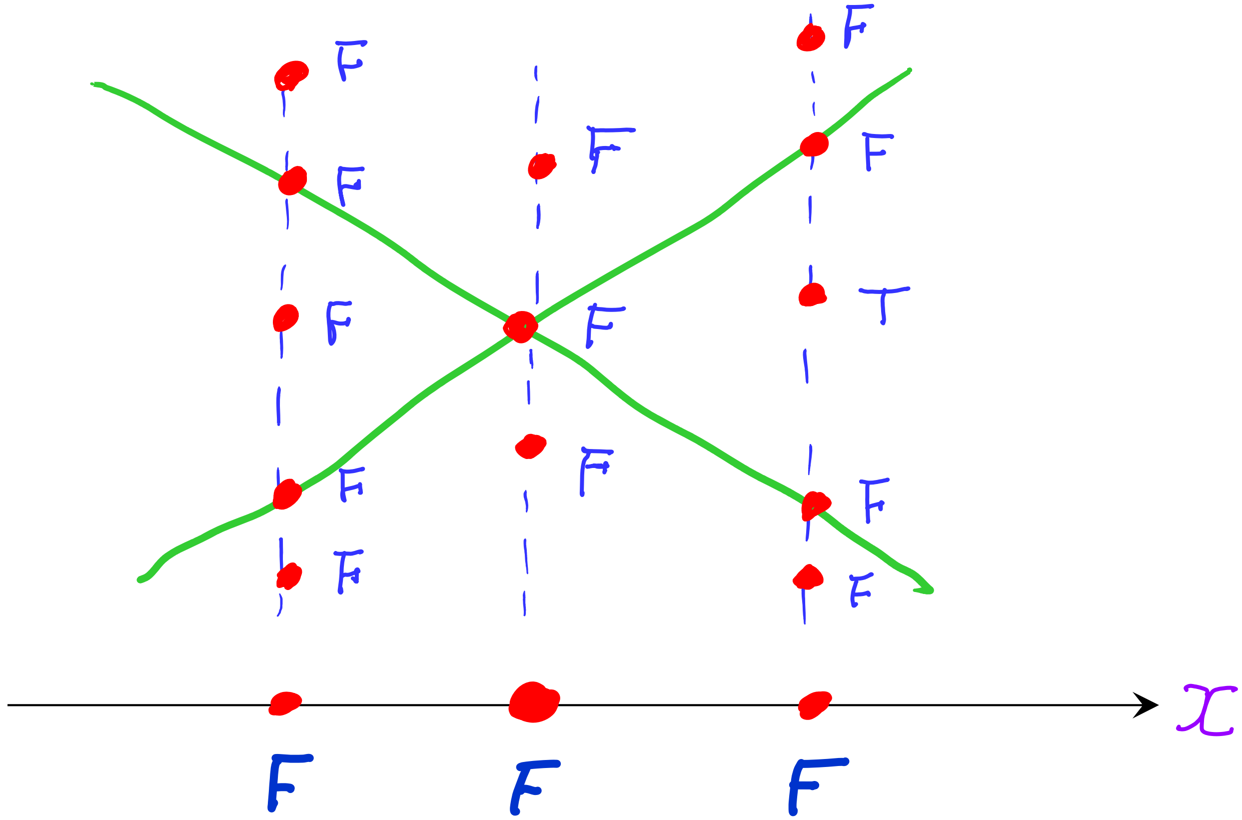
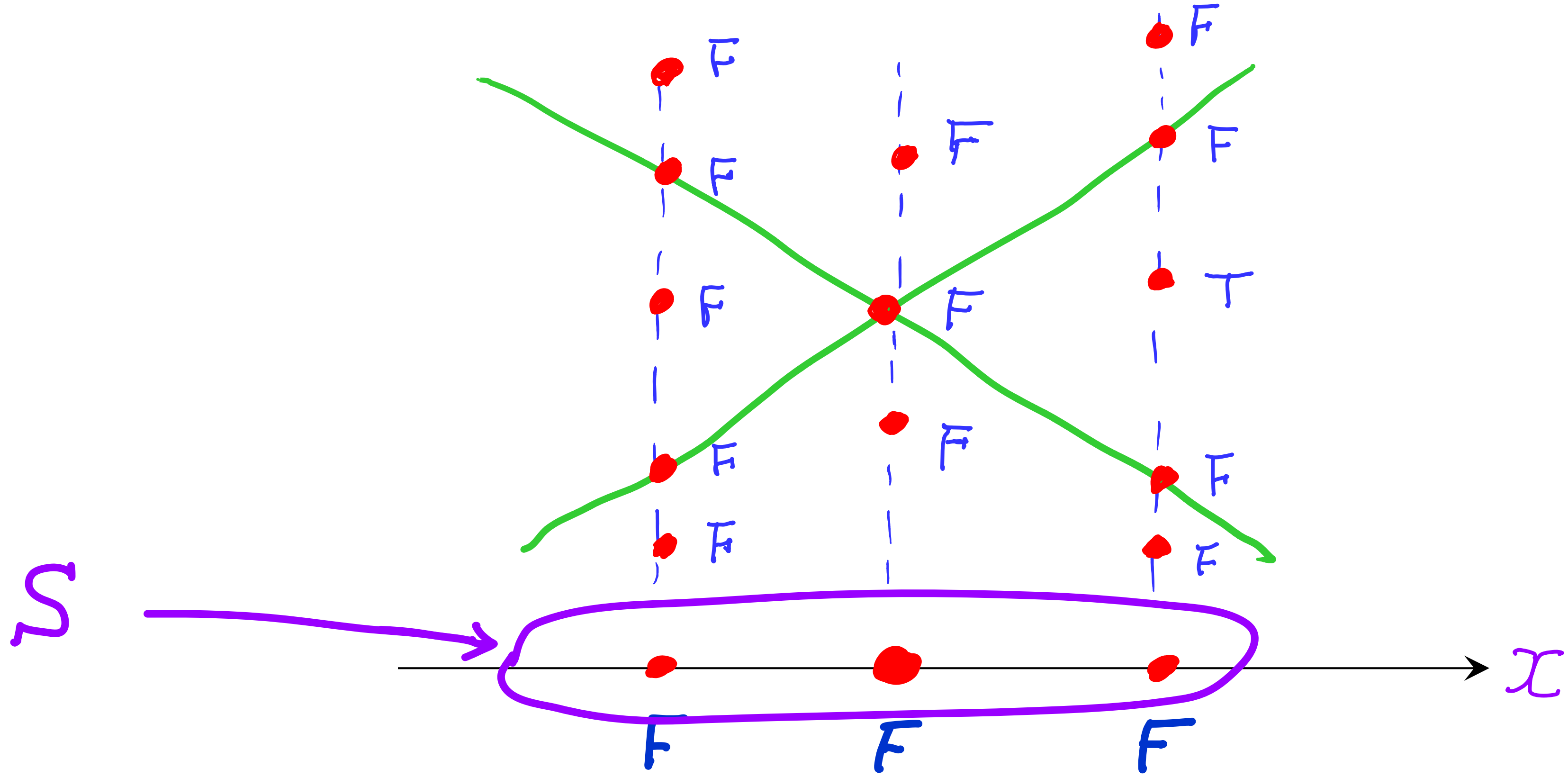


In: $\exists x \forall y \; x+y>0 \wedge x-y>0$



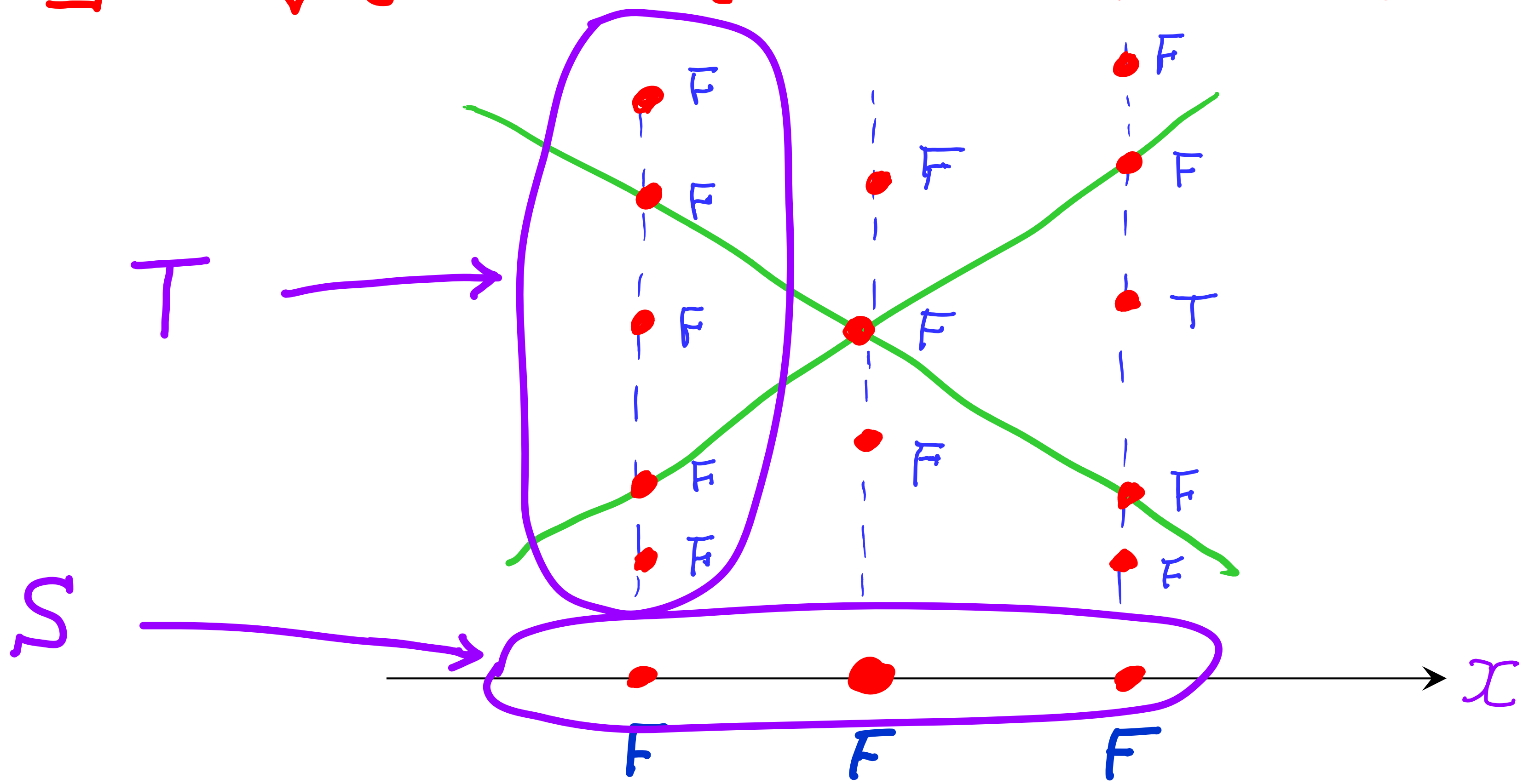
Out: False

In: $\exists x \, \forall y \;\; x + y > 0 \; \wedge \; x - y > 0$



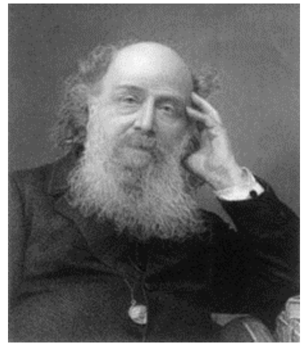
Out: False

In: $\exists x \, \forall y \; x + y > 0 \wedge x - y > 0$



Out: False

Resultant



Resultant

$$f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad a_3 \neq 0$$

$$g = b_2 x^2 + b_1 x + b_0 \qquad b_2 \neq 0$$

Resultant

$f = \underset{\neq 0}{a_3} x^3 + a_2 x^2 + a_1 x + a_0$

$g = \underset{\neq 0}{b_2} x^2 + b_1 x + b_0$

$\mathrm{res}_x(f, g) =$

Resultant

$$f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad a_3 \neq 0$$

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$$\mathrm{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 \\ & & & \\ & & & \\ & & & \end{vmatrix}$$

Resultant

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Resultant

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Resultant

$$f = \underset{\neq 0}{a_3} x^3 + a_2 x^2 + a_1 x + a_0$$

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Resultant

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$$g = b_2 x^2 + b_1 x + b_0 \qquad b_2 \neq 0$$

$$\operatorname{res}_x(f,g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

Resultant

$$f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad a_3 \neq 0$$

$$g = b_2 x^2 + b_1 x + b_0 \qquad b_2 \neq 0$$

$$h = \mathrm{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

Resultant

$$f = \underset{\neq 0}{a_3} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = \underset{\neq 0}{b_2} x^2 + b_1 x + b_0$$

$$h = \operatorname{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

Theorem

Resultant

$$f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad a_3 \neq 0$$

$$g = b_2 x^2 + b_1 x + b_0 \qquad b_2 \neq 0$$

$$h = \mathrm{res}_x(f,g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

Theorem

$$\exists x \in \mathbb{C}\ f(x) = 0 \wedge g(x) = 0$$

Resultant

$$f = \underset{\neq 0}{a_3} x^3 + a_2 x^2 + a_1 x + a_0$$

$$g = \underset{\neq 0}{b_2} x^2 + b_1 x + b_0$$

$$h = \mathrm{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

Theorem

$$\exists x \in \mathbb{C}\ f(x) = 0 \wedge g(x) = 0 \iff$$

Resultant

$$f = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \qquad a_3 \neq 0$$

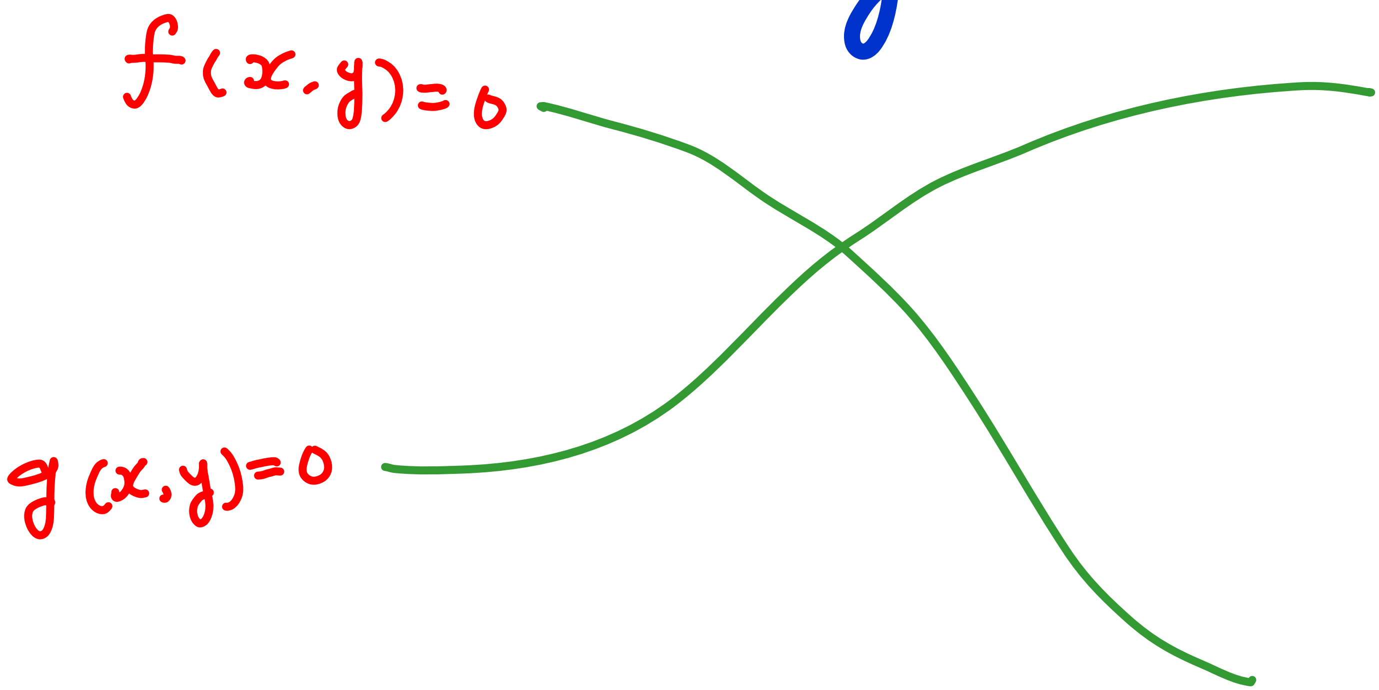
$$g = b_2 x^2 + b_1 x + b_0 \qquad b_2 \neq 0$$

$$h = \mathrm{res}_x(f, g) = \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & \\ & a_3 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \end{vmatrix}$$

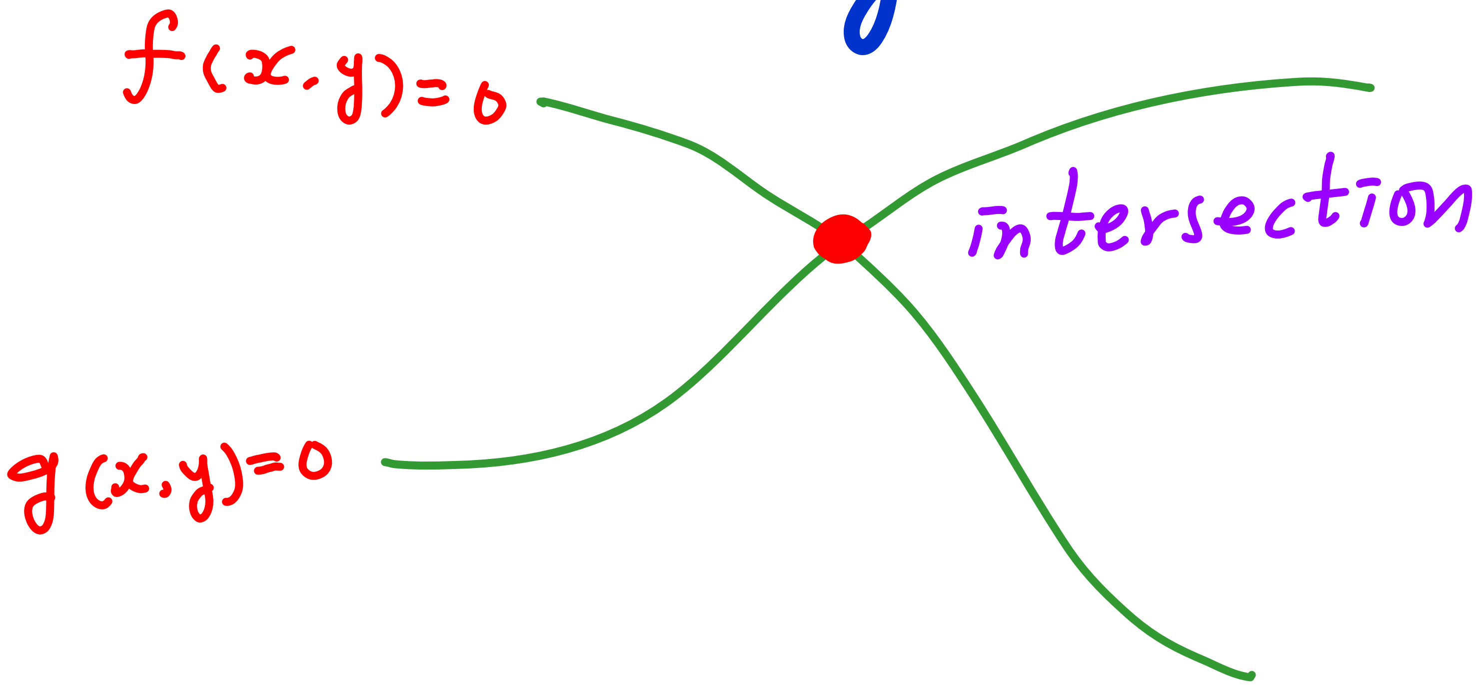
Theorem

$$\exists x \in \mathbb{C} \;\; f(x) = 0 \wedge g(x) = 0 \iff h = 0$$

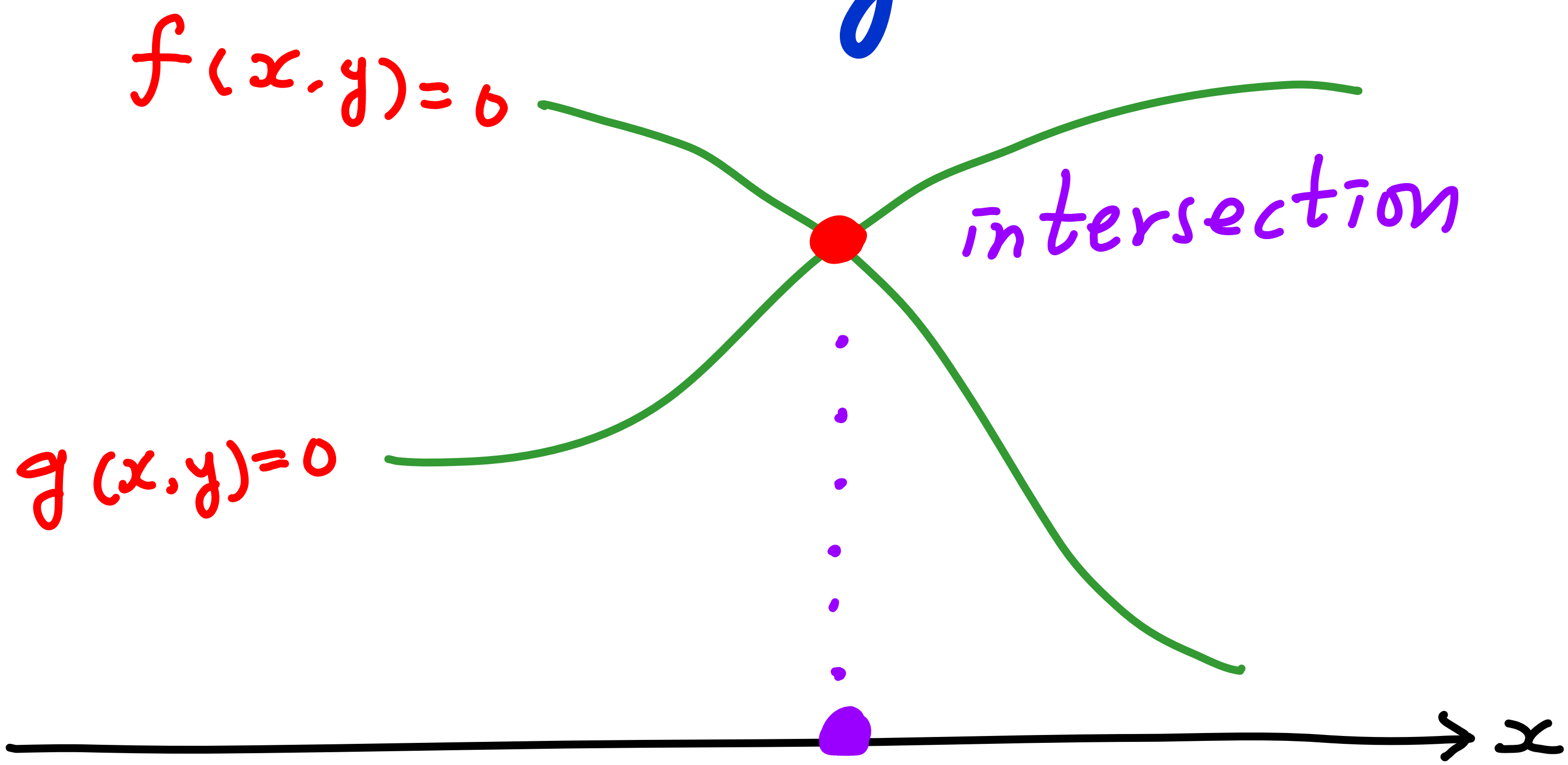
Interesting Points



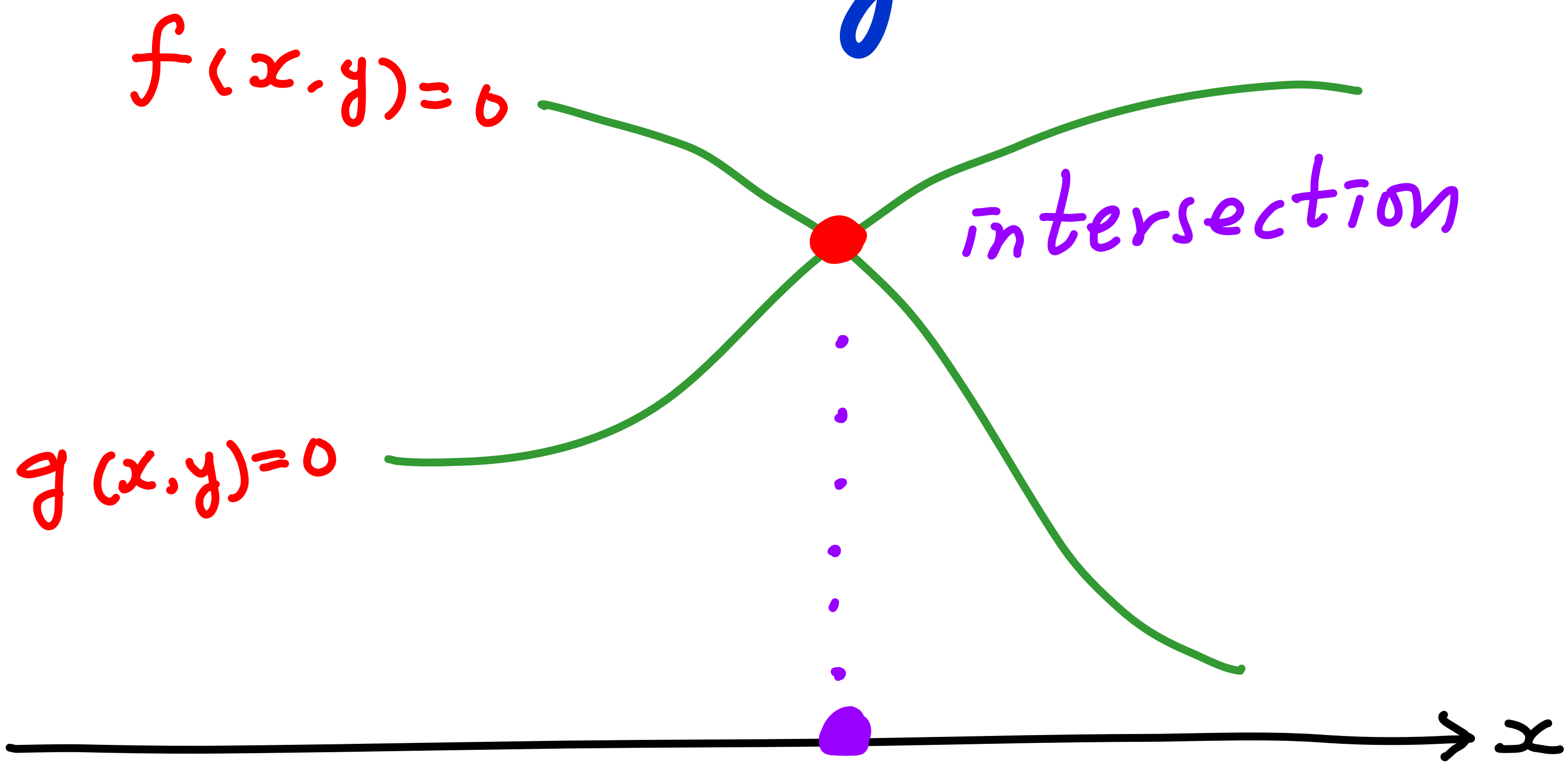
Interesting Points



Interesting Points



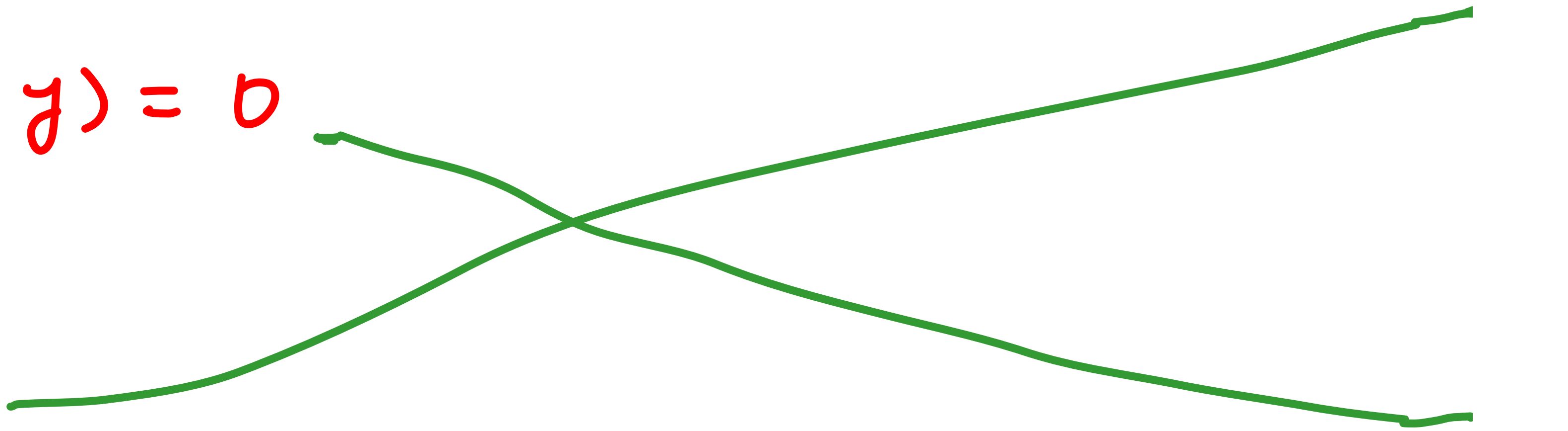
Interesting Points



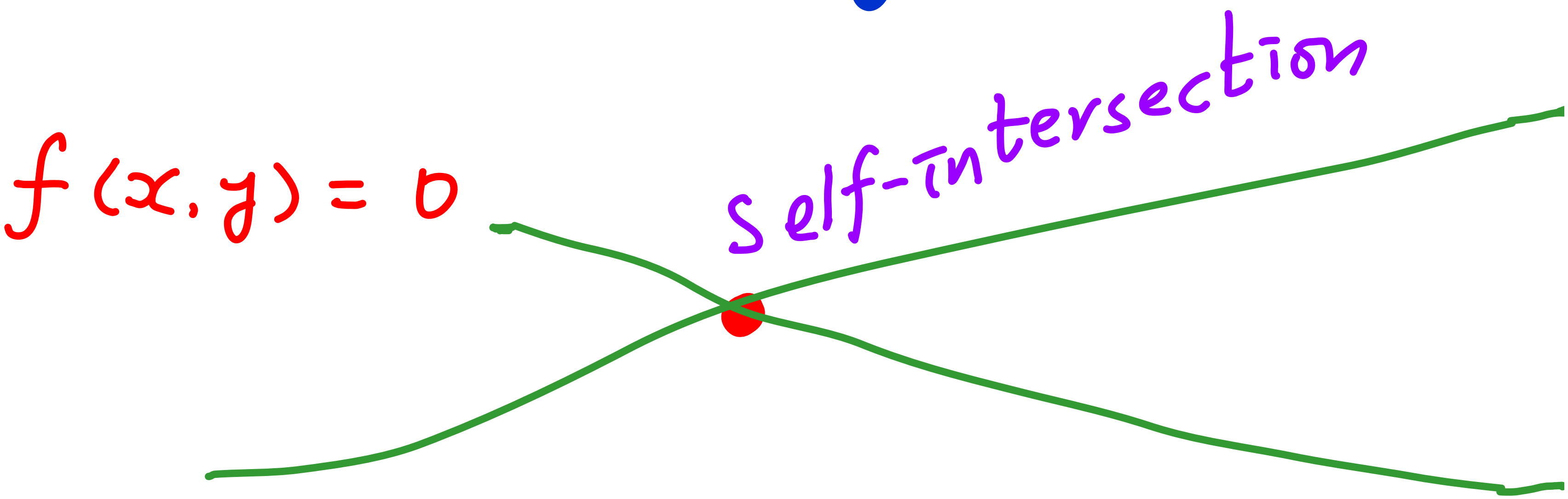
$$r(x) = \mathrm{res}_y(f, g)$$

Interesting Points

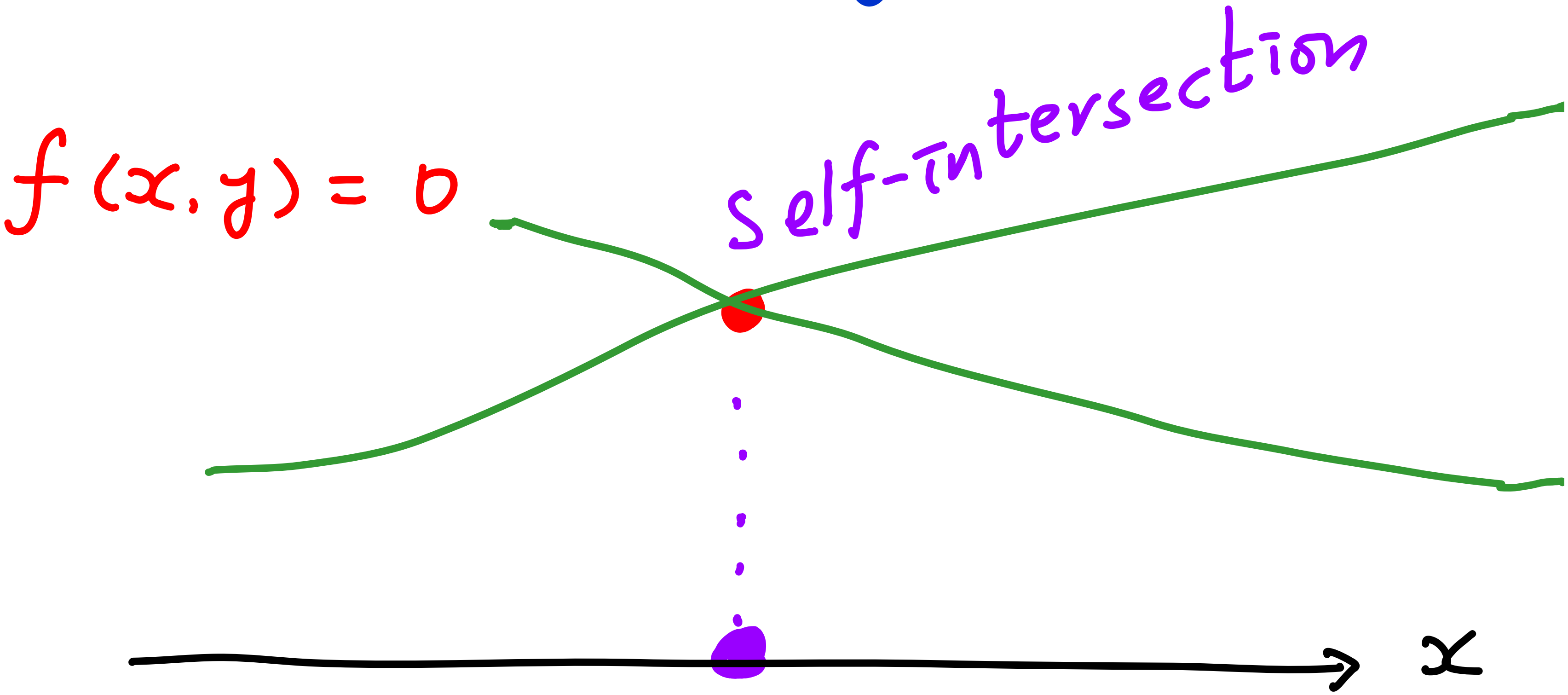
$f(x, y) = 0$



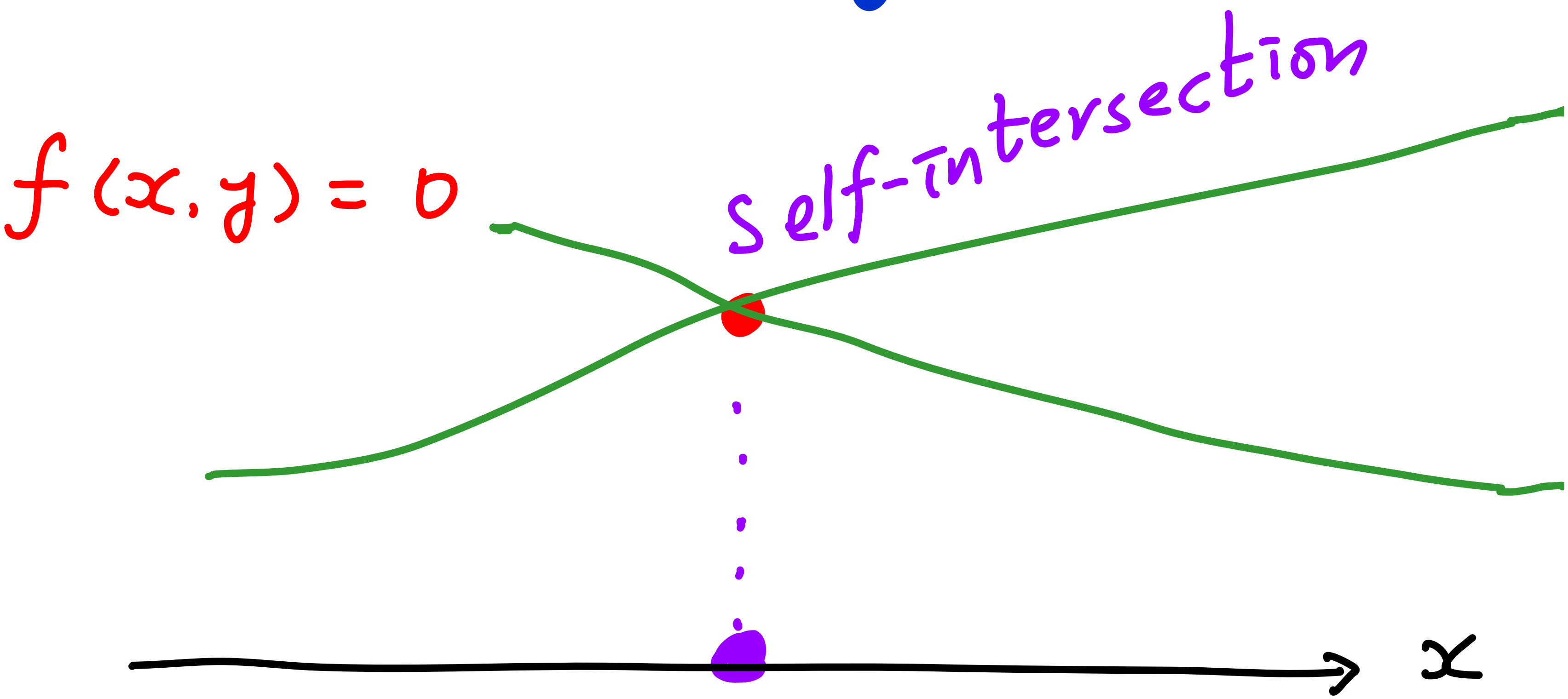
Interesting Points



Interesting Points



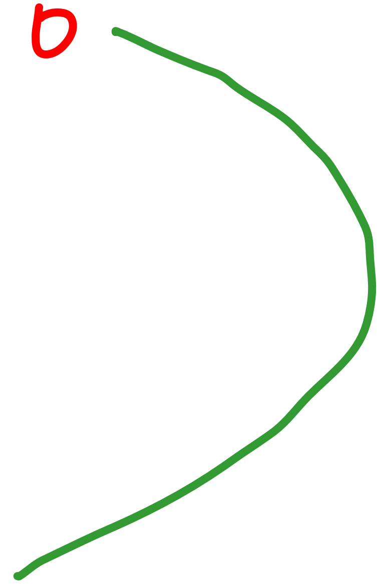
Interesting Points



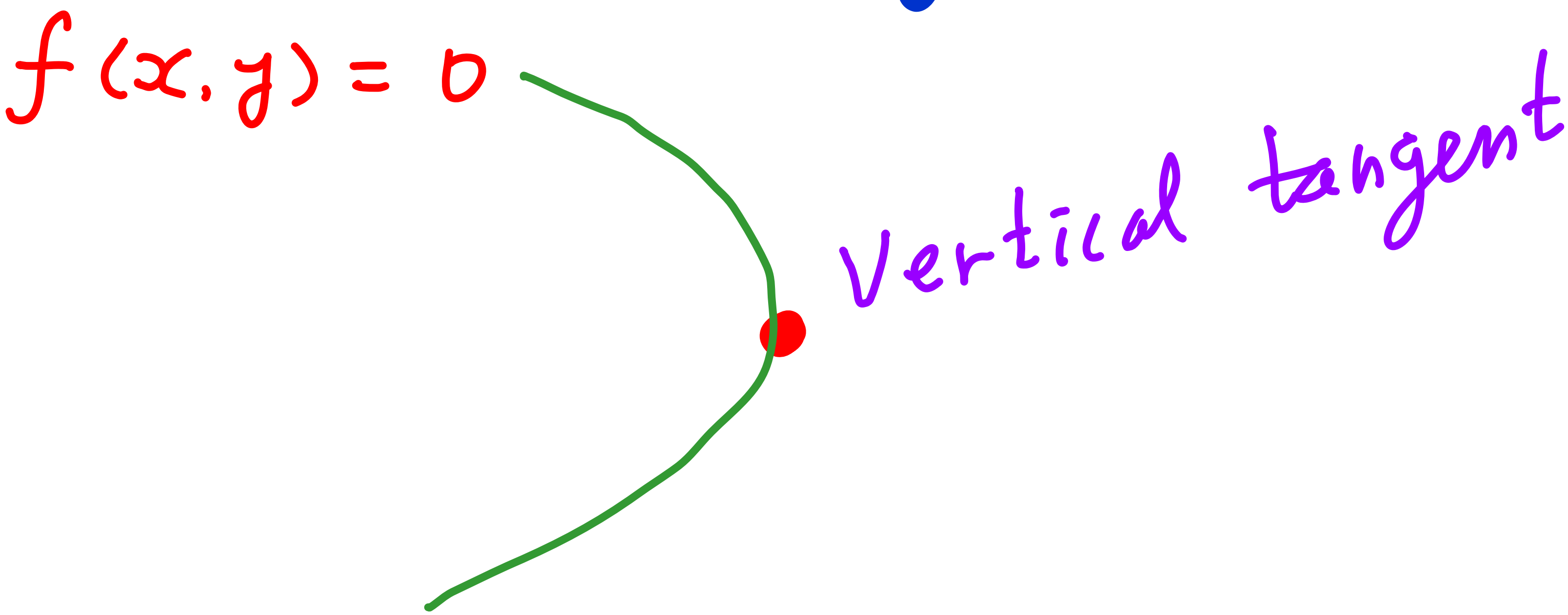
$$d(x) = \operatorname{res}_y\left(f, \frac{\partial f}{\partial y}\right)$$

Interesting Points

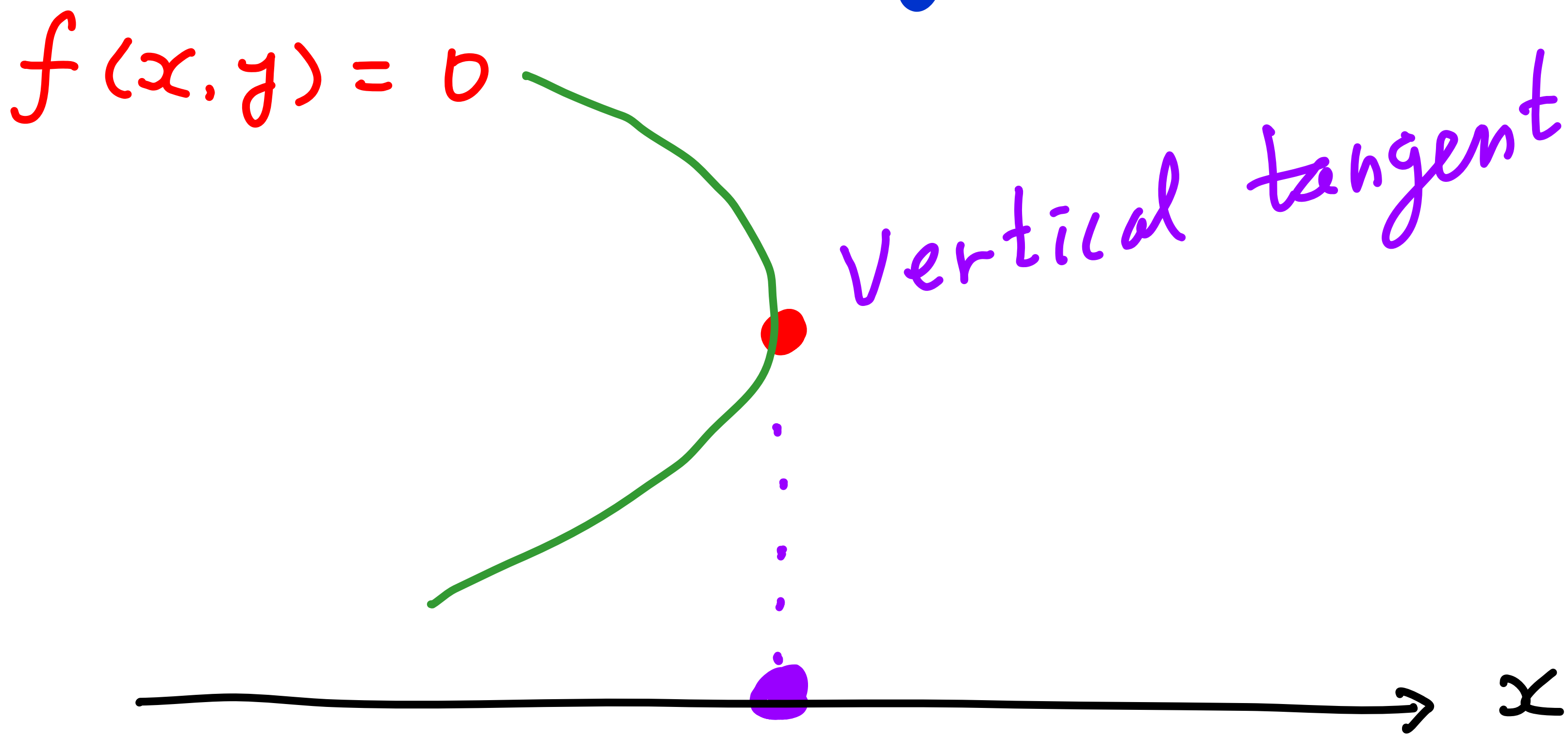
$f(x, y) = 0$



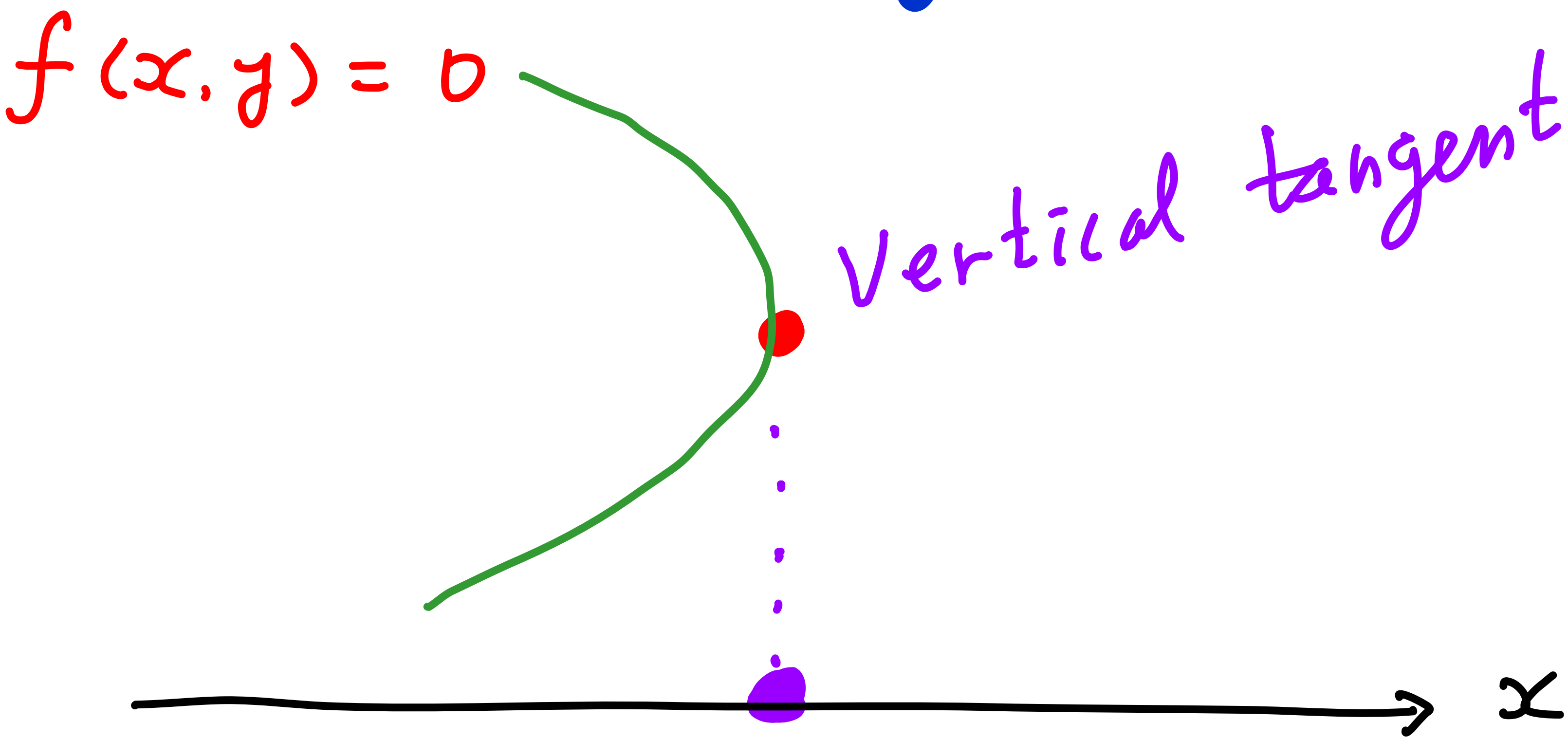
Interesting Points



Interesting Points



Interesting Points



$$d(x) = \operatorname{res}_y\left(f, \frac{\partial f}{\partial y}\right)$$

Interesting Points

projection polys

$$\left\{\begin{array}{l} \mathrm{res}_y(f, g) \\ \mathrm{res}_y\left(f, \frac{\partial f}{\partial y}\right) \\ \mathrm{res}_y\left(g, \frac{\partial g}{\partial y}\right) \end{array}\right.$$

Challenges

- Reduce the # of projection polys
- Projection "on demand"
- Simplify the output formula
- make incremental

Repackaging

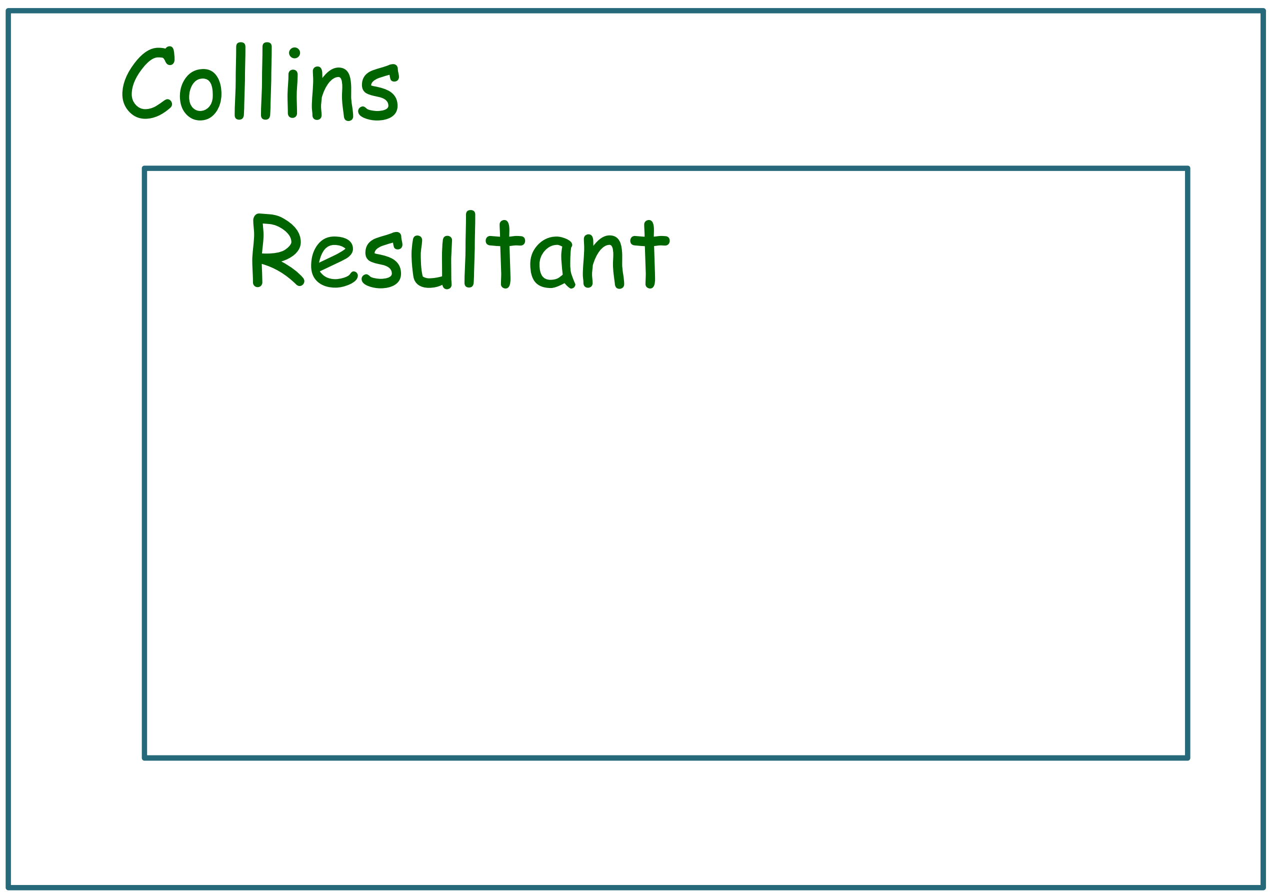
Prolongation - Relaxation

Observe

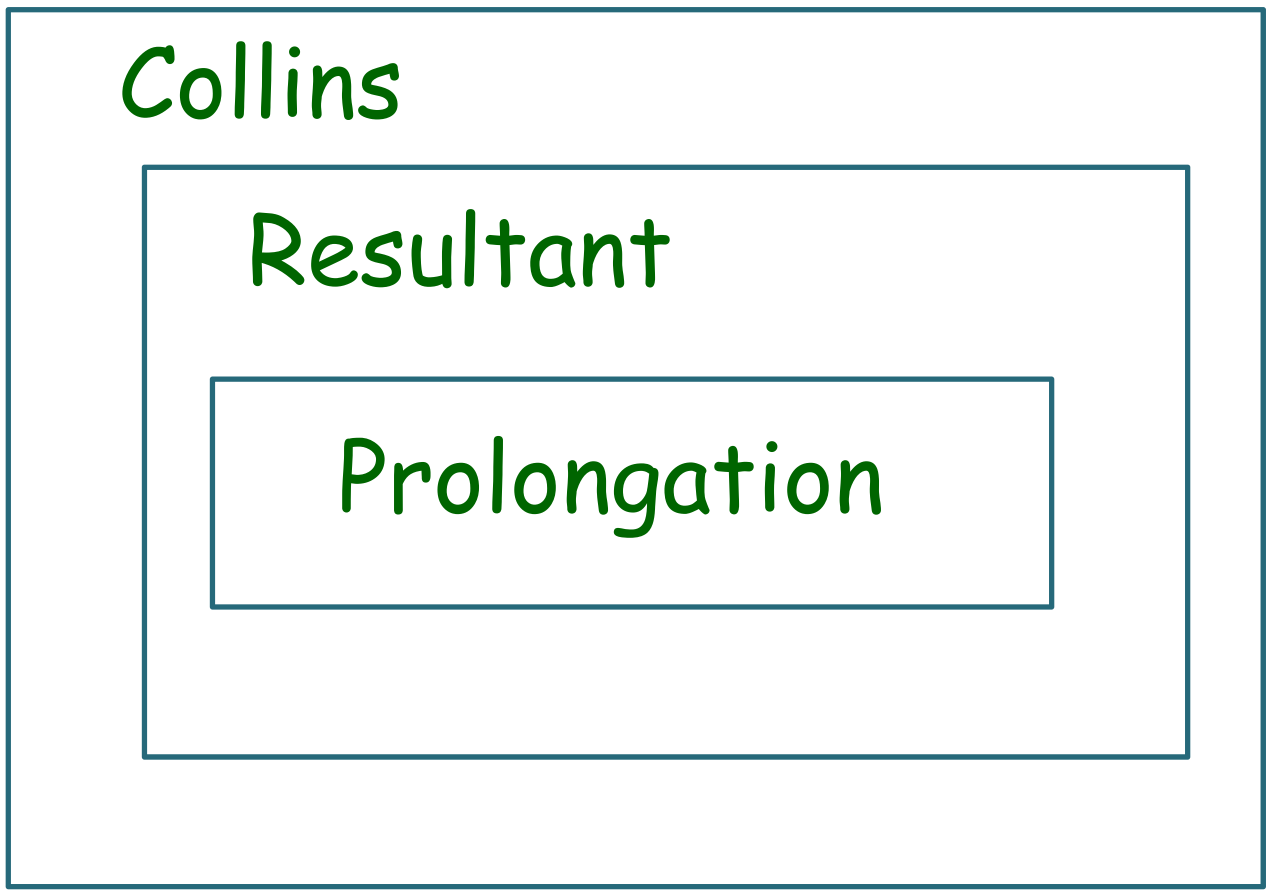
Observe

Collins

Observe



Observe



$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \qquad\qquad b_1 x + b_0 \end{cases}$$

prolongation

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \qquad\qquad b_1 x + b_0 \end{cases}$$

prolongation

$$\begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \qquad\qquad b_1 x + b_0 \end{cases}$$

prolongation

$$\begin{array}{ccc} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{array}$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = b_1 x + b_0 \end{cases}$$

$\longleftarrow$ prolongation

$$\Updownarrow$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \qquad\qquad b_1 x + b_0 \end{cases} \longleftarrow \text{prolongation}$$

$$\Updownarrow$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix}$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \phantom{b_1 x^2 +{}} b_1 x + b_0 \end{cases}$$

prolongation

$$\Updownarrow$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix}$$

$$\Downarrow \quad \Uparrow \ \mathbb{C}$$

$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix}$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = g = b_1 x + b_0 \end{cases}$$

$$\Updownarrow$$

$$\exists x \quad \begin{cases} 0 = f = a_2 x^2 + a_1 x + a_0 \\ 0 = xg = b_1 x^2 + b_0 x \\ 0 = g = \phantom{b_1 x^2 +{}} b_1 x + b_0 \end{cases}$$

prolongation

$$\Updownarrow$$

$$\exists x \quad 0 = \begin{bmatrix} f \\ xg \\ g \end{bmatrix} = \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} \qquad \begin{bmatrix} x_2 \\ x_1 \\ 1 \end{bmatrix}$$

$$\Downarrow \quad \Uparrow \; \mathbb{C}$$

$$0 = \det \begin{bmatrix} a_2 & a_1 & a_0 \\ b_1 & b_0 & \\ & b_1 & b_0 \end{bmatrix}$$

relaxation

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$ false

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \wedge x^3 - x^2 + x = 0$$

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \wedge x^3 - x^2 + x = 0 \leftarrow$ prolongation

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\Updownarrow$

$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \;\wedge\; x^3 - x^2 + x = 0$ ← prolongation

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$ false

$$\Updownarrow$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \;\wedge\; x^3 - x^2 + x = 0$$ ← prolongation

$$\exists \vec{x} \in {\mathbb{R}^+}^3 \quad x_2 - x_1 + 1 = 0 \;\wedge\; x_3 - x_2 + x_1 = 0$$

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\Updownarrow$

$$\exists x \in \mathbb{R}^+ \;\wedge\; \begin{array}{l} x^2 - x - 1 = 0 \\ x^3 - x^2 + x = 0 \end{array}$$ ← prolongation

$$\exists \vec{x} \in \mathbb{R}^{+3} \;\wedge\; \begin{array}{l} x_2 - x_1 + 1 = 0 \\ x_3 - x_2 + x_1 = 0 \end{array}$$ ← Relaxaction

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$ false

$$\Updownarrow$$

$$\exists x \in \mathbb{R}^+ \quad x^2 - x - 1 = 0 \;\wedge\; x^3 - x^2 + x = 0$$ ← prolongation

$$\Downarrow$$

$$\exists \vec{x} \in \mathbb{R}^{+3} \quad x_2 - x_1 + 1 = 0 \;\wedge\; x_3 - x_2 + x_1 = 0$$ ← Relaxaction

$$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$$ false

$$\Updownarrow$$

$$\exists x \in \mathbb{R}^+ \wedge \begin{array}{l} x^2 - x - 1 = 0 \\ x^3 - x^2 + x = 0 \end{array}$$ ← prolongation

$$\Downarrow$$

$$\exists \vec{x} \in \mathbb{R}^{+3} \wedge \begin{array}{l} x_2 - x_1 + 1 = 0 \\ x_3 - x_2 + x_1 = 0 \end{array}$$ ← Relaxaction

$$\Updownarrow$$ ← Linear method

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\Updownarrow$

$\exists x \in \mathbb{R}^+ \wedge \begin{array}{l} x^2 - x - 1 = 0 \\ x^3 - x^2 + x = 0 \end{array}$ $\leftarrow$ prolongation

$\Downarrow$

$\exists \vec{x} \in \mathbb{R}^{+3} \wedge \begin{array}{l} x_2 - x_1 + 1 = 0 \\ x_3 - x_2 + x_1 = 0 \end{array}$ $\leftarrow$ Relaxaction

$\Updownarrow$ $\leftarrow$ Linear method

false

$\exists x \in \mathbb{R}^+ \quad x^2 - x + 1 = 0$ false

$\Updownarrow$

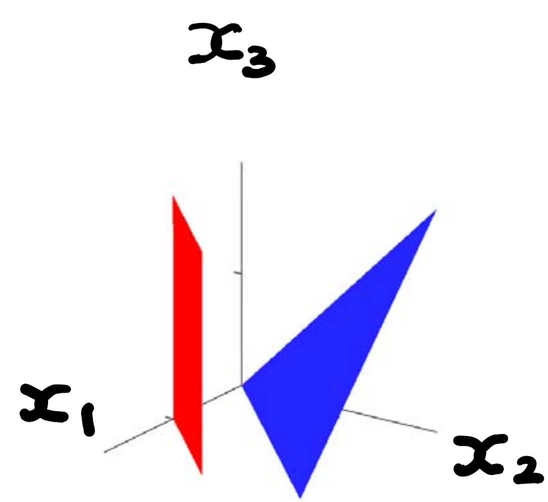
$\exists x \in \mathbb{R}^+ \wedge \begin{array}{l} x^2 - x - 1 = 0 \\ x^3 - x^2 + x = 0 \end{array}$ ← prolongation

$\Downarrow$

$\exists \vec{x} \in \mathbb{R}^{+3} \wedge \begin{array}{l} x_2 - x_1 + 1 = 0 \\ x_3 - x_2 + x_1 = 0 \end{array}$ ← Relaxaction

$\Updownarrow$ ← Linear method

false



Cany, Grigor'ev, Roy,Renegar

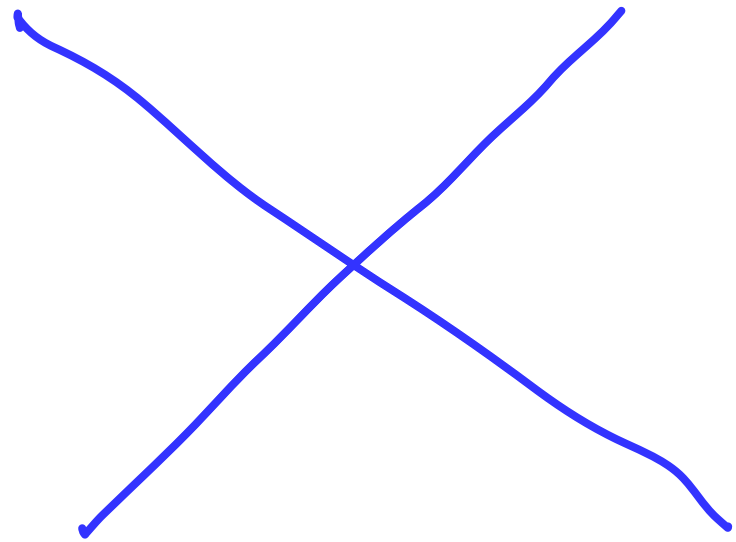
$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geqslant 0 \;\wedge\; \underbrace{x-y}_{g} \geqslant 0$$

$$\exists x \, \exists y \; \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

$$f \cdot g = 0$$

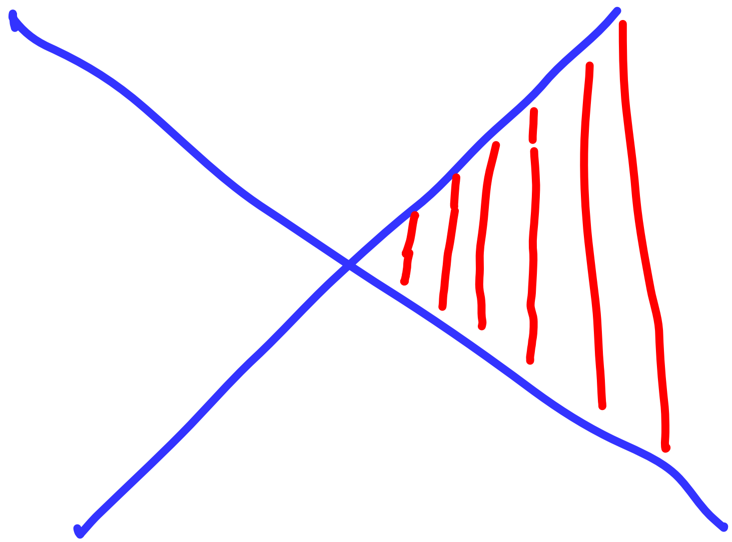
$$\exists x \exists y \quad \underbrace{x+y}_{f} \geq 0 \ \wedge \ \underbrace{x-y}_{g} \geq 0$$

$$f \cdot g = 0$$



$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

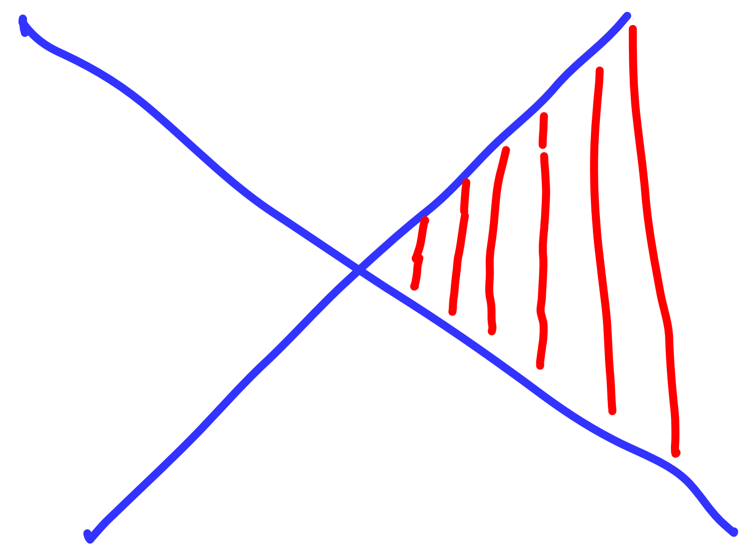
$$f \cdot g = 0$$



$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

$$f \cdot g = 0$$

$$(f+\varepsilon)(g+\varepsilon) = \varepsilon^2$$



$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geq 0 \ \wedge \ \underbrace{x-y}_{g} \geq 0$$

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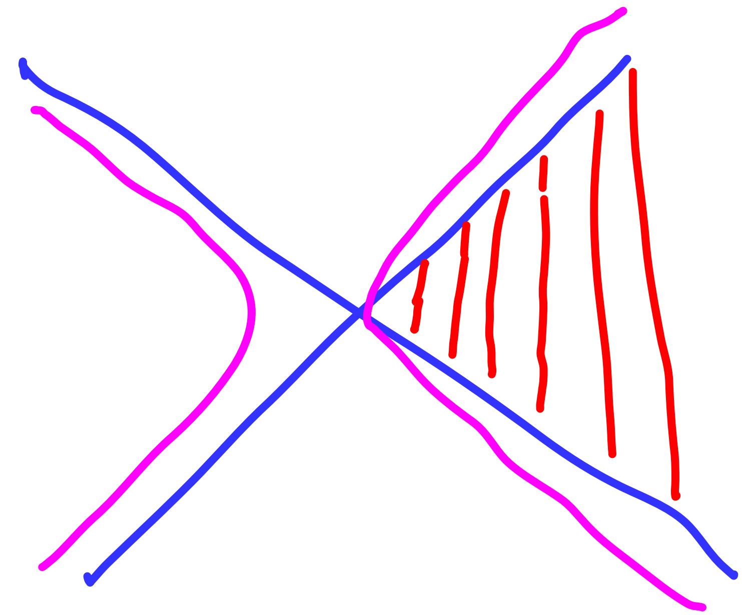

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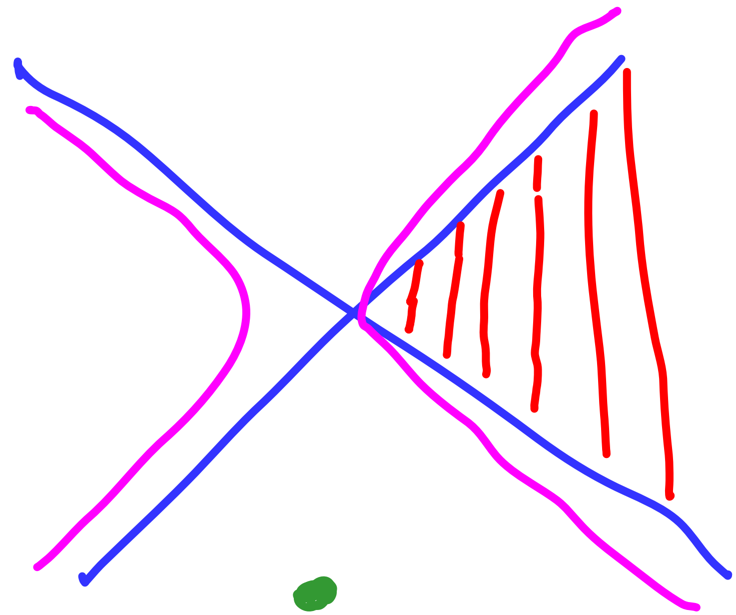

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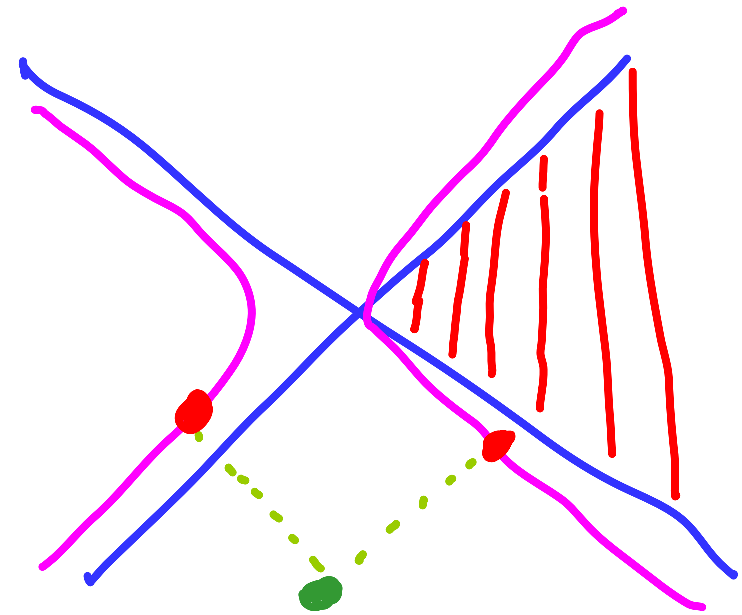

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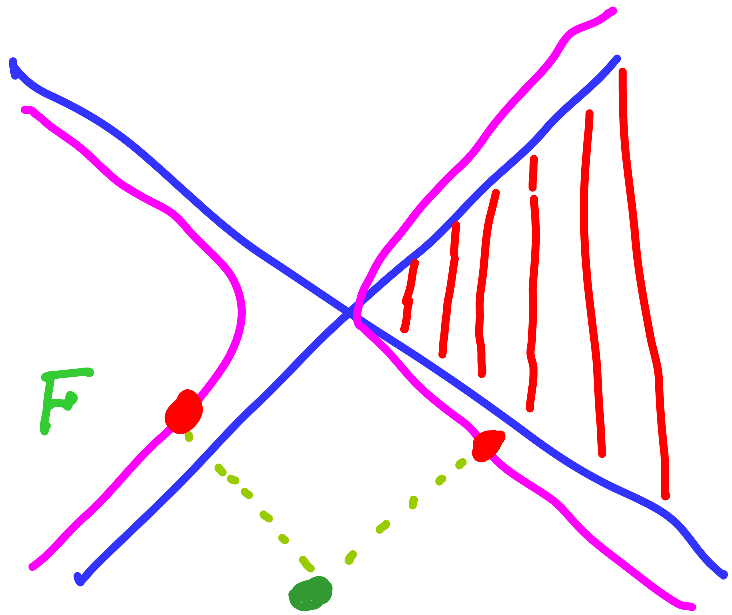

$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

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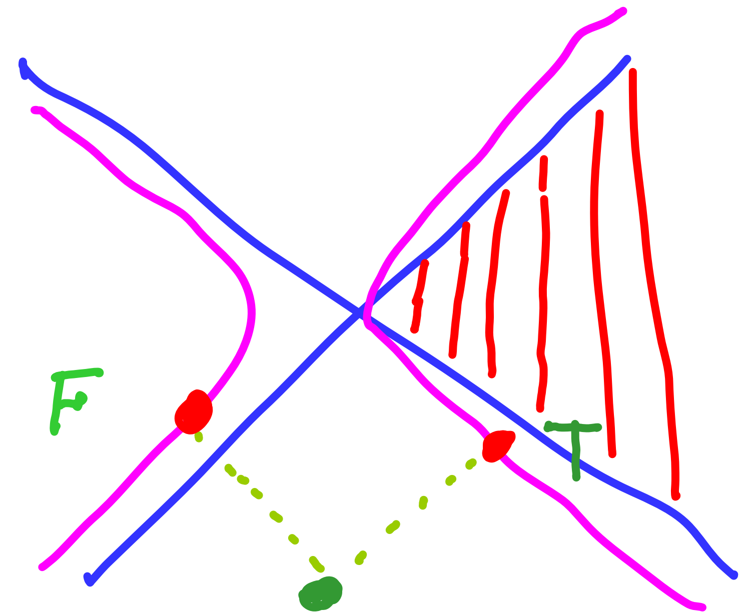
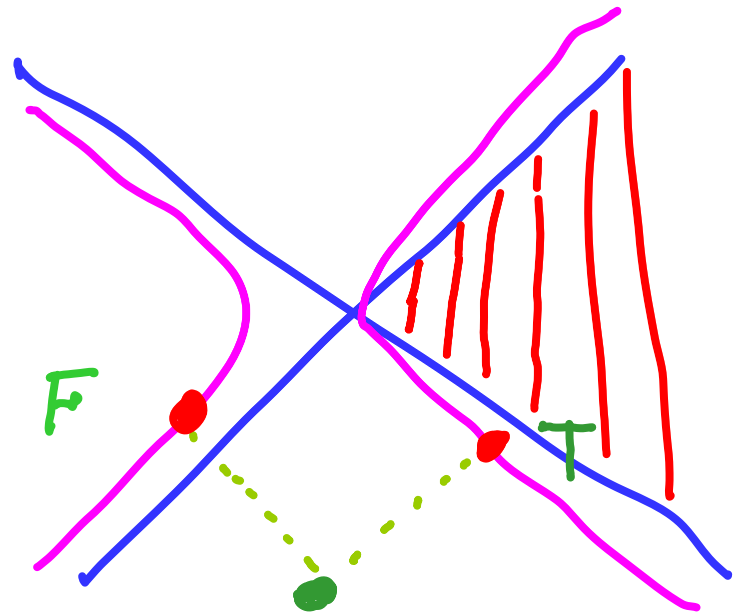


$$\exists x \, \exists y \;\; \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

True!

$$f \cdot g = 0$$

$$(f+\varepsilon)(g+\varepsilon) = \varepsilon^2$$

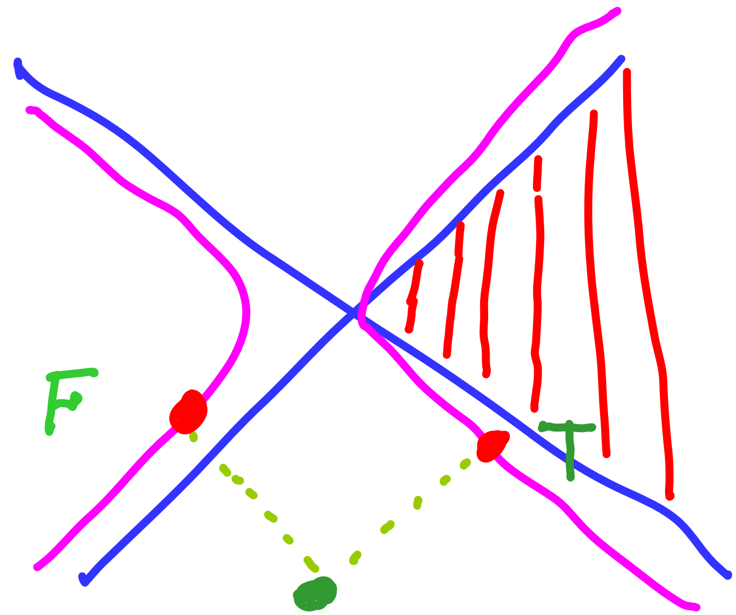


$$\exists x \, \exists y \quad \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

True!

$$f \cdot g = 0$$

$$(f+\varepsilon)(g+\varepsilon) = \varepsilon^2 + \delta$$



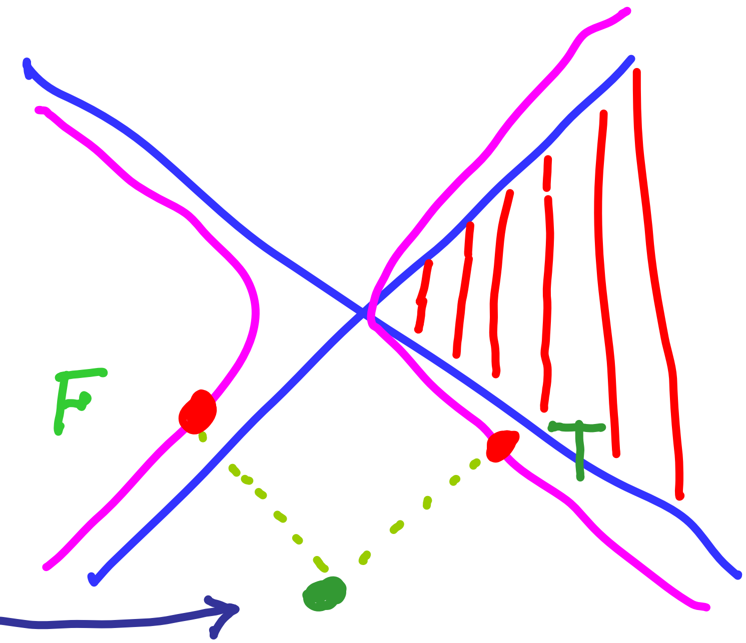
$$\exists x \, \exists y \;\; \underbrace{x+y}_{f} \geq 0 \;\wedge\; \underbrace{x-y}_{g} \geq 0$$

True!

$$f \cdot g = 0$$

$$(f+\varepsilon)(g+\varepsilon) = \varepsilon^2 + \delta$$

"generically"



$$\exists x \in \mathbb{R}^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\Uparrow$$

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in R^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{R(\varepsilon)}$$

real algebraic closure

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\varepsilon)}$$

real algebraic closure

$$K = \overline{L(\delta)}$$

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\varepsilon)}$$

real algebraic closure

$$K = \overline{L(\delta)}$$

$$g = \prod_{i=1}^{r} (f_i + \varepsilon) - \varepsilon^r - \delta$$

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\varepsilon)}$$

real algebraic closure

$$K = \overline{L(\delta)}$$

$$g = \prod_{i=1}^{n} (f_i + \varepsilon) - \varepsilon^r - \delta$$

T = Solutions $\in K^n$ of Lagrange mul. eq. for "distance" from a point p.

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$\exists x \in \mathbb{R}^n \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

$$L = \overline{\mathbb{R}(\varepsilon)} \quad \text{real algebraic closure}$$

$$K = \overline{L(\delta)}$$

$$g = \prod_{i=1}^{n} (f_i + \varepsilon) - \varepsilon^r - \delta$$

T = Solutions $\in K^n$ of Lagrange mul. eq. for "distance" from a point p.

S = standard part of T.

$$\exists x \in S \quad f_1(x) \geq 0 \wedge \cdots \wedge f_r(x) \geq 0$$

Challenges

- Efficient computation with infinitismals
- Reducing the size of sample set.

Repackaging

Critical points - Morse complex

Problem

Problem

Input

Problem

Input

$$f \in R[x_1, \ldots, x_n]$$

Problem

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Toy Example

Problem

Input

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Toy Example

$$f = x^3 - x^2 + y^2$$

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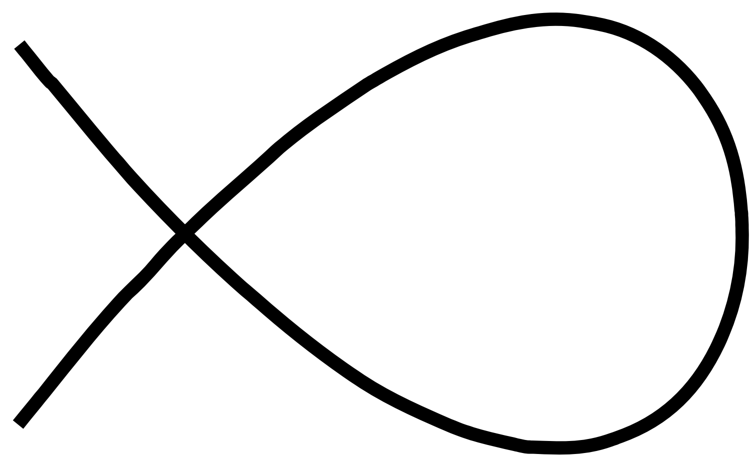

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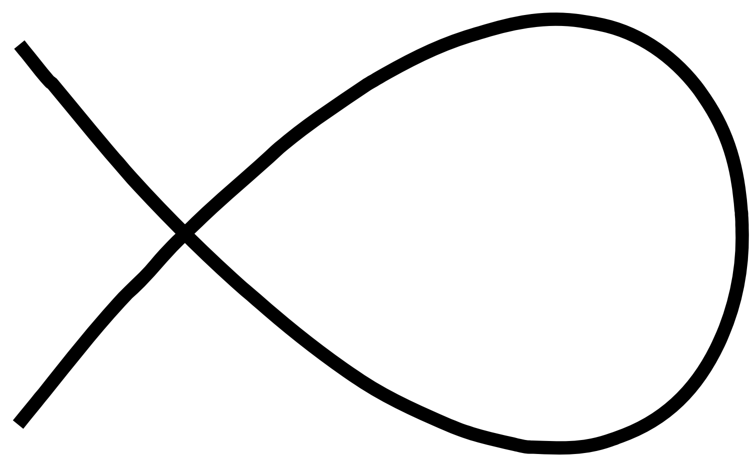

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$$f \in R[x_1,\ldots,x_n]$$

Output

A point in each connected component

Toy Example

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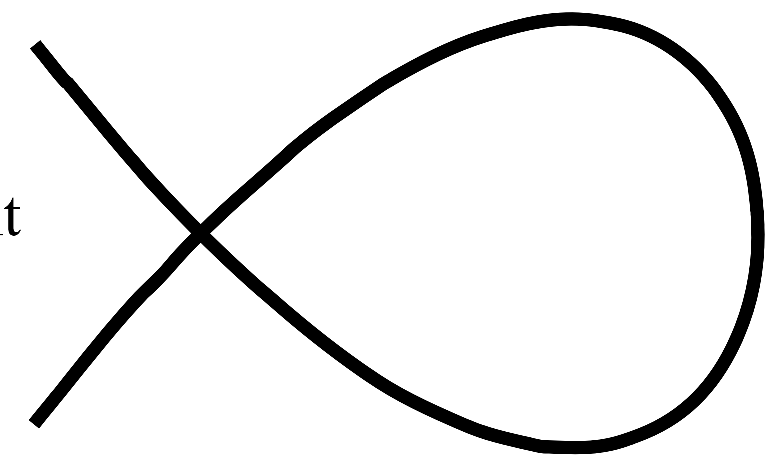

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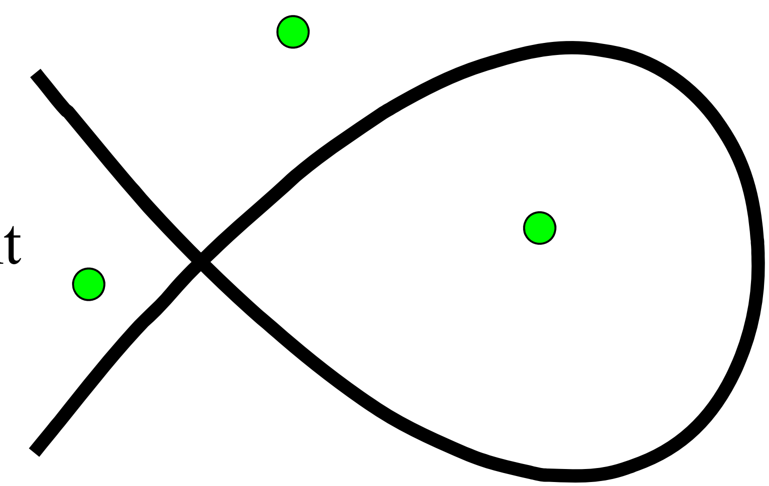

Idea

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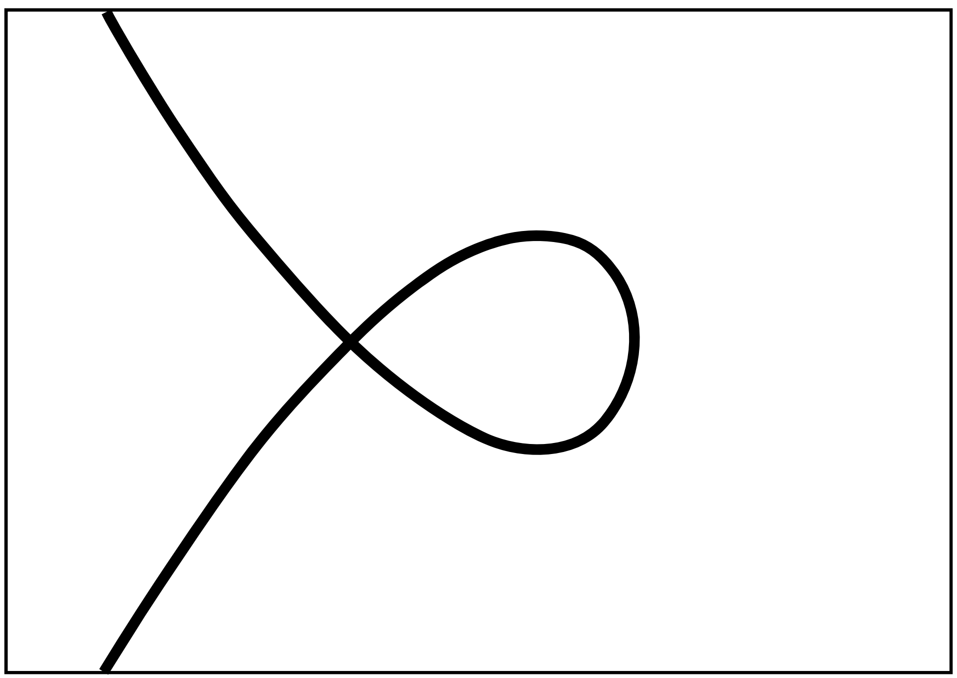
f

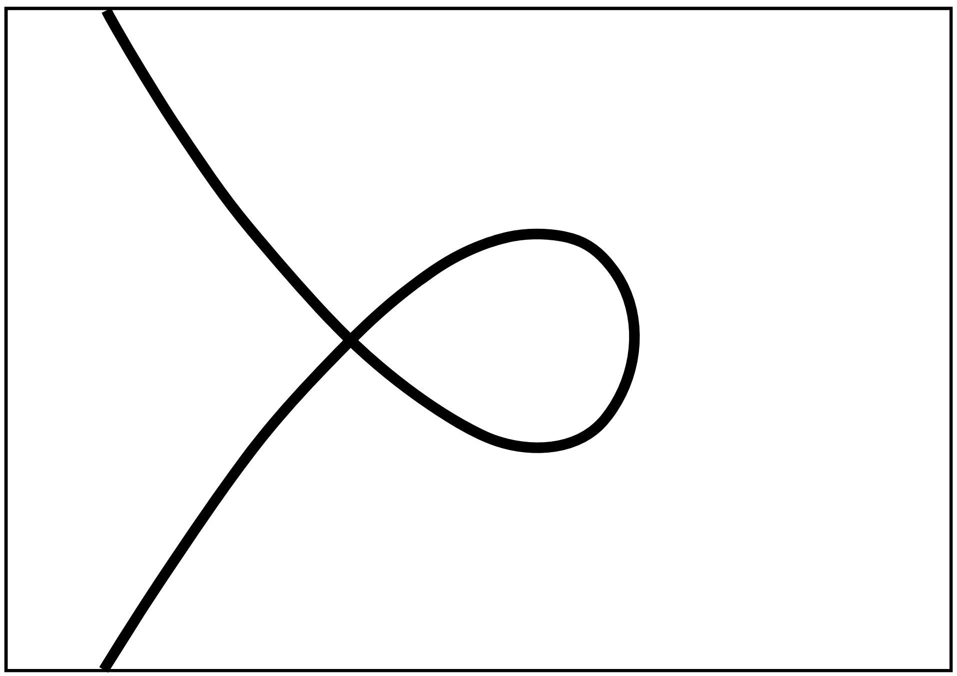
Idea

f



Idea

$$g = \frac{f^2}{\left(x^2 + y^2 + 1\right)^{d+1}}$$



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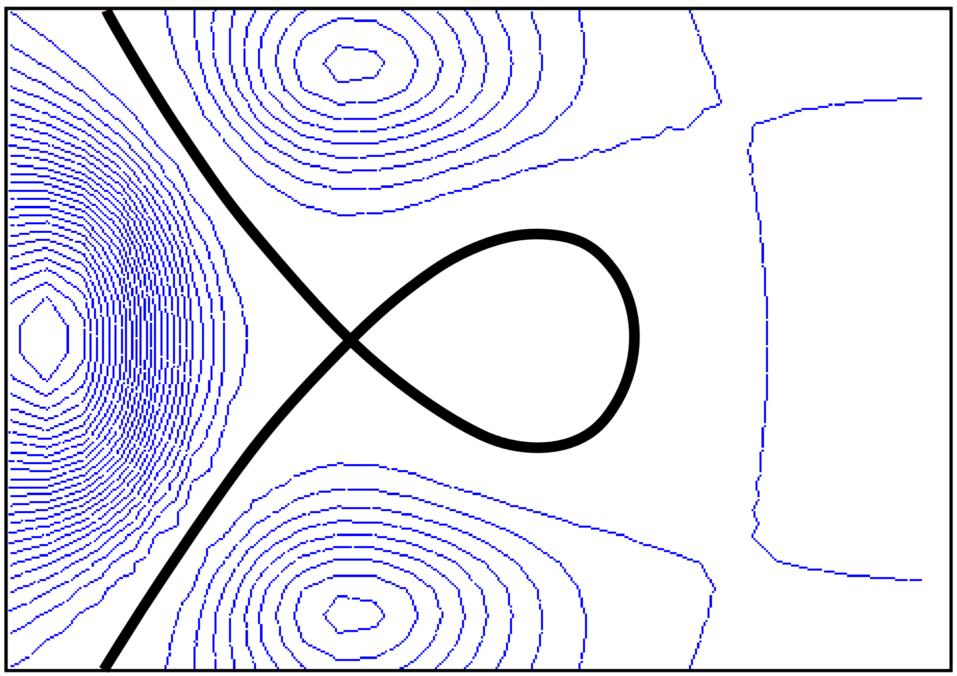

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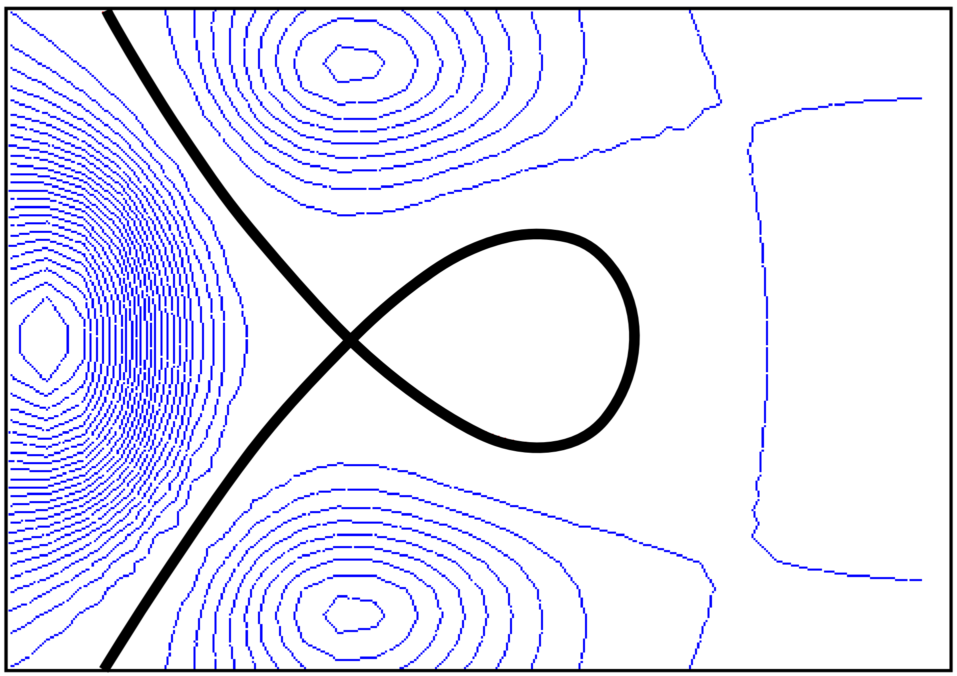

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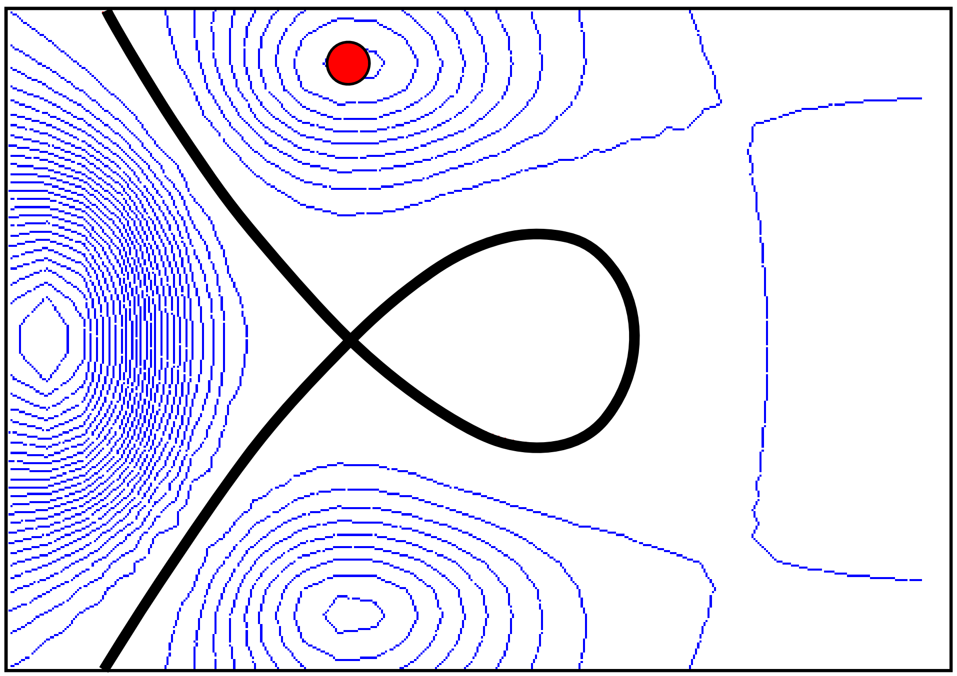

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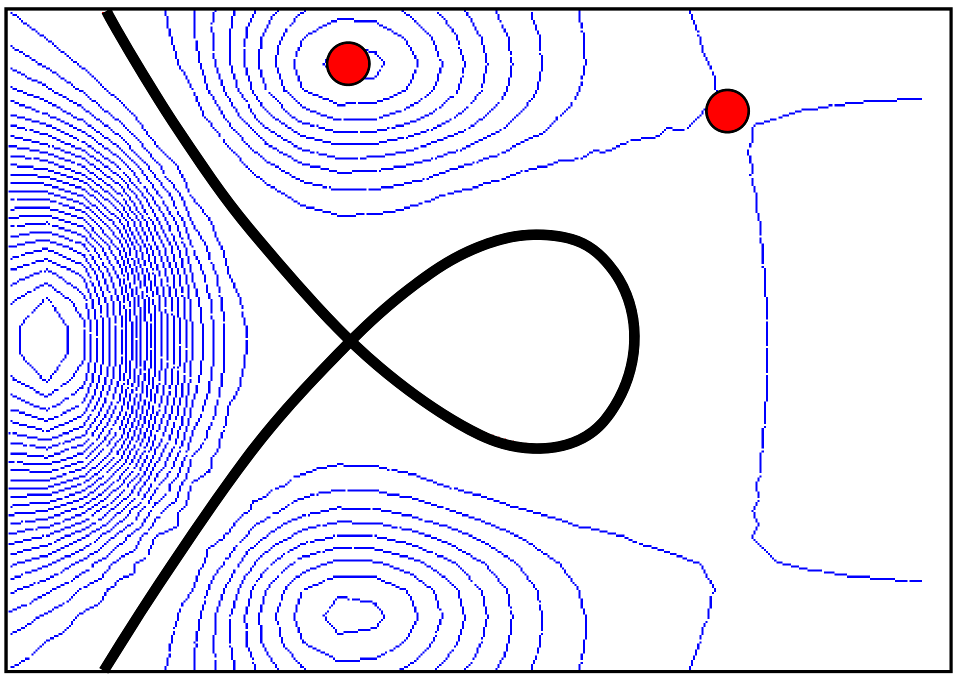

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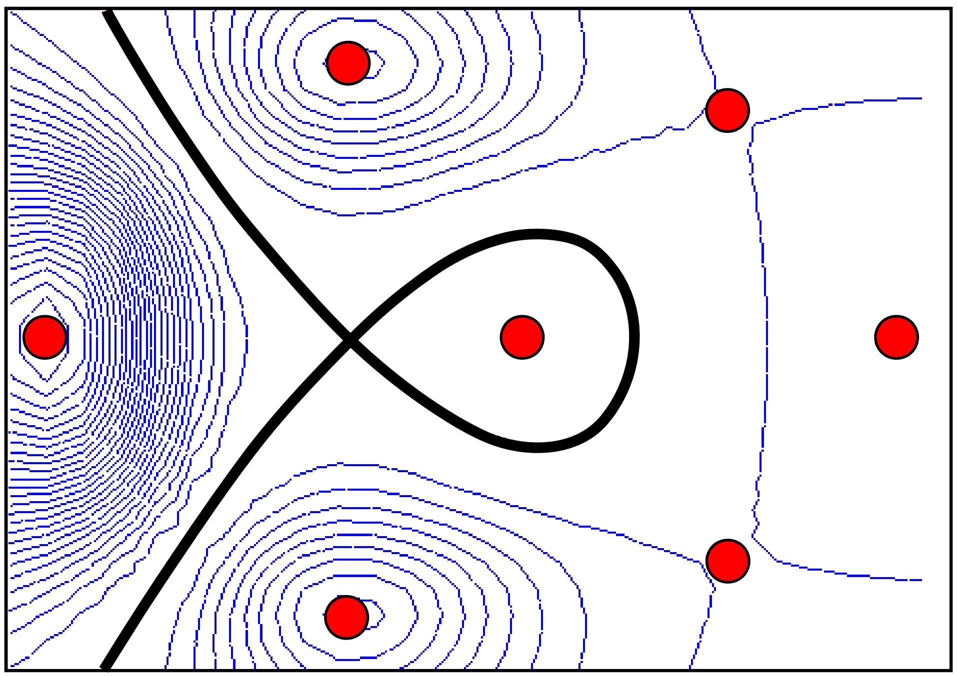

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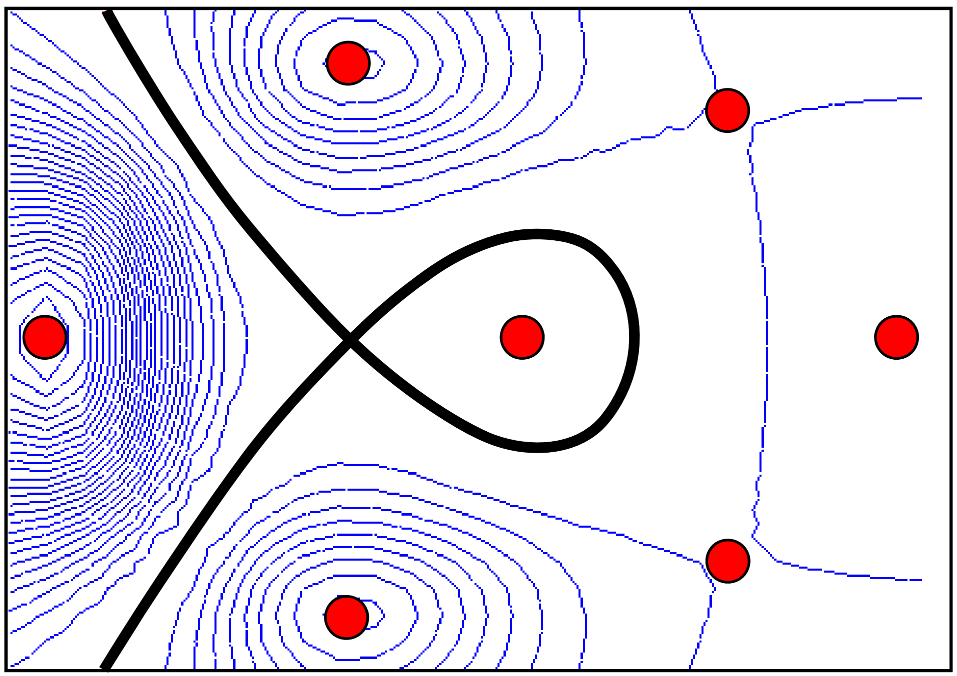

Hybrid method

Idea

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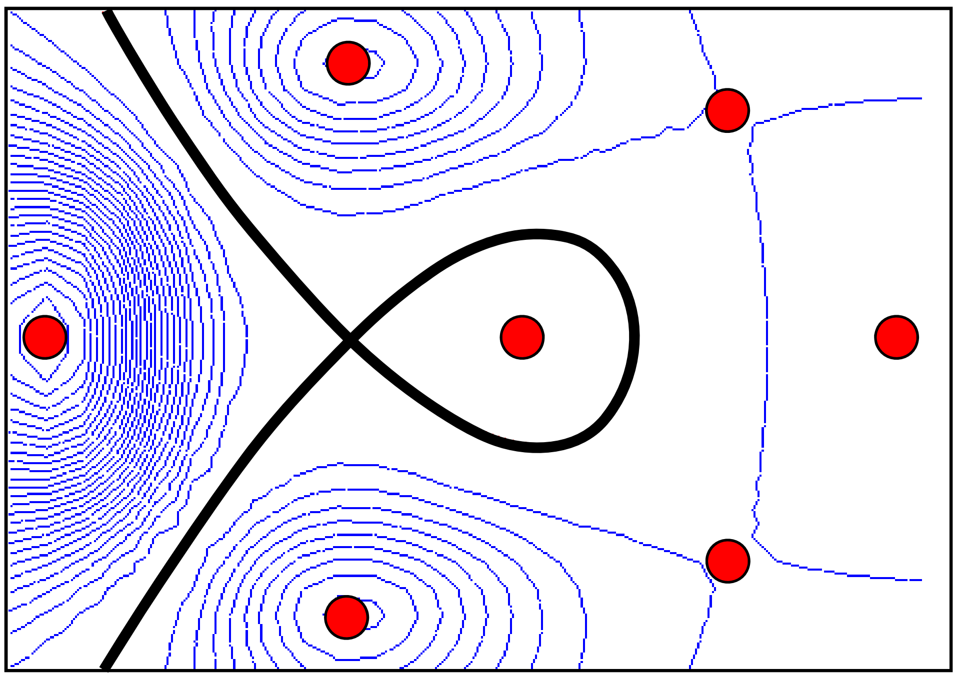

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Hybrid method

- Symbolic: Resultant

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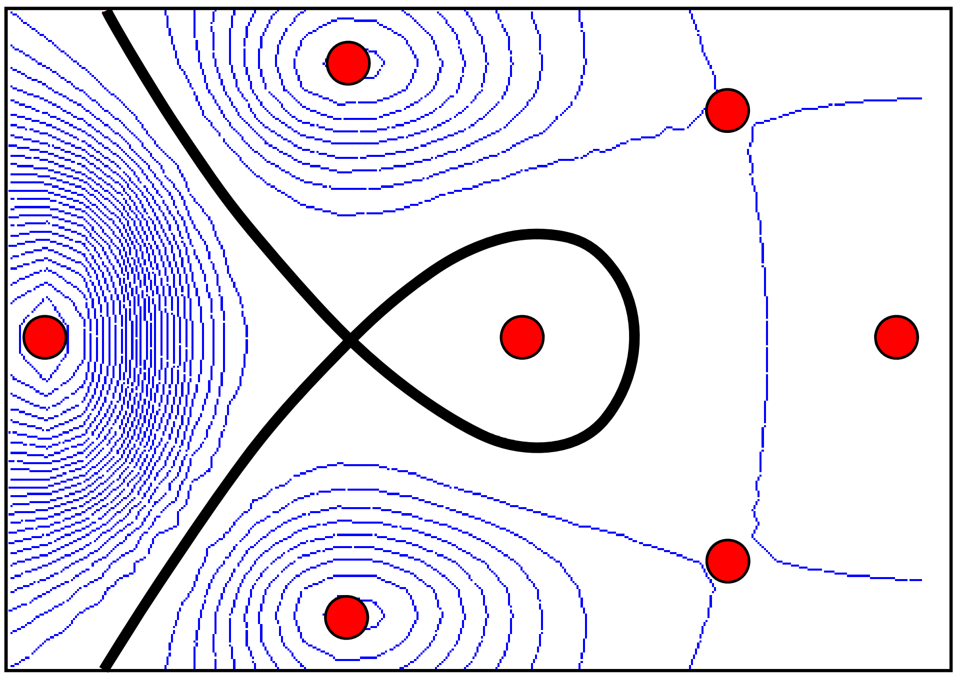

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- Symbolic: Resultant
- Numeric : Interval (HSO)

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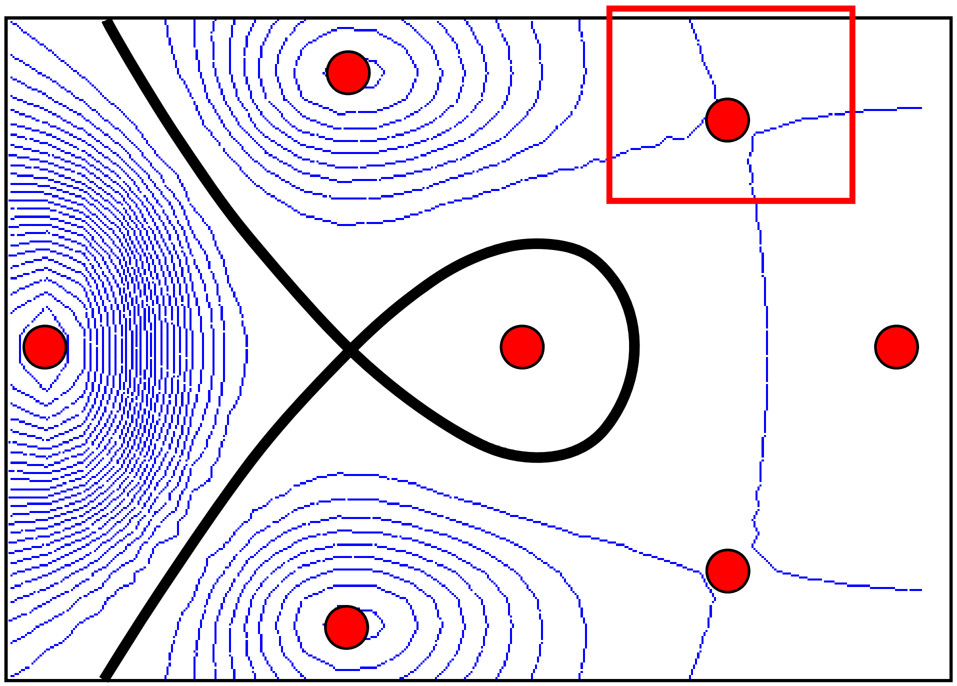

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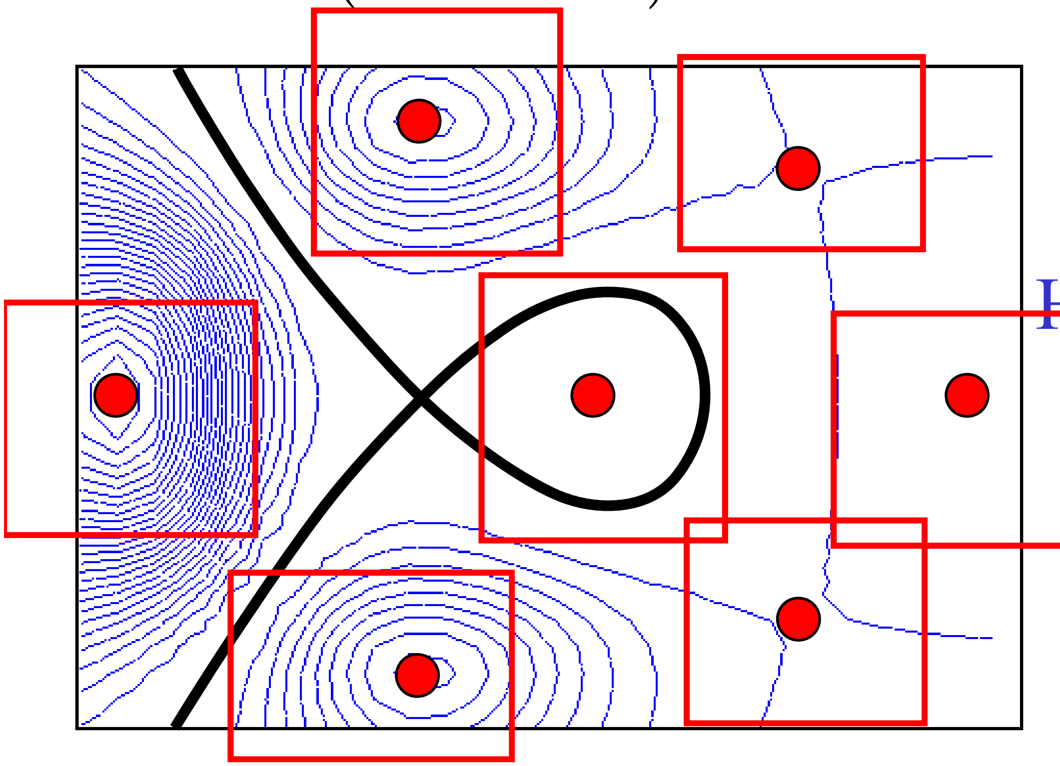

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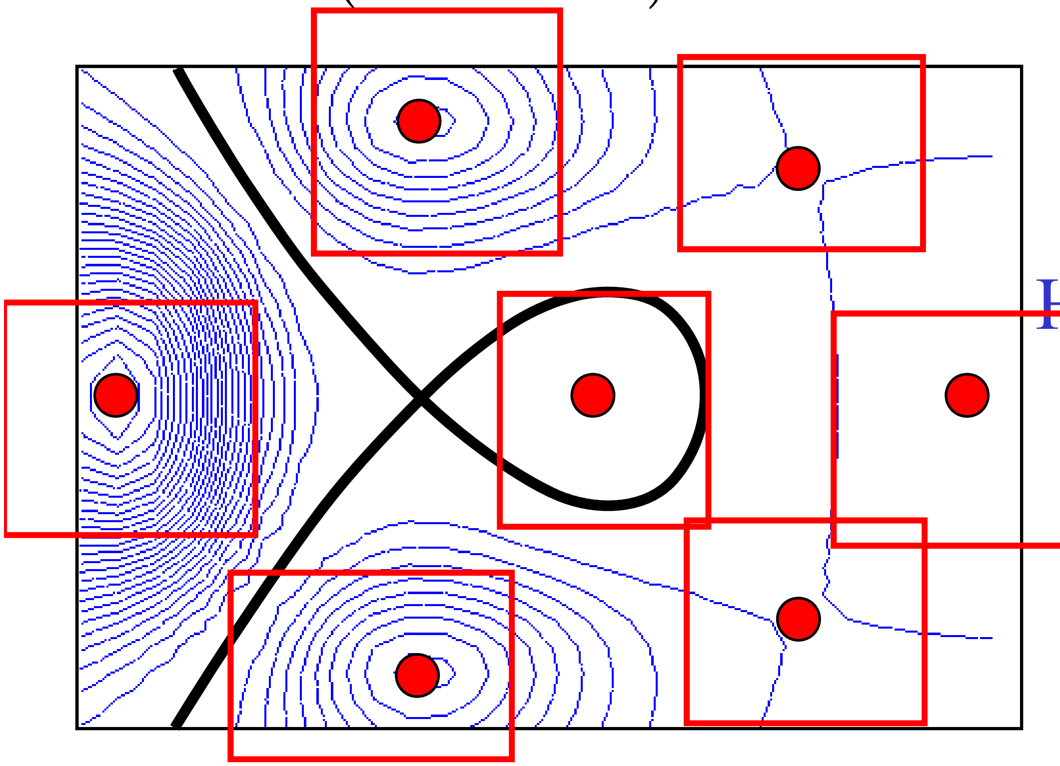

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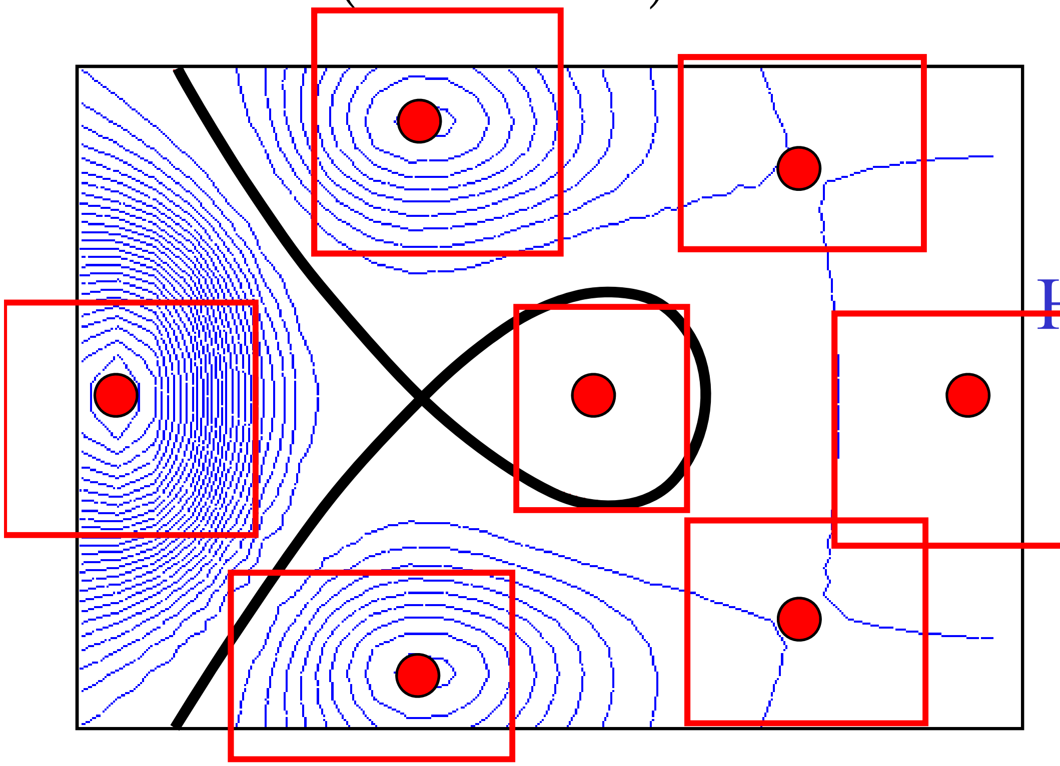

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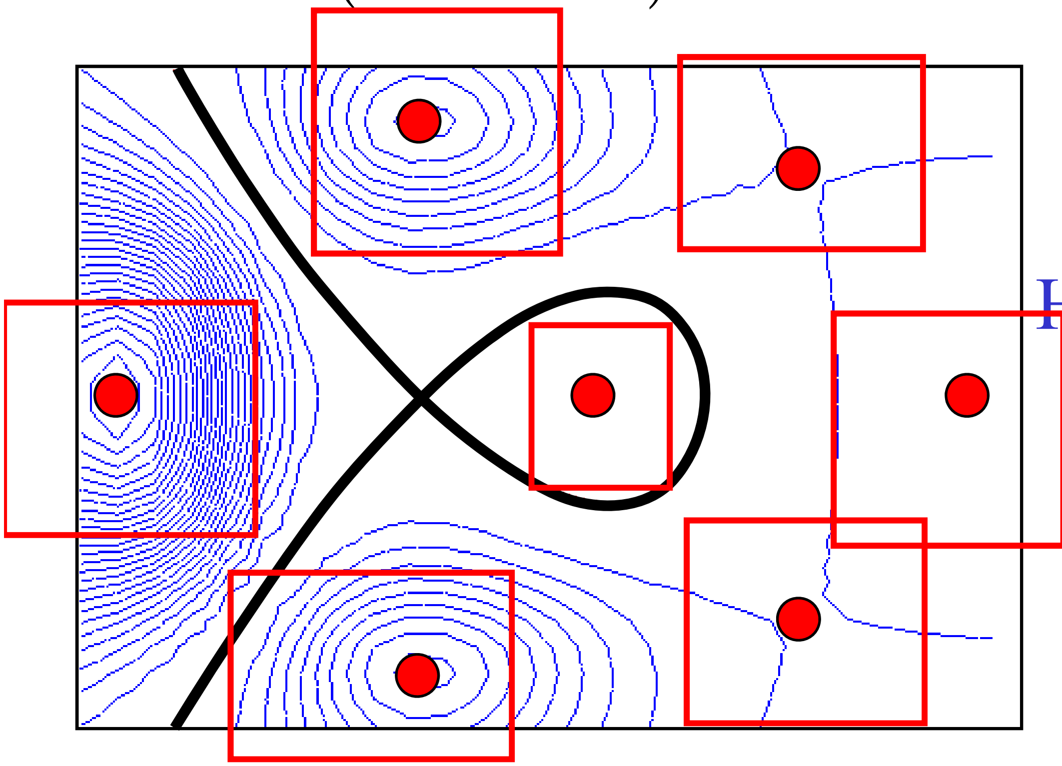

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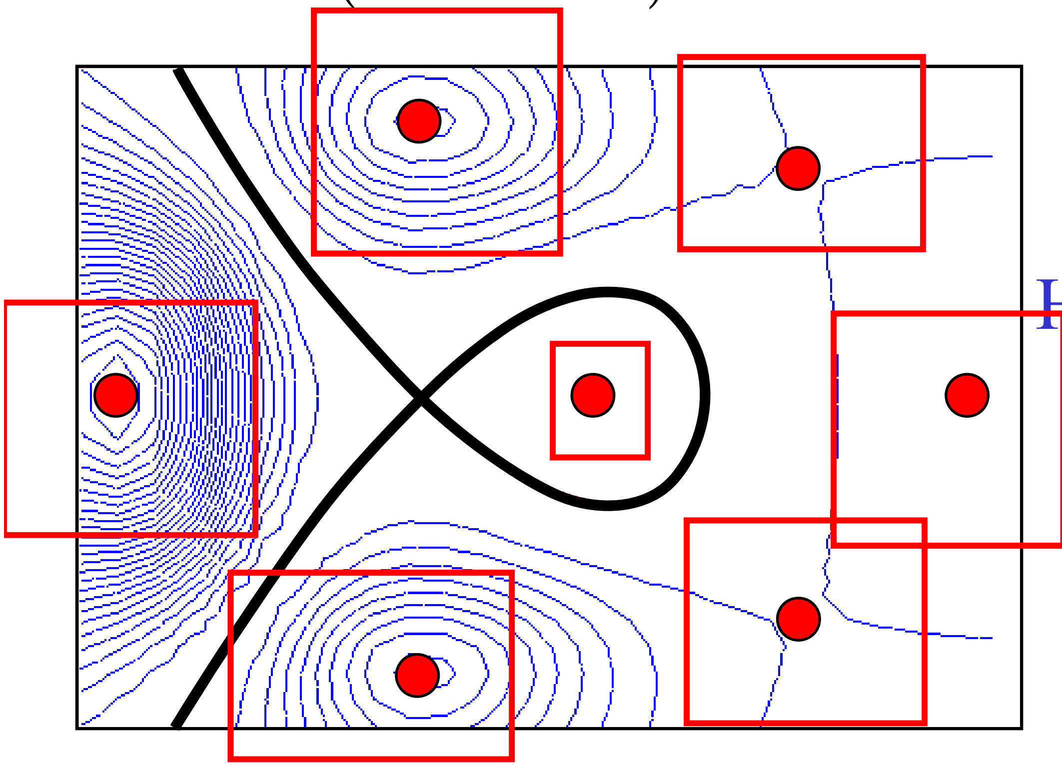

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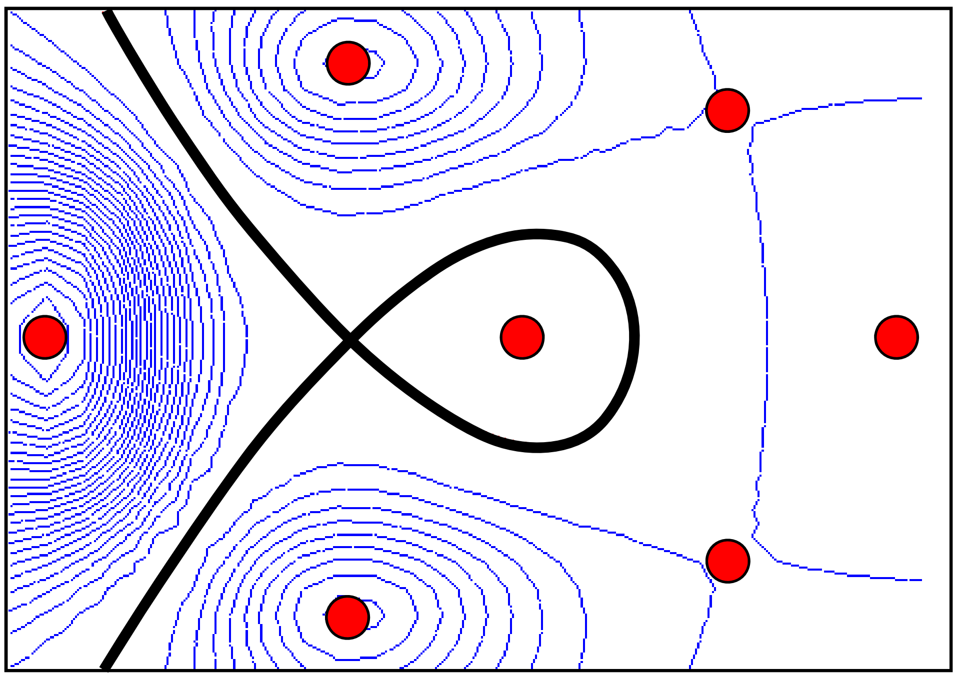

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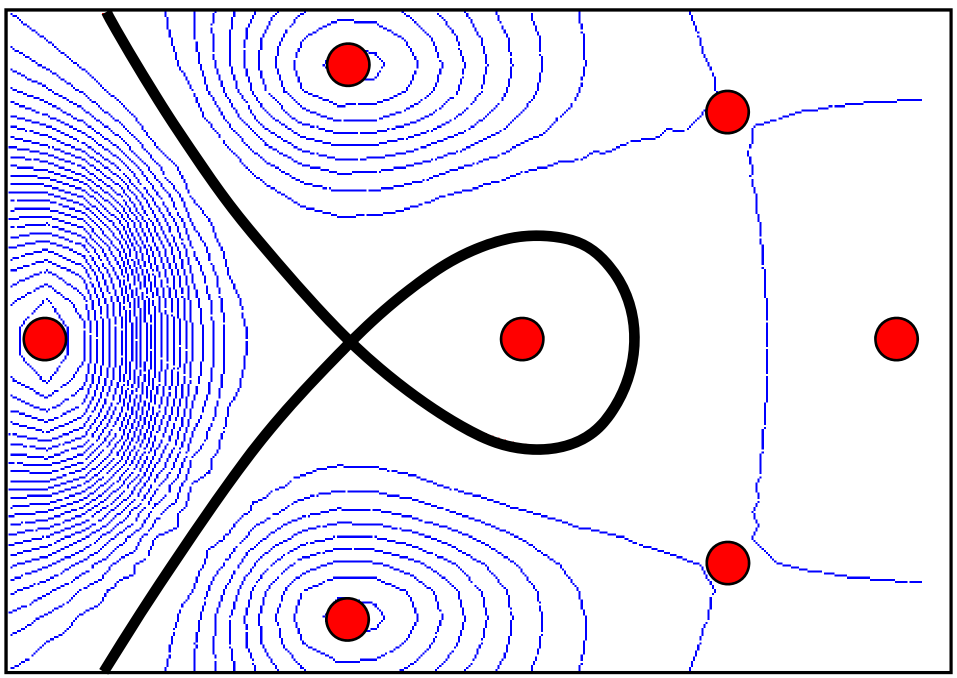

$$g_x = g_y = 0$$

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Hessian of g

$$H = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix}$$

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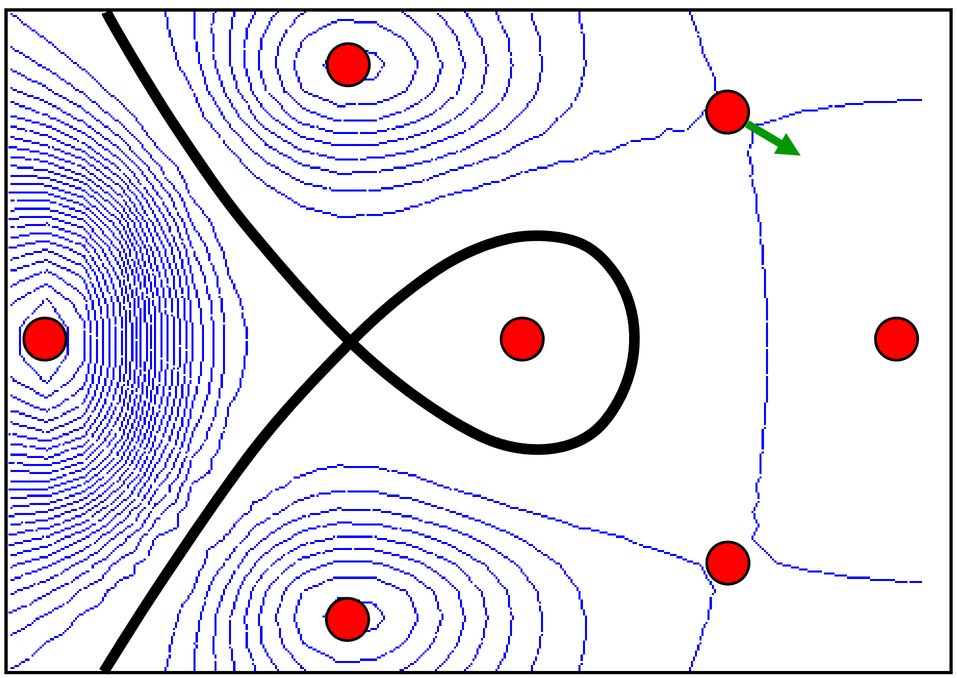

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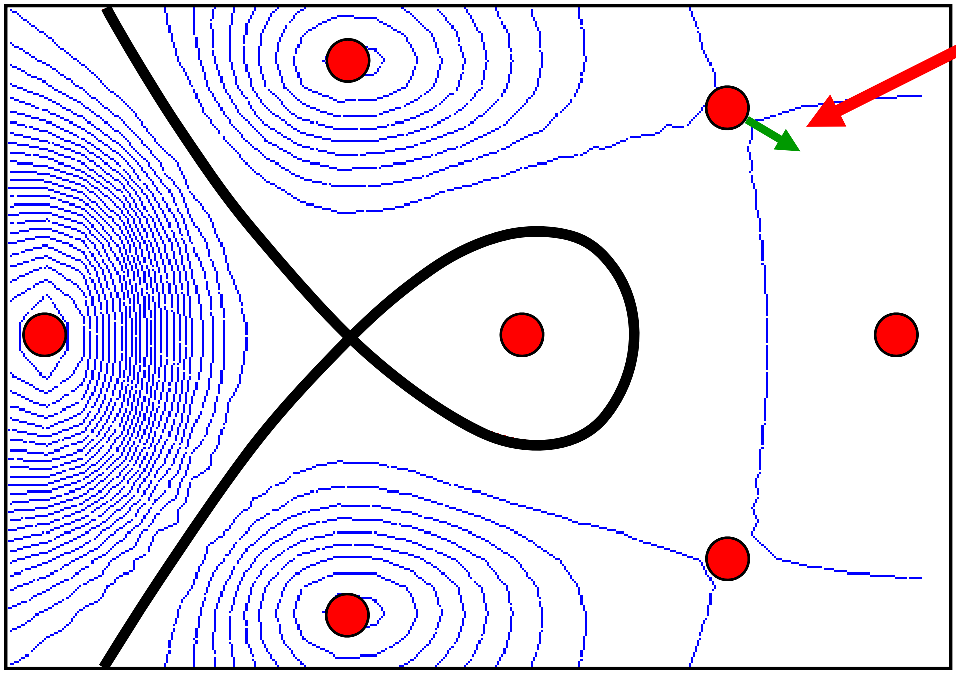

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eigenvector of H
with positive eigenvalue

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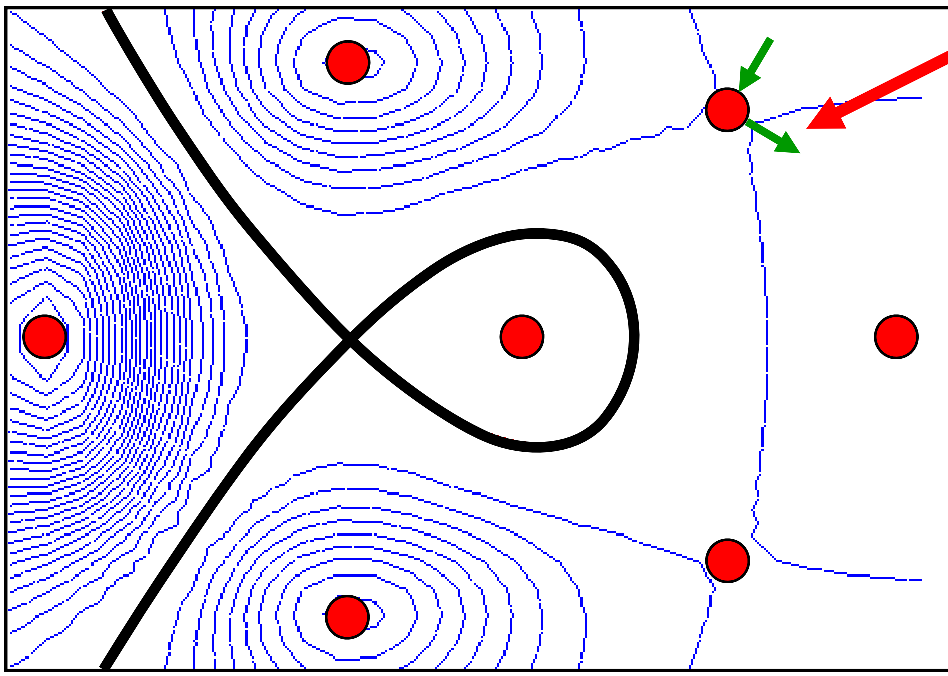

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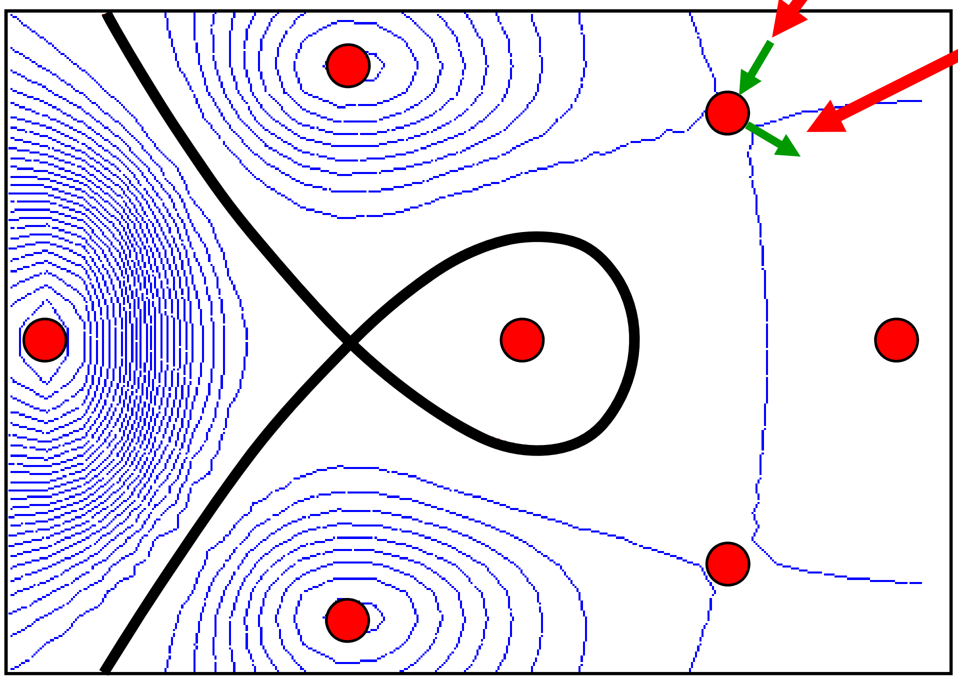

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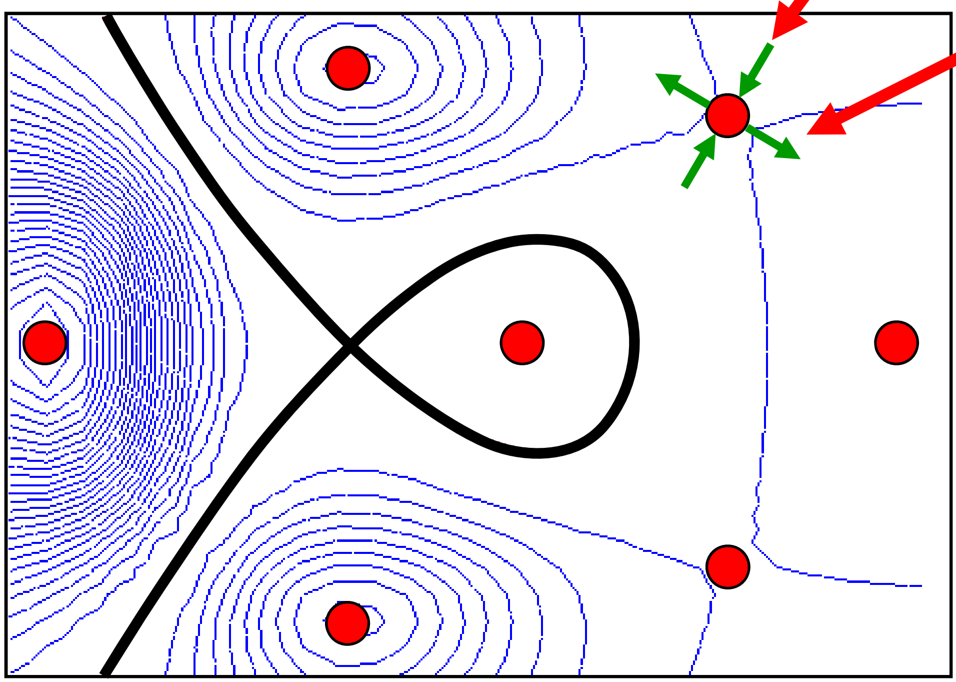

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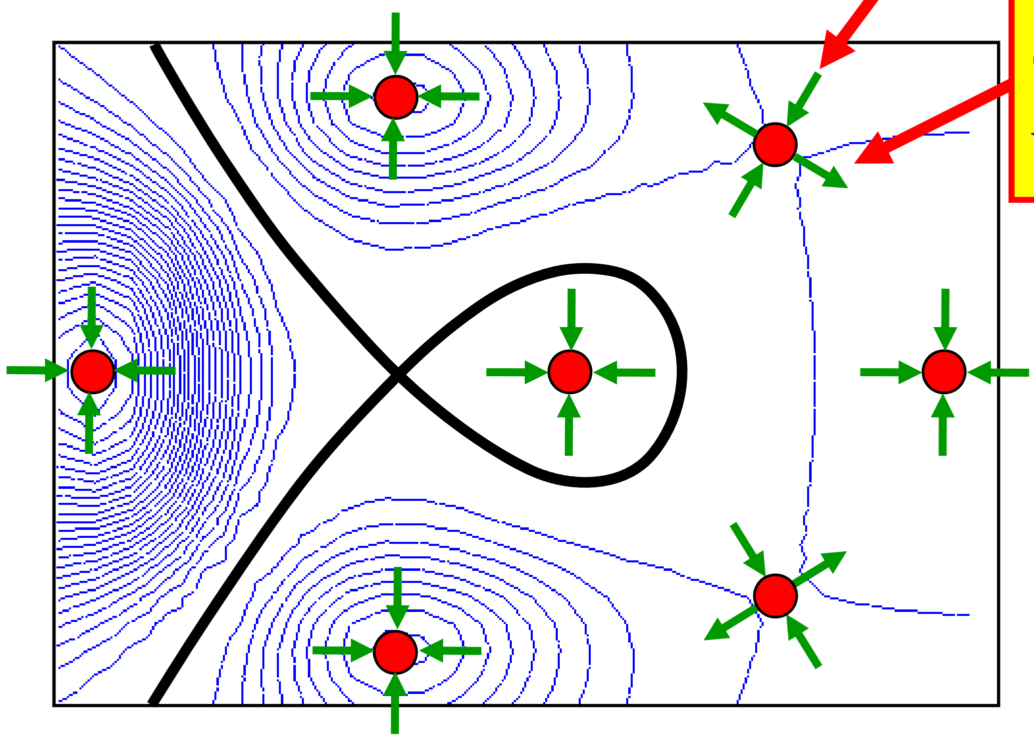

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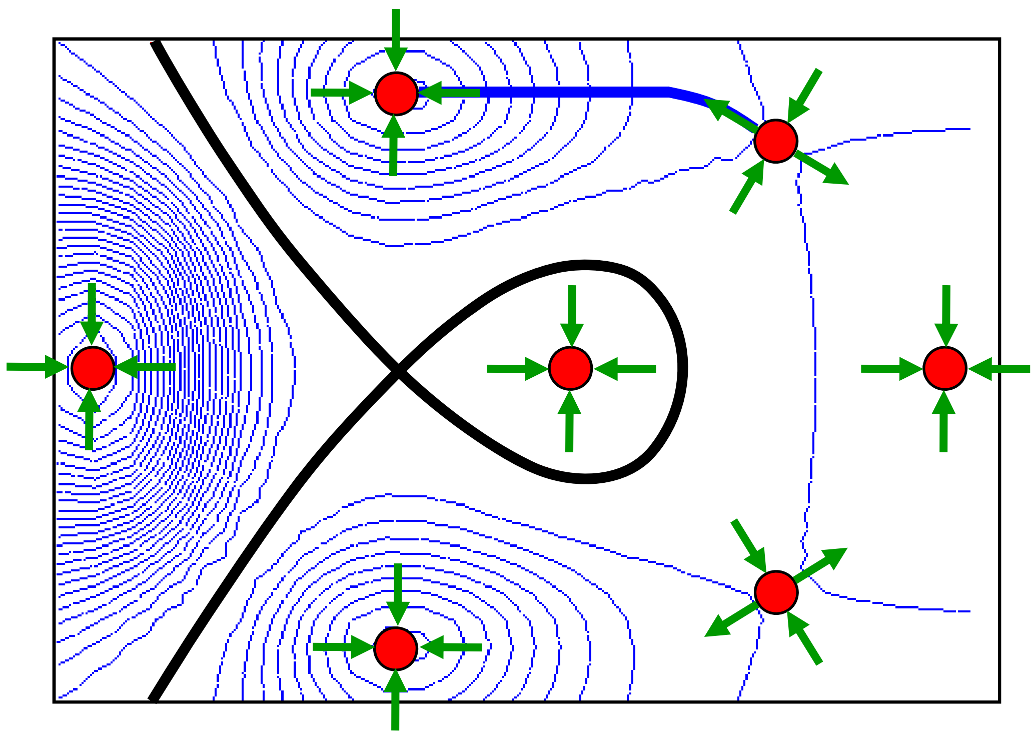

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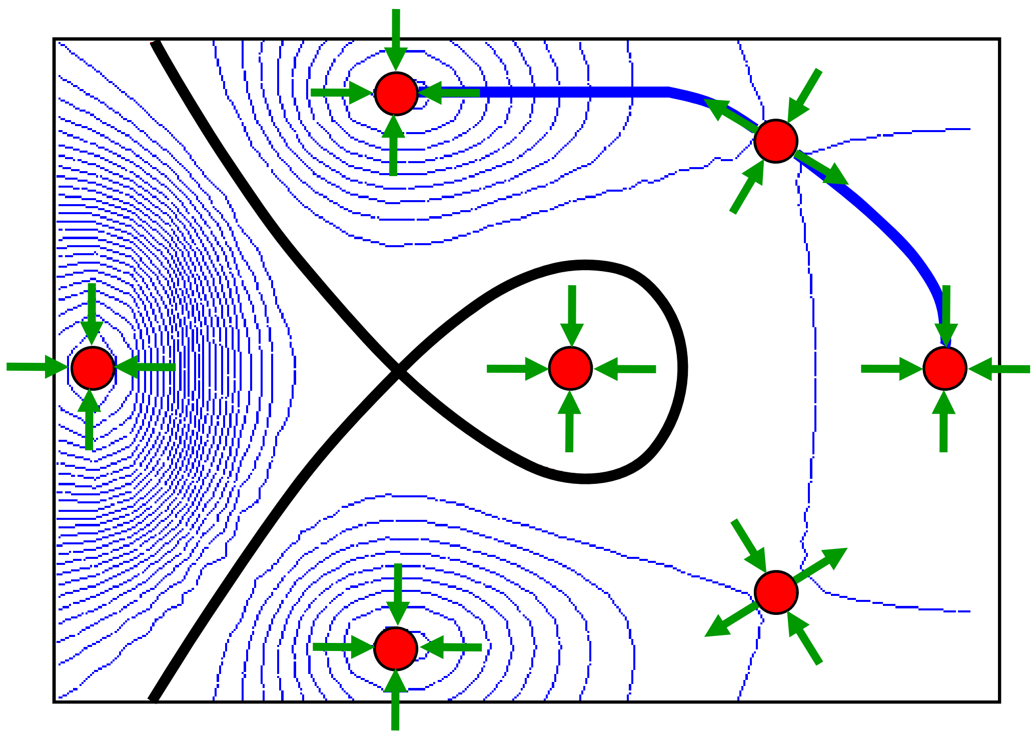

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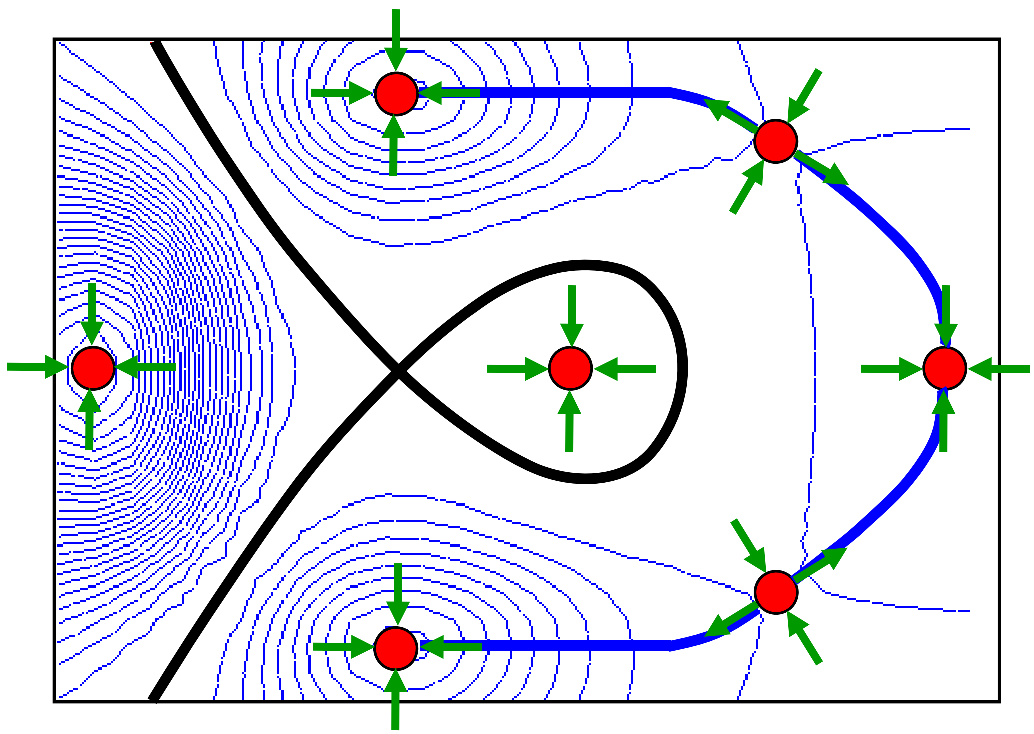

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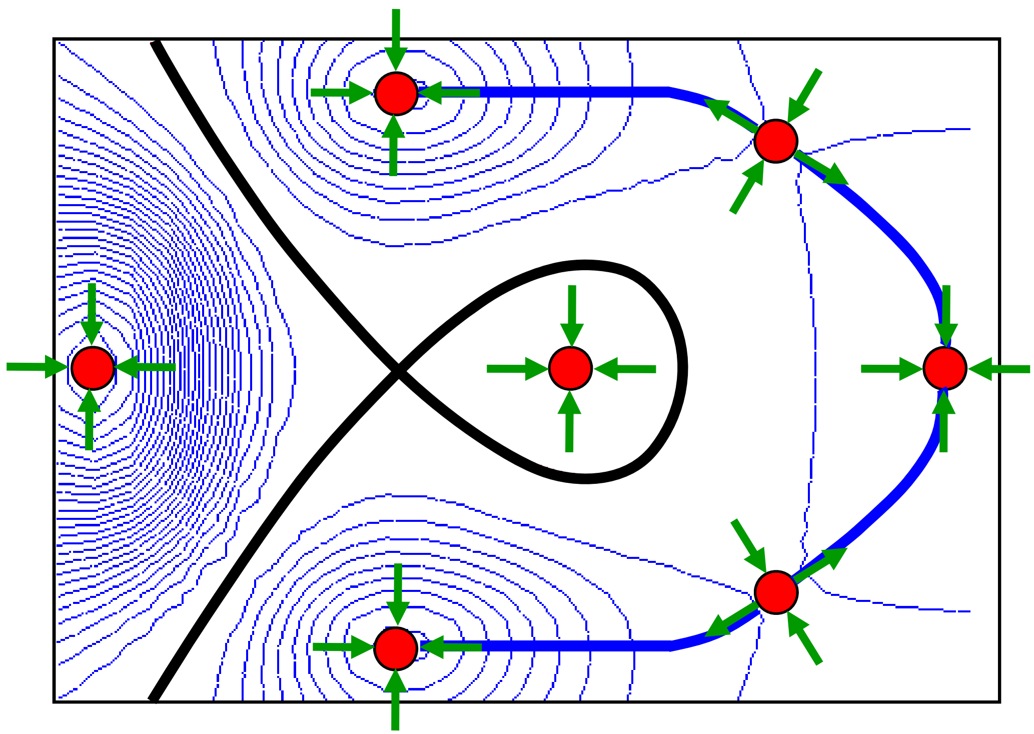

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Interval ODE solver

Idea

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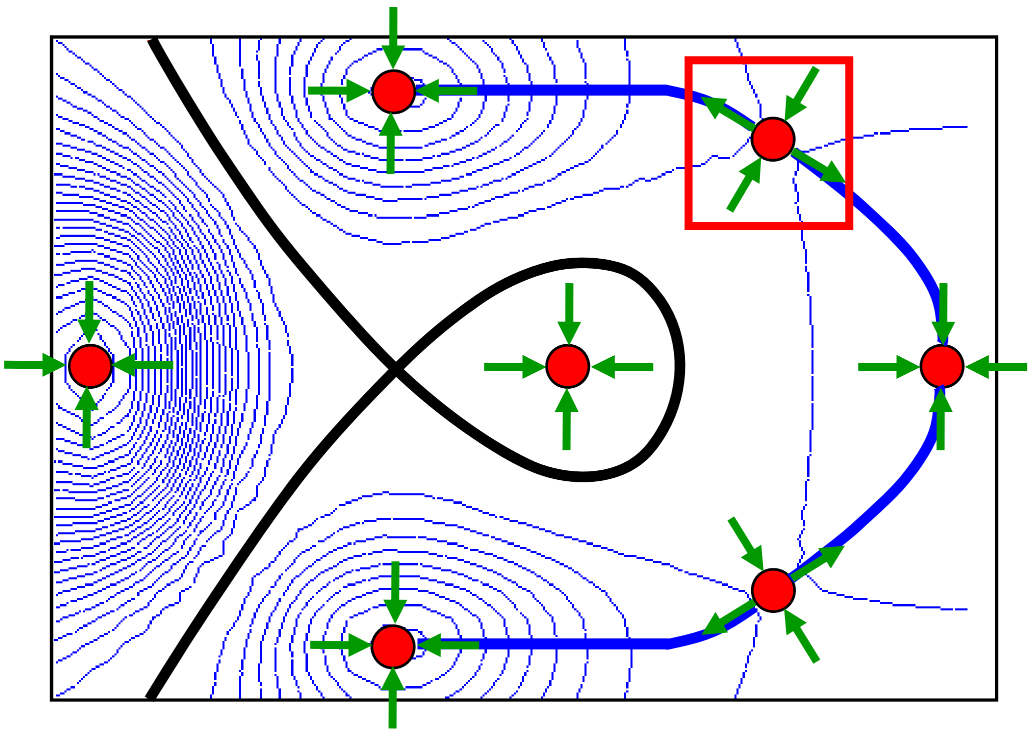

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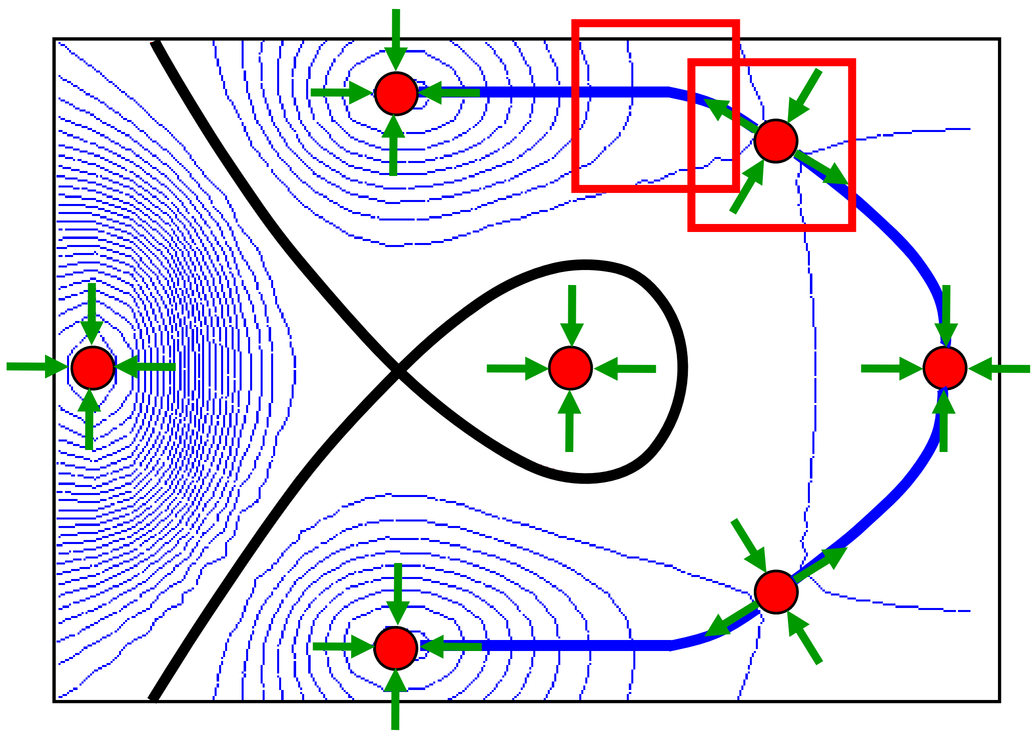

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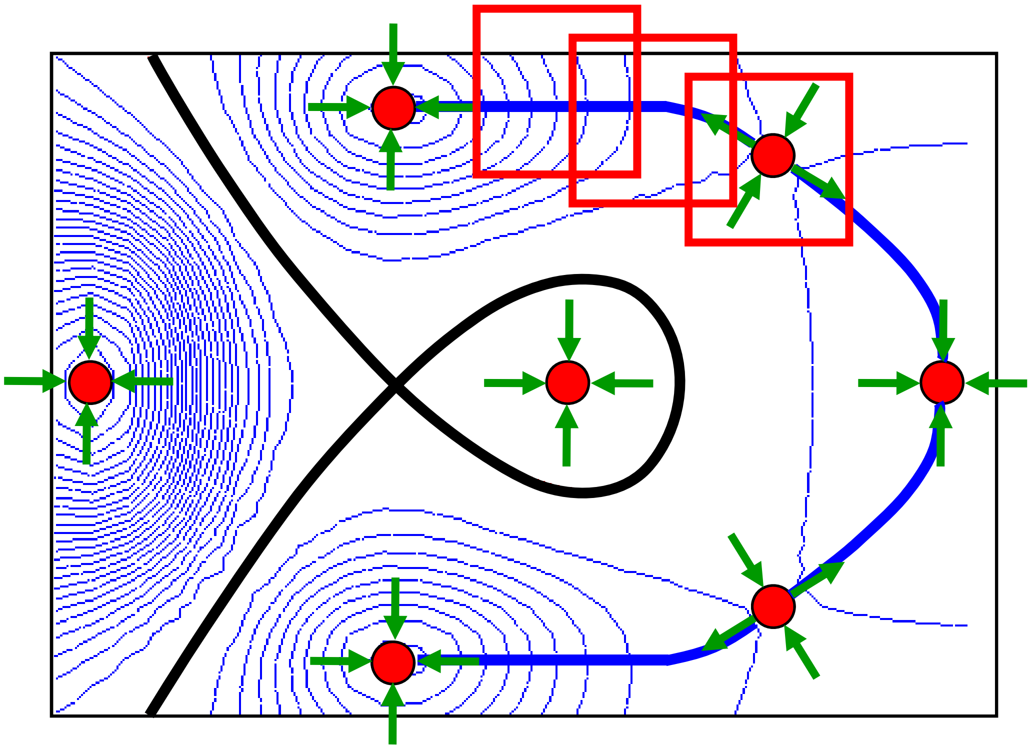

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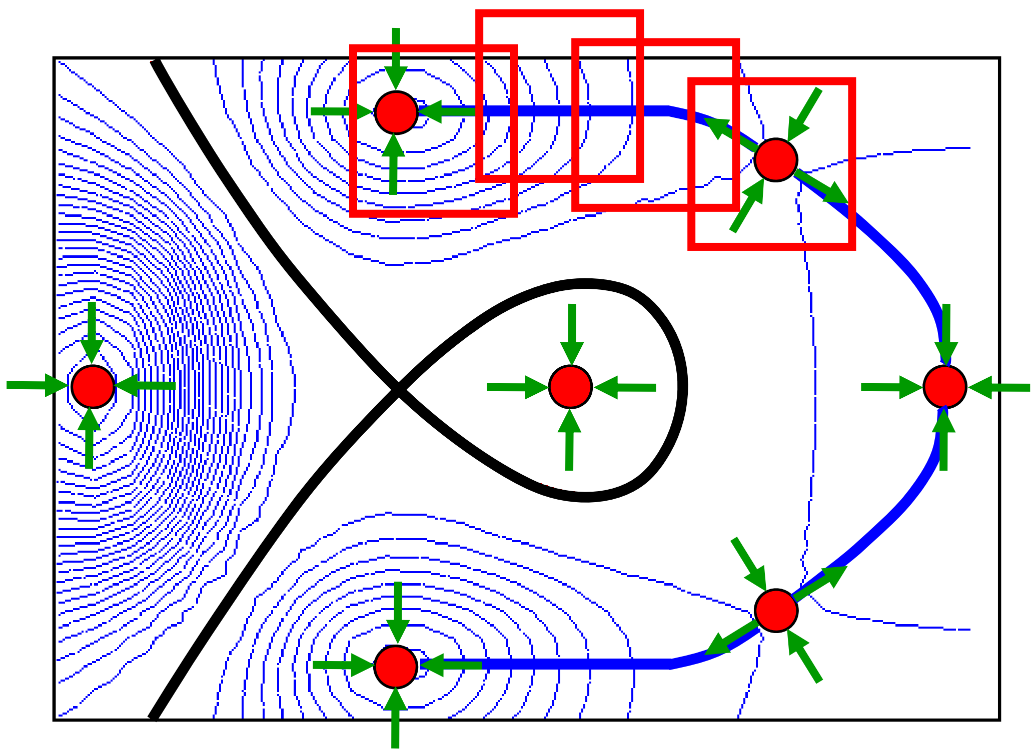

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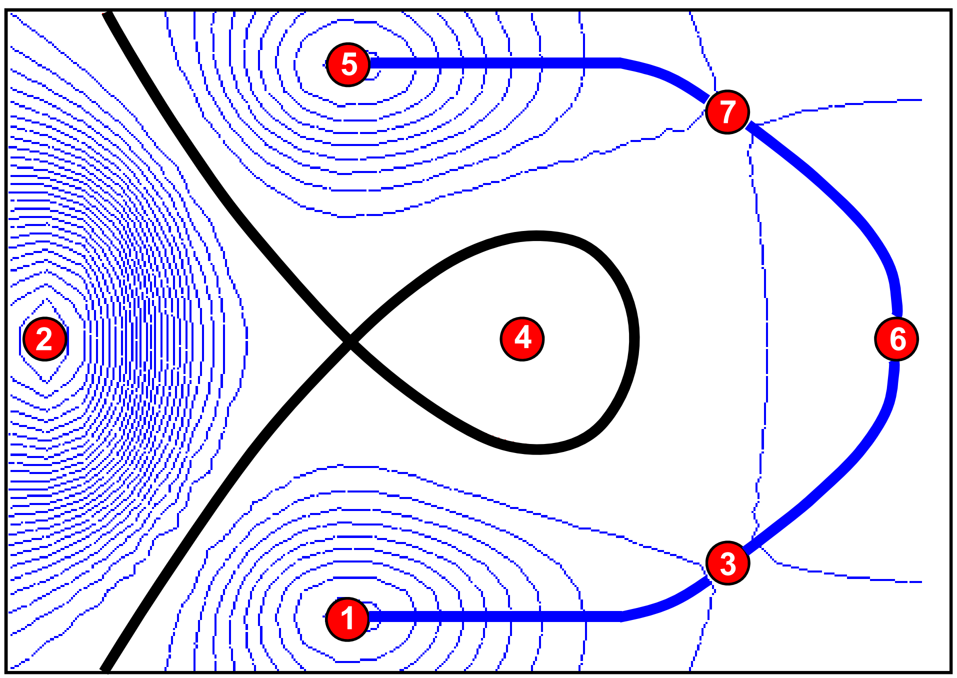

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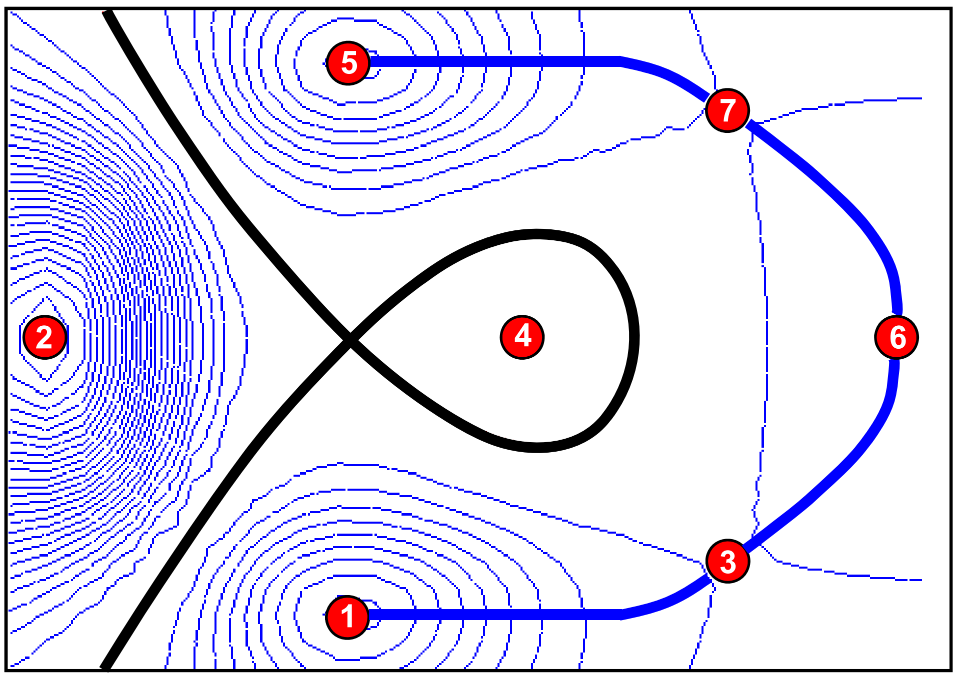
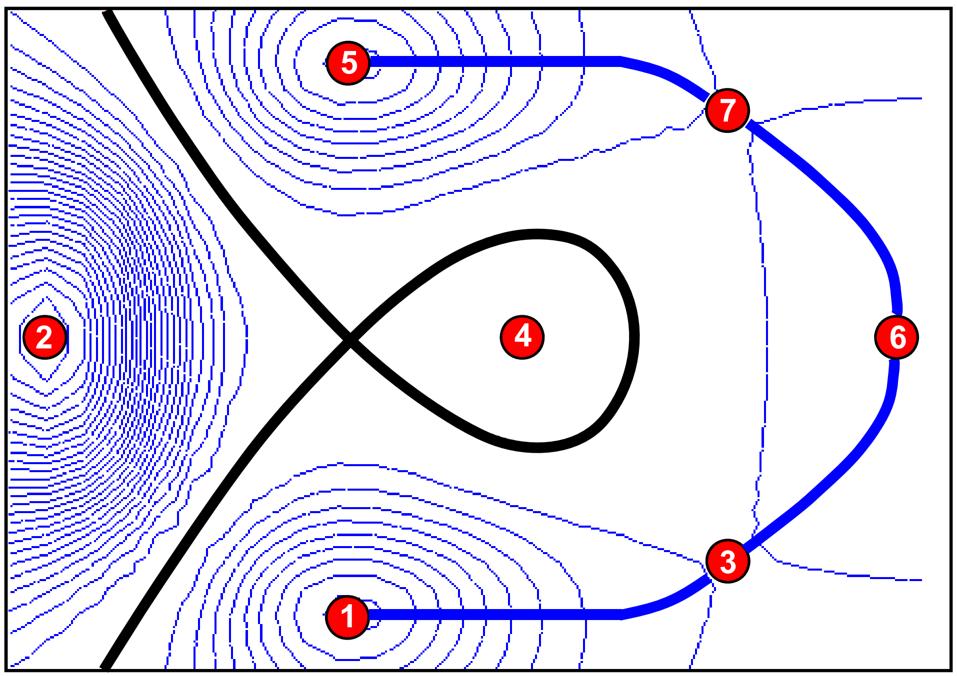


Adjacency

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

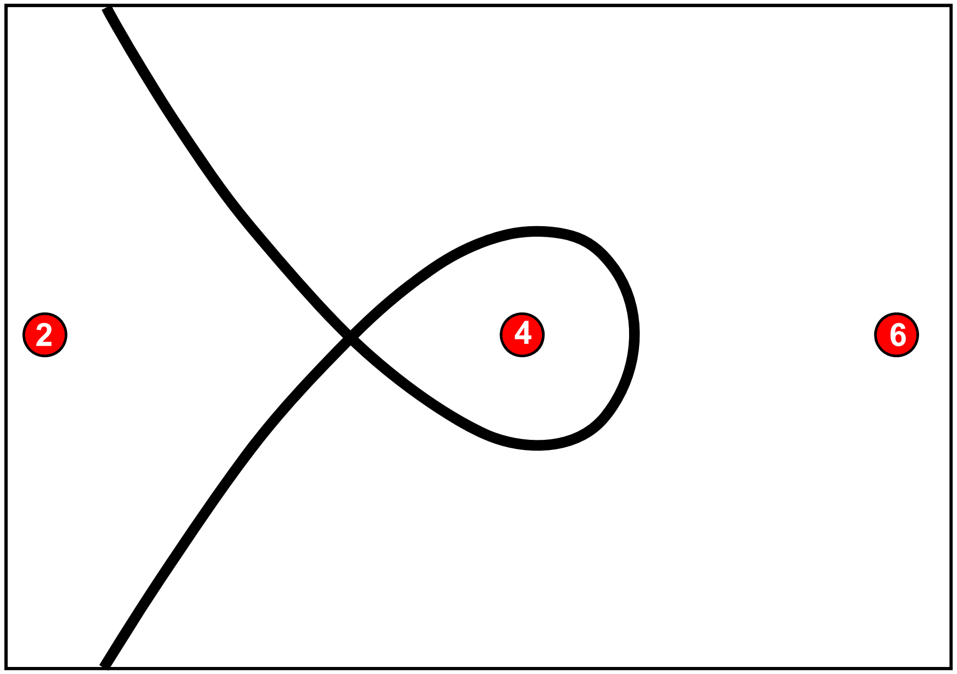
Idea



Closure

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Idea



Closure

$$
\begin{array}{c|c}
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\end{bmatrix}
\end{array}
$$

Error function

Problem

Input: $f(x)$

Output: $\exists x$ s.t. $f(x) > 0$

$$x = (x_1, \ldots, x_n)$$

$$f = (f_1, \ldots, f_m)$$

Ex. 1

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2$

Output :

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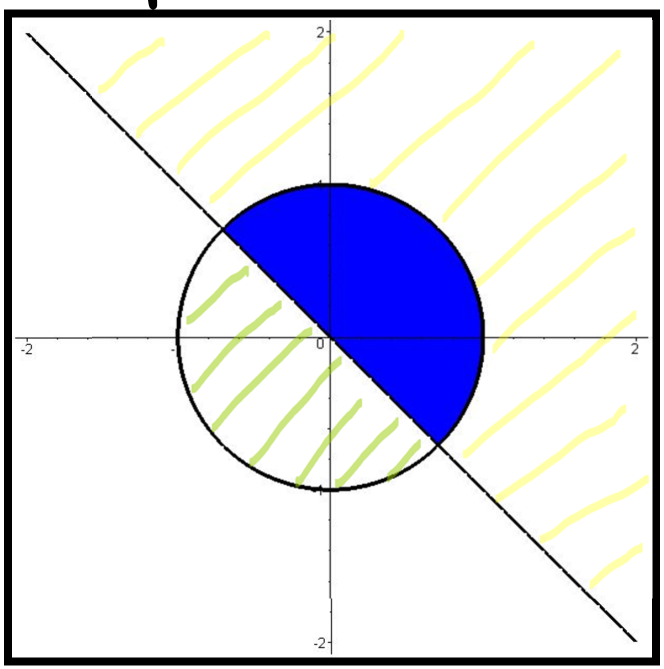
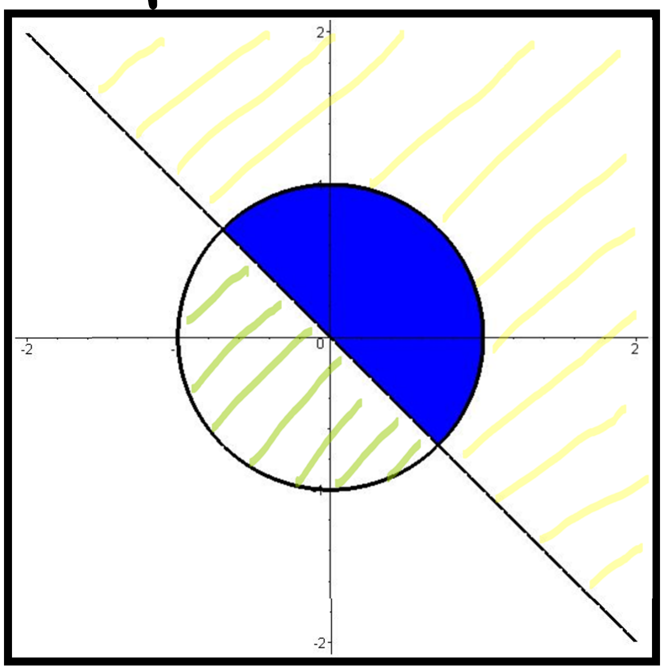


Ex. 1

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2$

Output : True

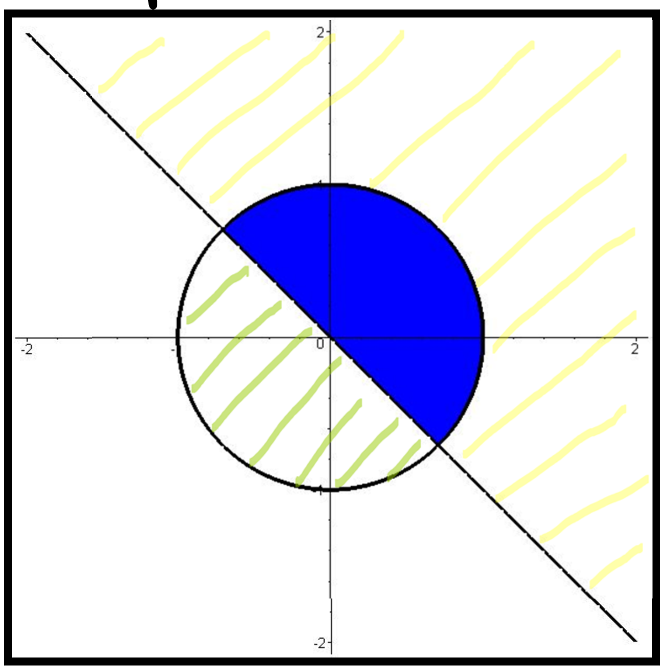


Ex. 1

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2$

Output: True



Ex. 2

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2 - 2$

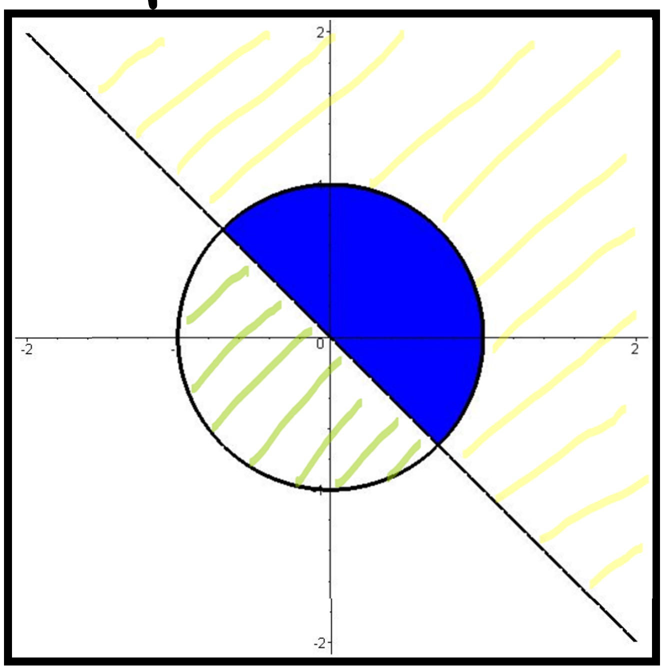
Output:

Ex. 1

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2$

Output: True

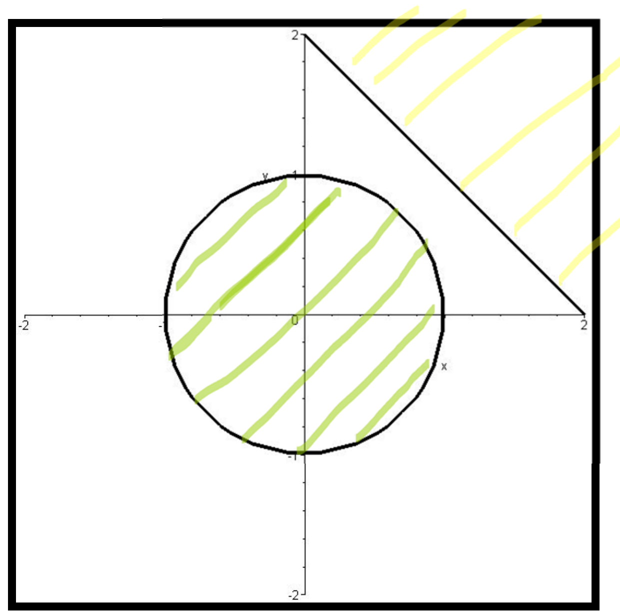


Ex. 2

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2 - 2$

Output:

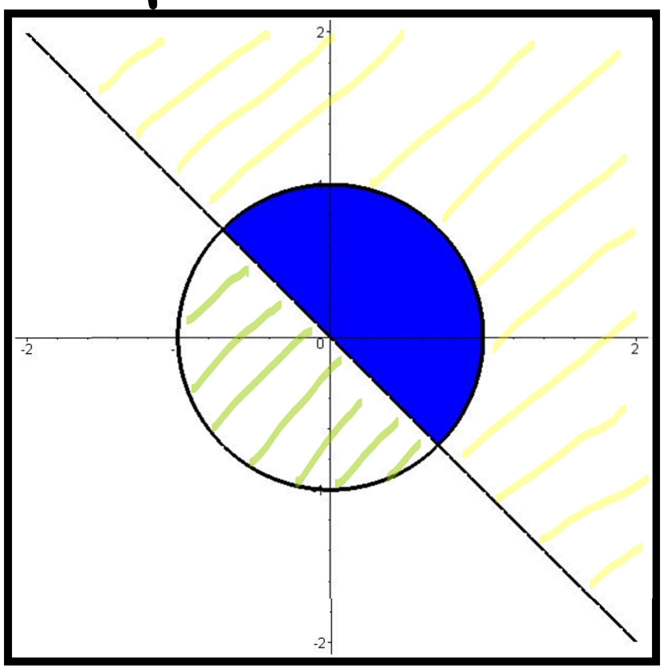


Ex. 1

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2$

Output: True

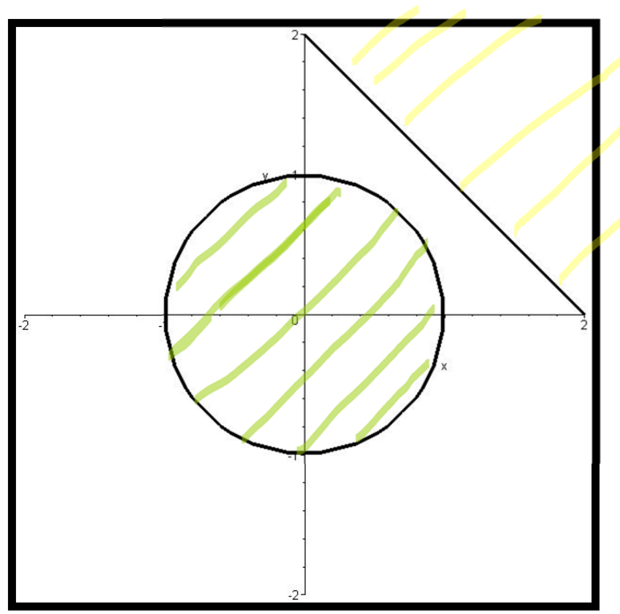


Ex. 2

Input: $-x_1^2 - x_2^2 + 1$

$x_1 + x_2 - 2$

Output: False



Ex. 3

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$

$x_1^2+x_2^2-3$

$-x_1-x_2$

Output :

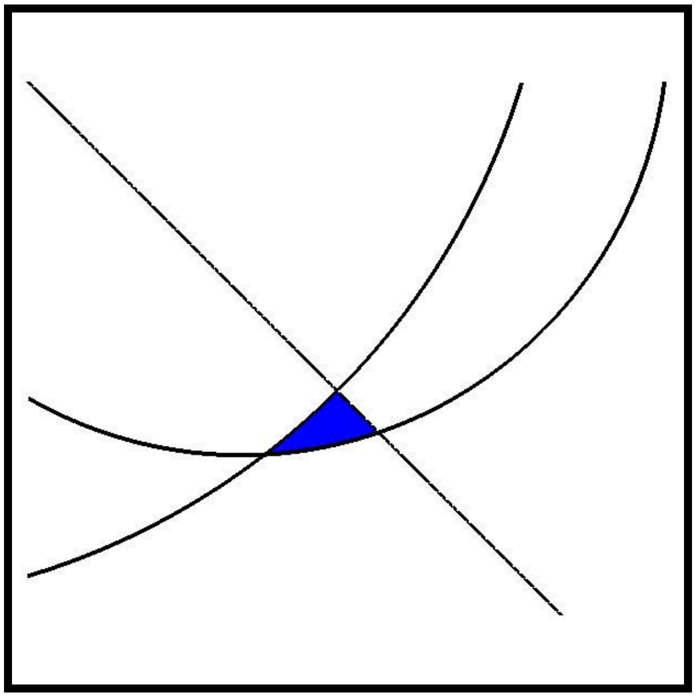
Ex. 3

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$

$x_1^2+x_2^2-3$

$-x_1-x_2$

Output :



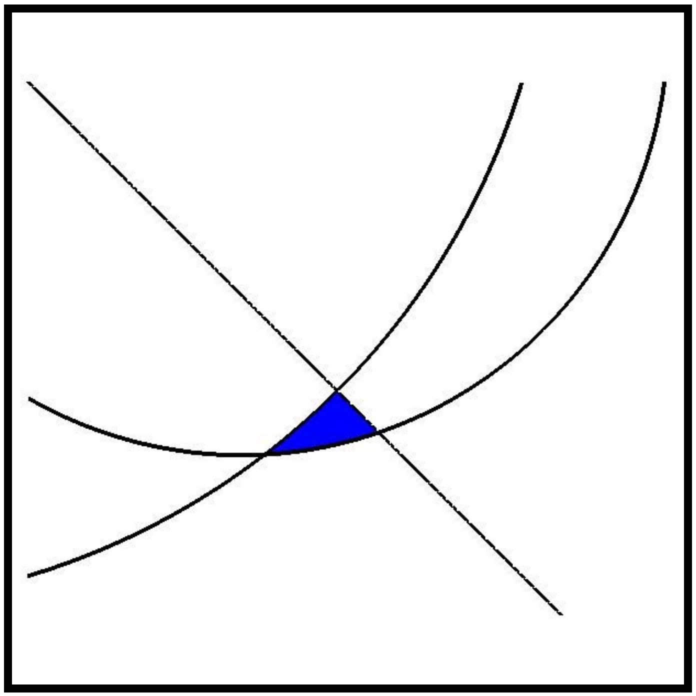
Ex. 3

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$

$x_1^2+x_2^2-3$

$-x_1-x_2$

Output : True



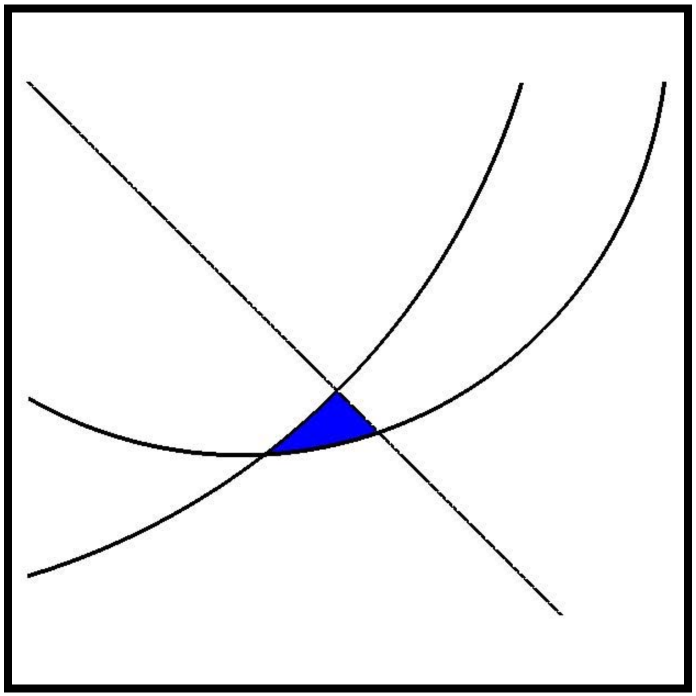
Ex. 3

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$

$x_1^2+x_2^2-3$

$-x_1-x_2$

Output: True



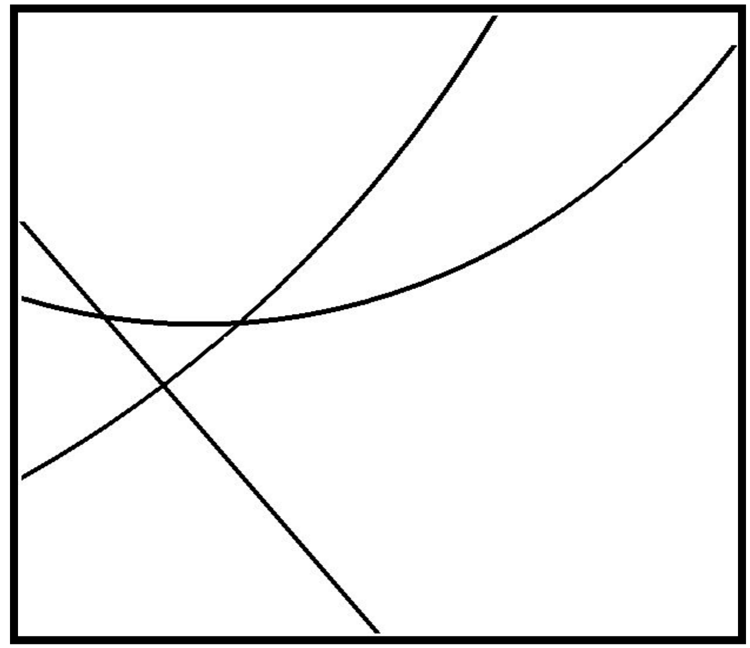
Ex. 4

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$

$x_1^2+x_2^2-3$

$-x_1-x_2-\frac{1}{2}$

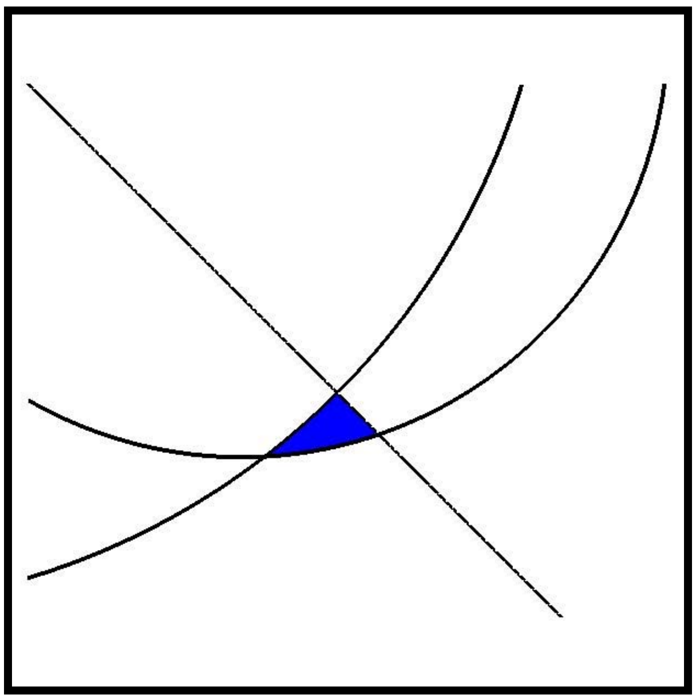
Output:



Ex. 3

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$
$x_1^2+x_2^2-3$
$-x_1-x_2$

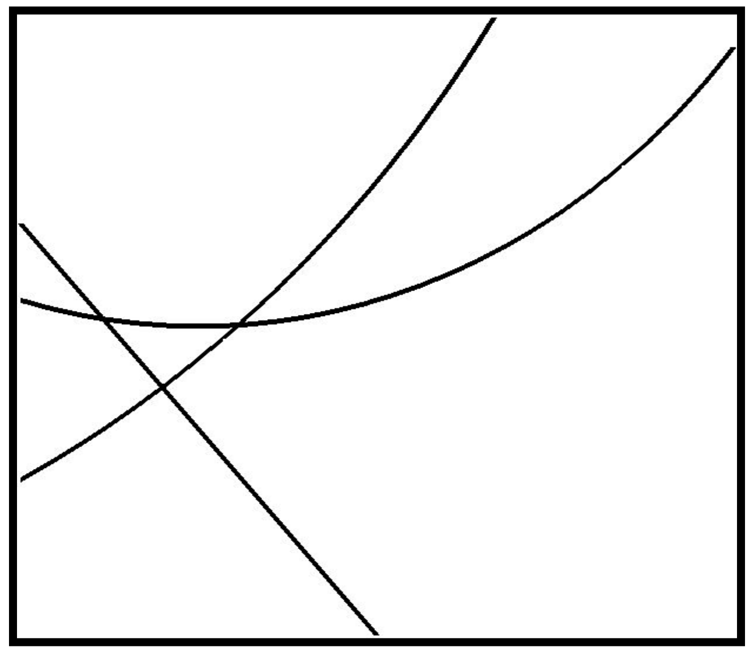
Output: True



Ex. 4

Input: $-(x_1-1)^2-(x_2-\frac{3}{8})^2+1$
$x_1^2+x_2^2-3$
$-x_1-x_2-\frac{1}{2}$

Output: False



Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3
x3^2 + 2 x 1 x3^4 x4 - 2
x1 x3 x4^4 - 8

4 x1 x4 + 3 x2 x3 x4 + 5
x1^2 x3 x4 + 7 x1 x2 x3^3
- 1

5 x1 x2 x3- 5 x1 x2 x3^2 +
7 x1^3 x3^2 - 5
```

\# vars = 4

\# polys = 3

degree = 6

Output:

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3
x3^2 + 2 x 1 x3^4 x4 - 2
x1 x3 x4^4 - 8

4 x1 x4 + 3 x2 x3 x4 + 5
x1^2 x3 x4 + 7 x1 x2 x3^3
- 1

5 x1 x2 x3- 5 x1 x2 x3^2 +
7 x1^3 x3^2 - 5
```

# vars = 4
# polys = 3
degree = 6

Output: True

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3
x3^2 + 2 x 1 x3^4 x4 - 2
x1 x3 x4^4 - 8

4 x1 x4 + 3 x2 x3 x4 + 5
x1^2 x3 x4 + 7 x1 x2 x3^3
- 1

5 x1 x2 x3- 5 x1 x2 x3^2 +
7 x1^3 x3^2 - 5
```

vars = 4

polys = 3

degree = 6

Output: True

Ex. 6

Input:

```
5 x3^4 + x1^2 x3^3 - 7 x1 x2
x3^2 x4 - 5 x1^3 x3^2 x4 + 4
x1 x2^2 x4^3 - 3

 -6 x2 x3 - 2 x1^3 x2^2-8 x2^4
x3 + x2^2 x3^3 + x1 x2^3 x3^2
- 4

x3^3 - x1^2 x2^2 -7 x1^3 x4^2
+ 8 x2 x3^4 + 7 x1^2 x2^2 x4^2
- 9
```

vars = 4

polys = 3

degree = 6

Output:

Ex. 5

Input:

```
-x1^2 x3^2 x4 5 x1 x2^3
x3^2 + 2 x 1 x3^4 x4 - 2
x1 x3 x4^4 - 8

4 x1 x4 + 3 x2 x3 x4 + 5
x1^2 x3 x4 + 7 x1 x2 x3^3
- 1

5 x1 x2 x3- 5 x1 x2 x3^2 +
7 x1^3 x3^2 - 5
```

\# vars = 4

\# polys = 3

degree = 6

Output: True

Ex. 6

Input:

```
5 x3^4 + x1^2 x3^3 - 7 x1 x2
x3^2 x4 - 5 x1^3 x3^2 x4 + 4
x1 x2^2 x4^3 - 3

 -6 x2 x3 - 2 x1^3 x2^2-8 x2^4
x3 + x2^2 x3^3 + x1 x2^3 x3^2
- 4

x3^3 - x1^2 x2^2 -7 x1^3 x4^2
+ 8 x2 x3^4 + 7 x1^2 x2^2 x4^2
- 9
```

\# vars = 4

\# polys = 3

degree = 6

Output: False

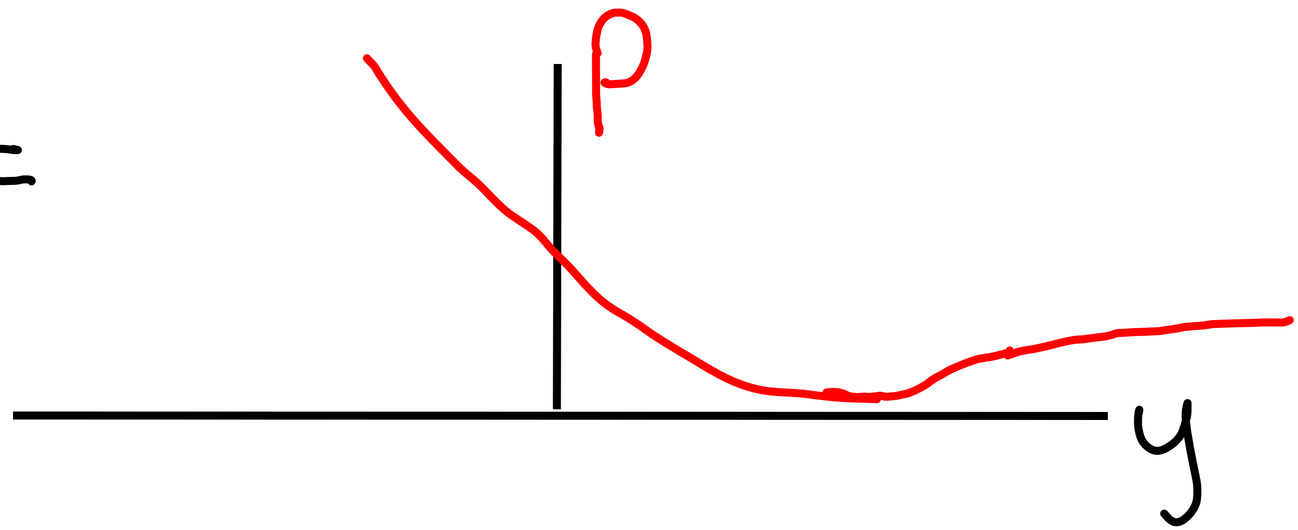
Error function

$$e(x) = p(f_1(x)) + \cdots + p(f_m(x))$$

Error function

$$e(x) = p(f_1(x)) + \cdots + p(f_m(x))$$

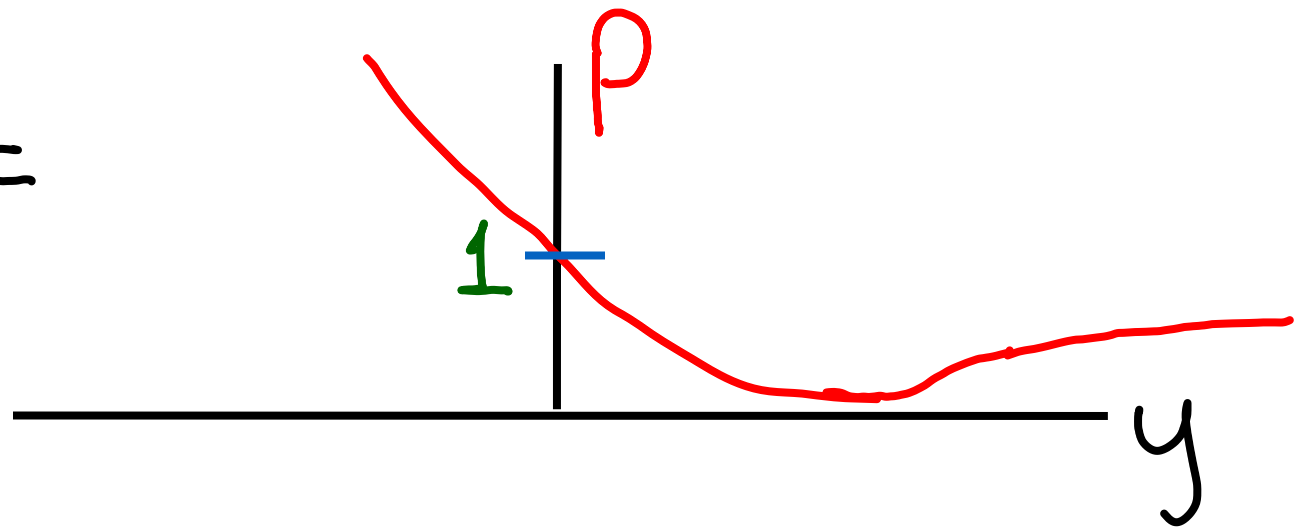
$p(y) =$



Error function

$$e(x) = p(f_1(x)) + \cdots + p(f_m(x))$$

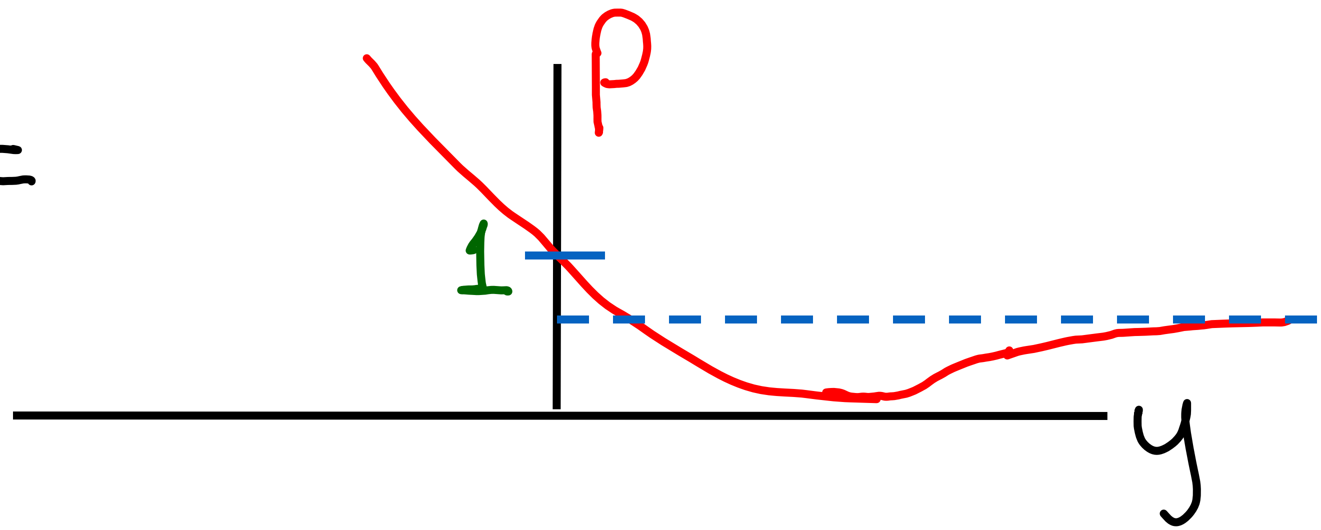
$p(y) =$



Error function

$$e(x) = p(f_1(x)) + \cdots + p(f_m(x))$$

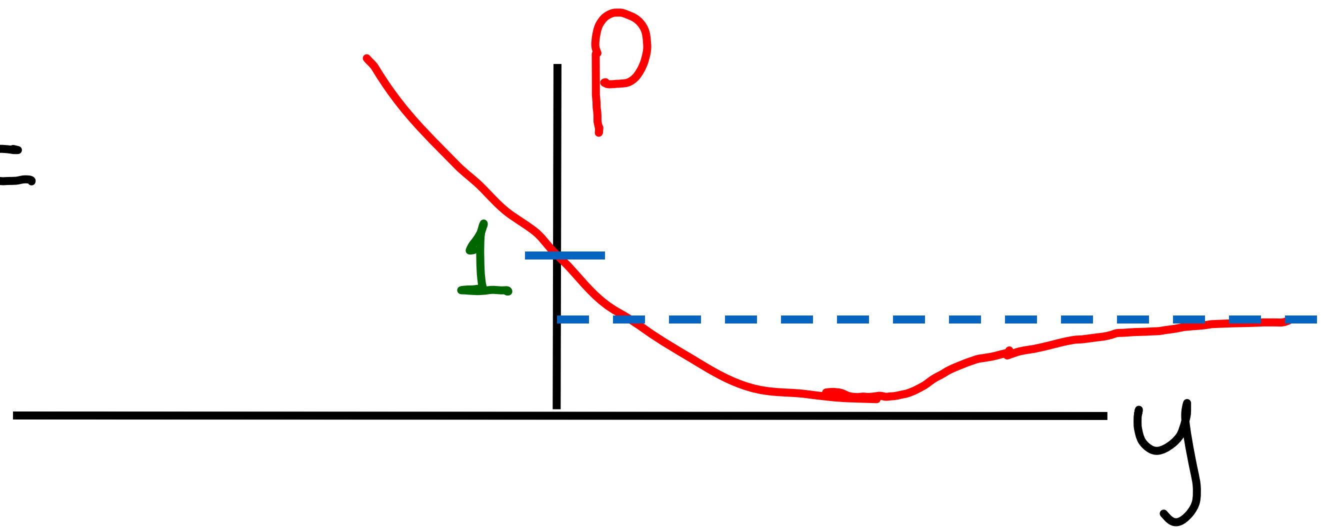
$p(y) =$



Error function

$$e(x) = p(f_1(x)) + \cdots + p(f_m(x))$$

$$p(y) =$$



Current Choice

$$p(y) = \frac{-y+\sqrt{y^2+4}}{2} - \left(\frac{-.05y+\sqrt{.05y^2+4}}{2m}\right) + \frac{1}{m}$$

Error function

$$f = \left(-x_1^2 - x_2^2 + 1,\ x_1 + x_2\right)$$

Error function

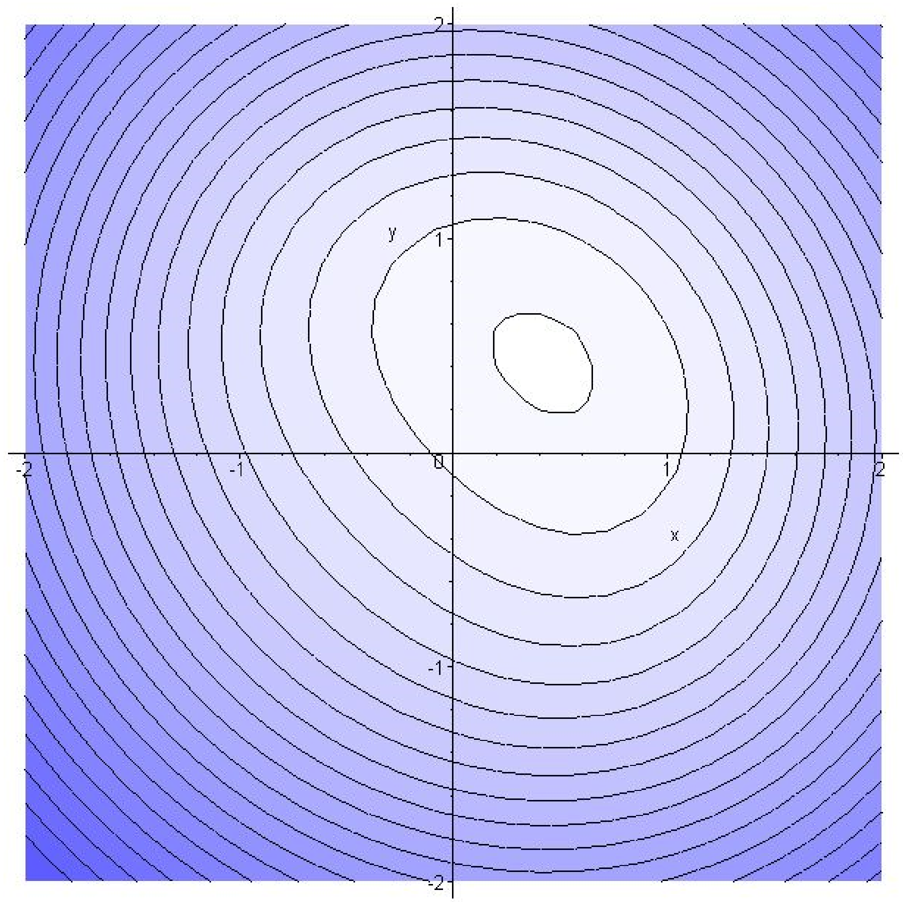
$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$

$$e(x) =$$

Error function

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$

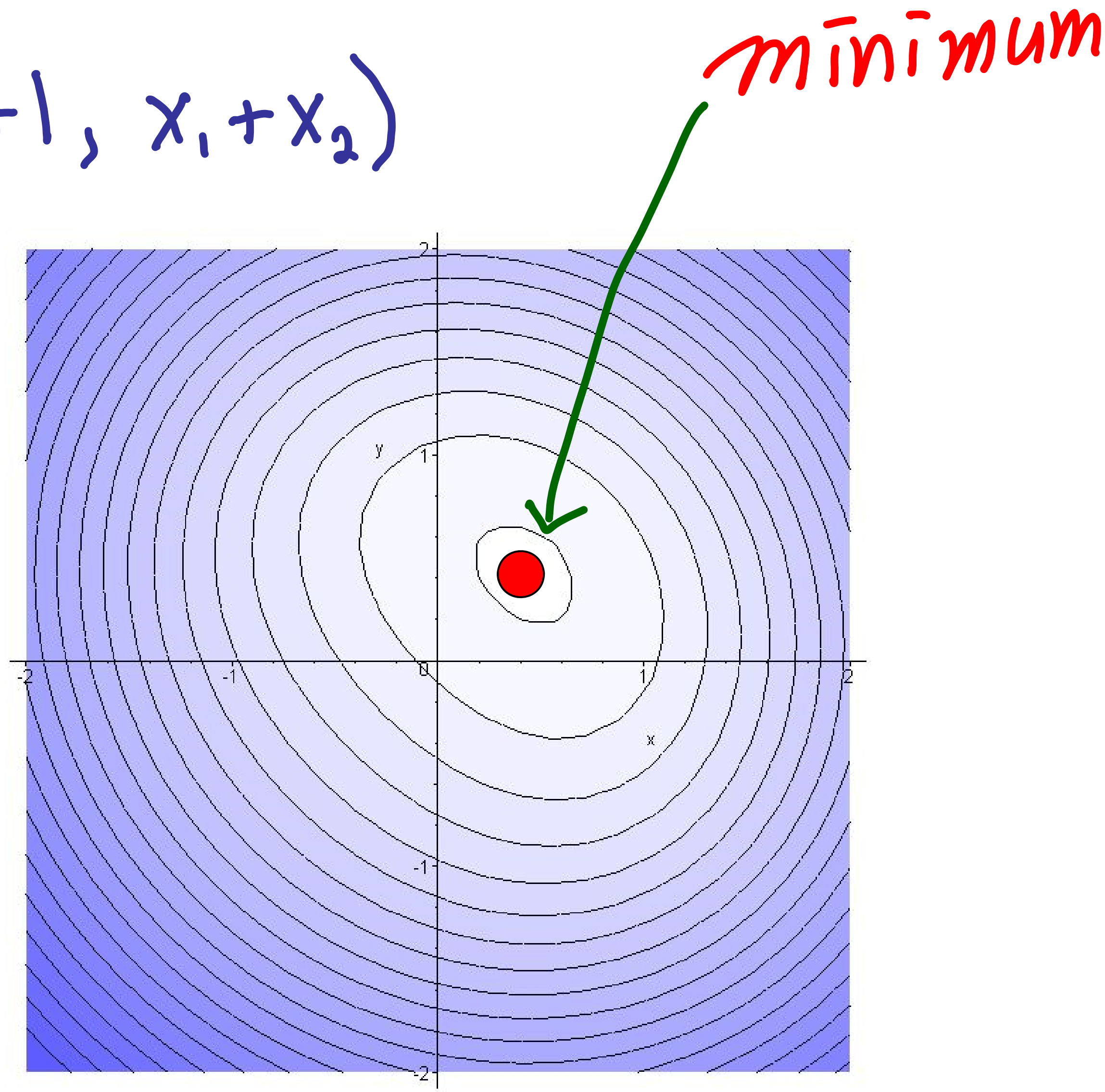
$$e(x) =$$



Error function

$$f = (-x_1^2 - x_2^2 + 1,\; x_1 + x_2)$$

$e(x) =$



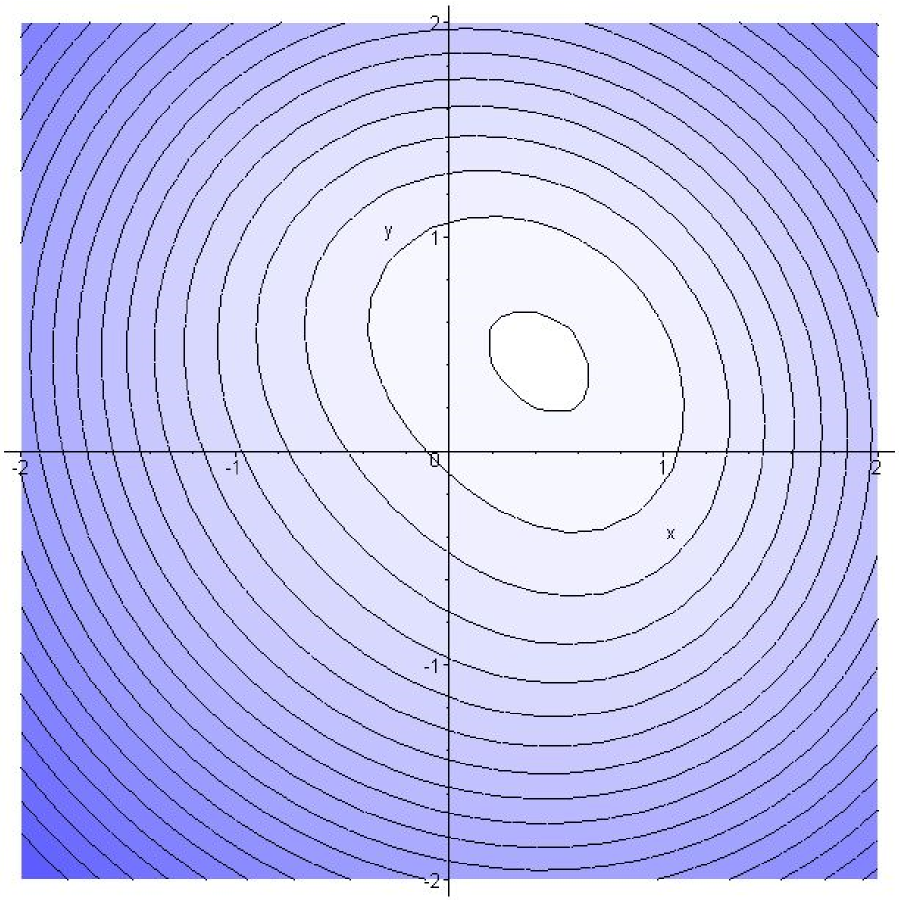
Sample and Test

Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$

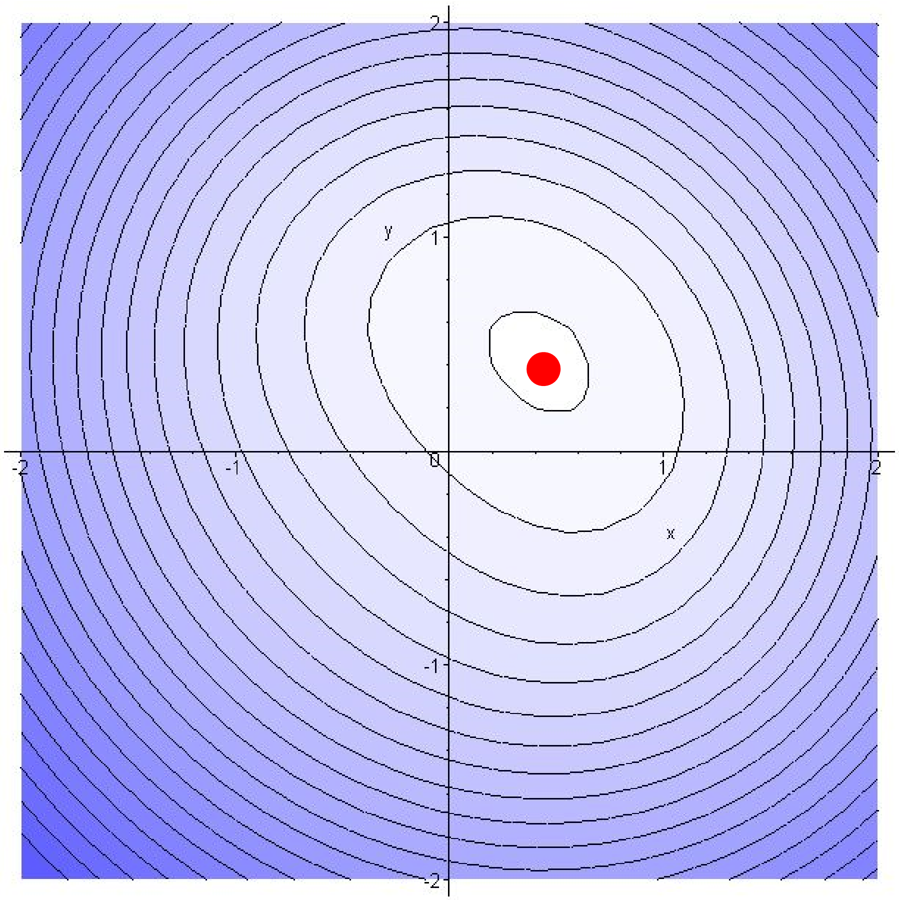
Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



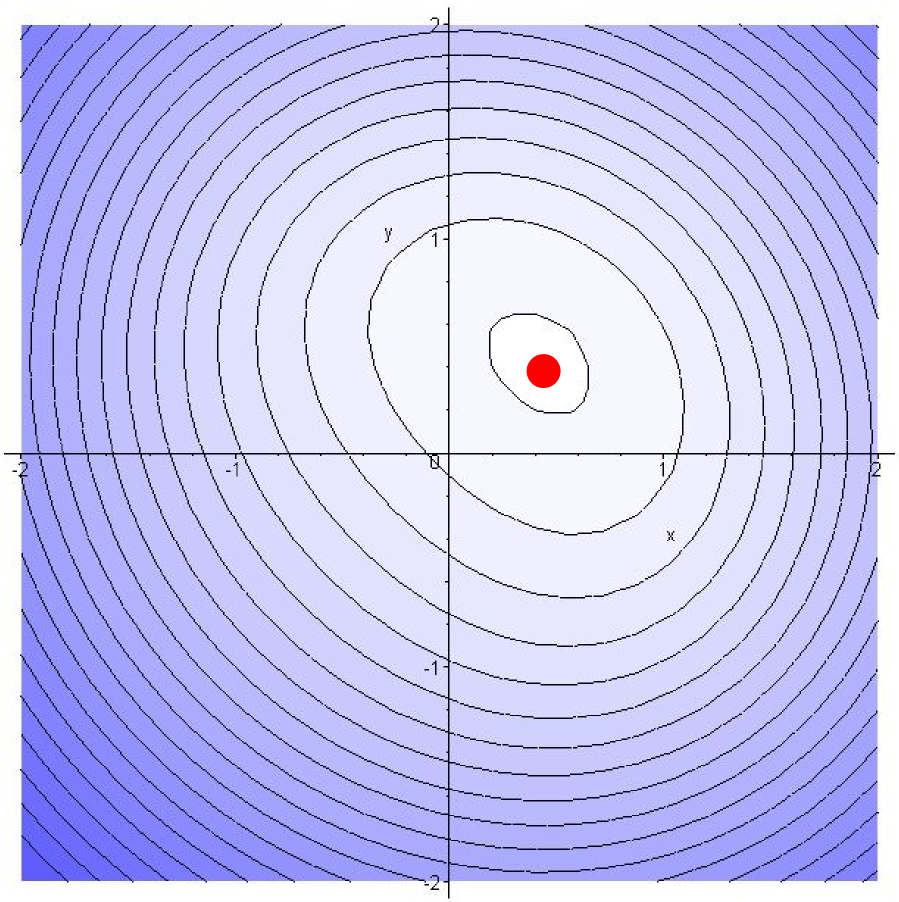
Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



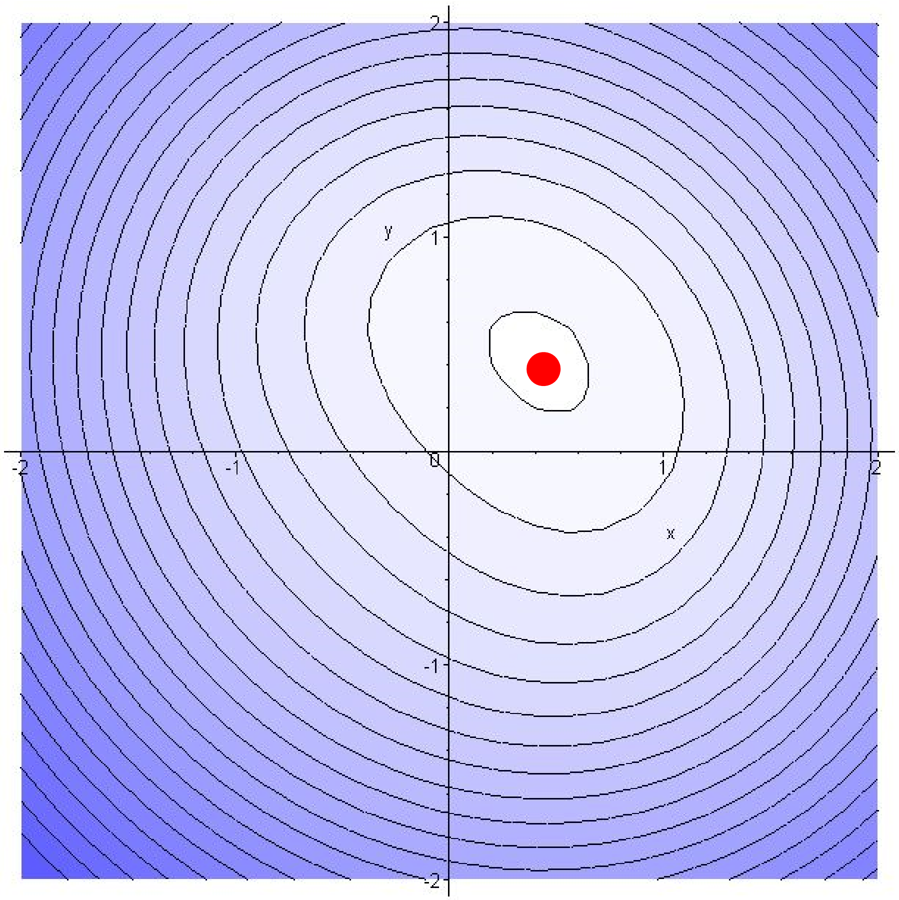
Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



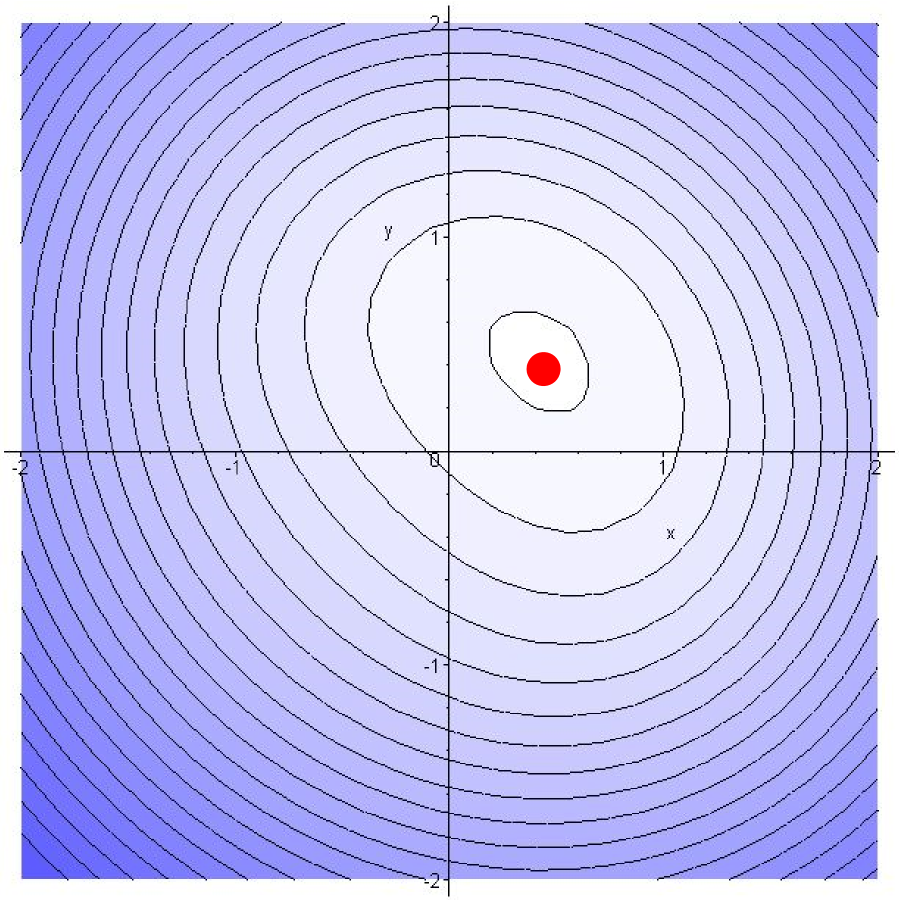
Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



Sample and Test Ex. 1

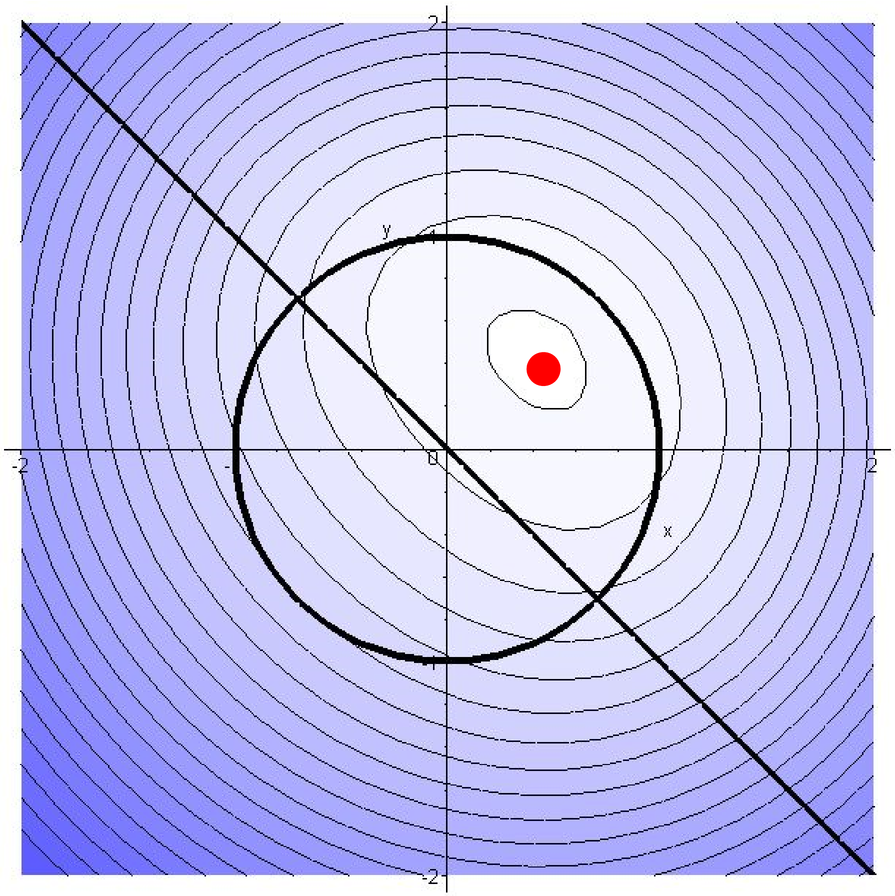
$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



True!

Sample and Test Ex. 1

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2)$$



True!

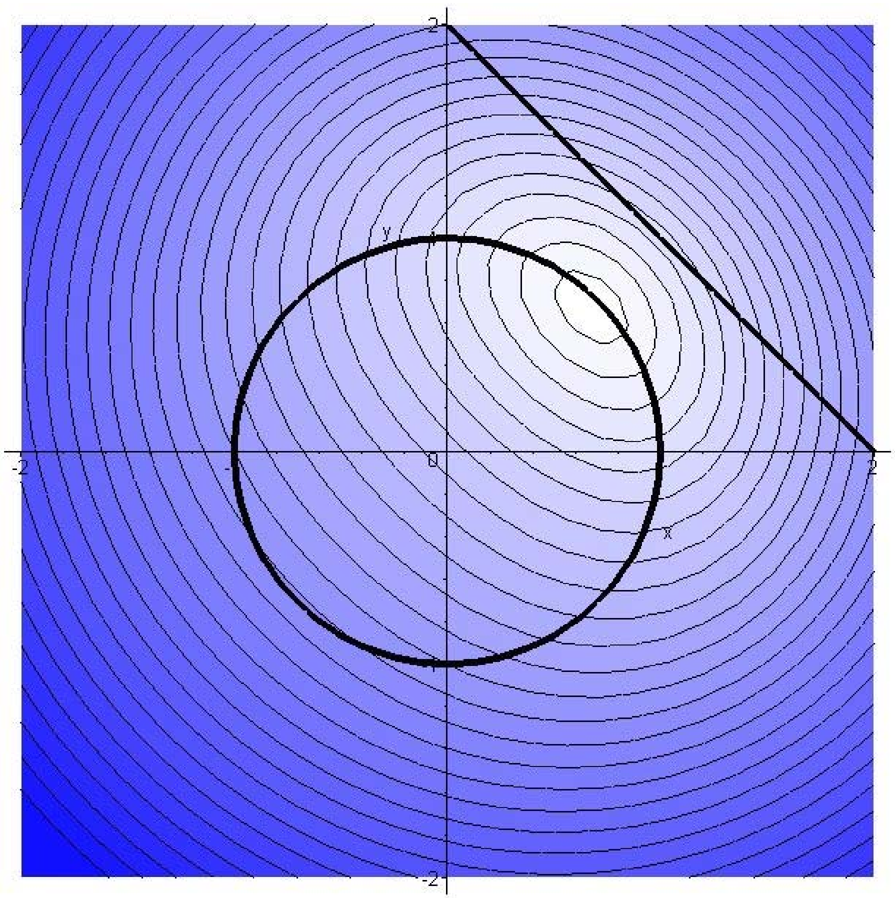
Sample and Test

Sample and Test Ex. 2

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2 - 2)$$

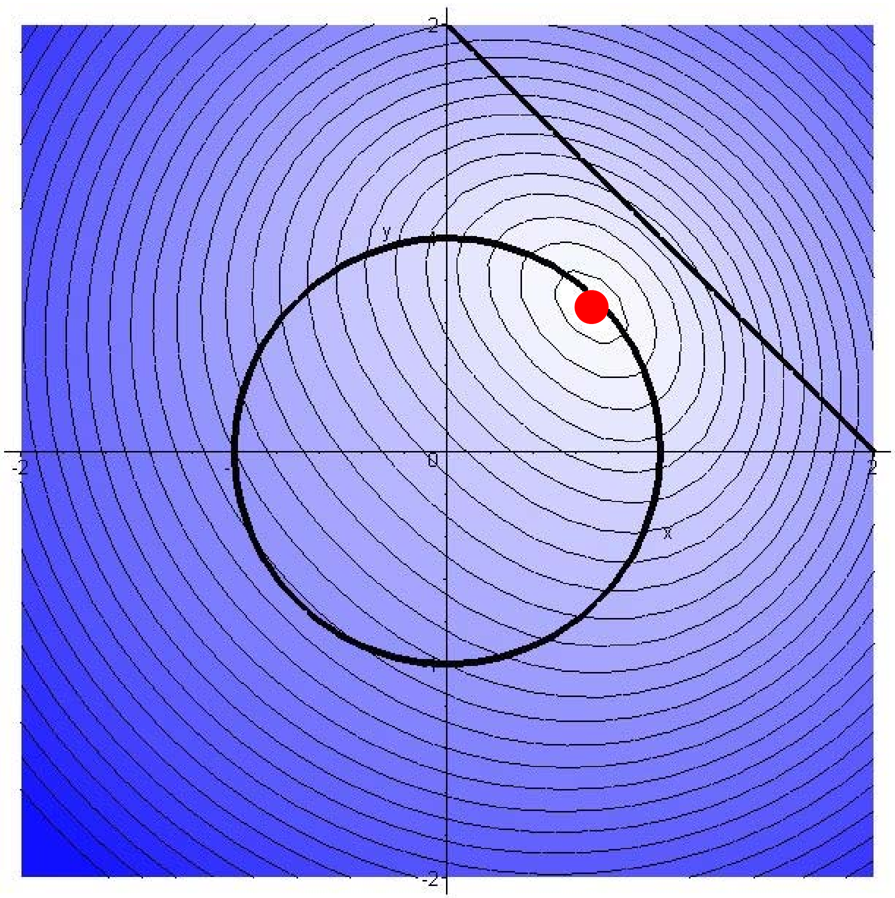
Sample and Test Ex. 2

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2 - 2)$$



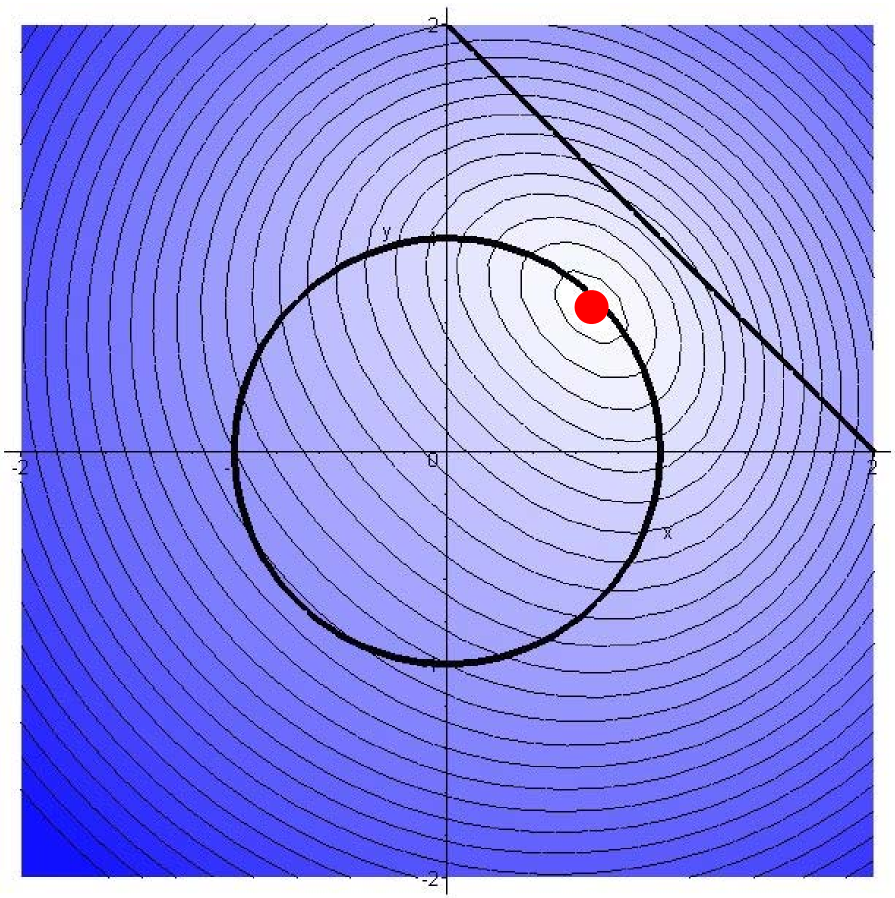
Sample and Test Ex. 2

$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2 - 2)$$



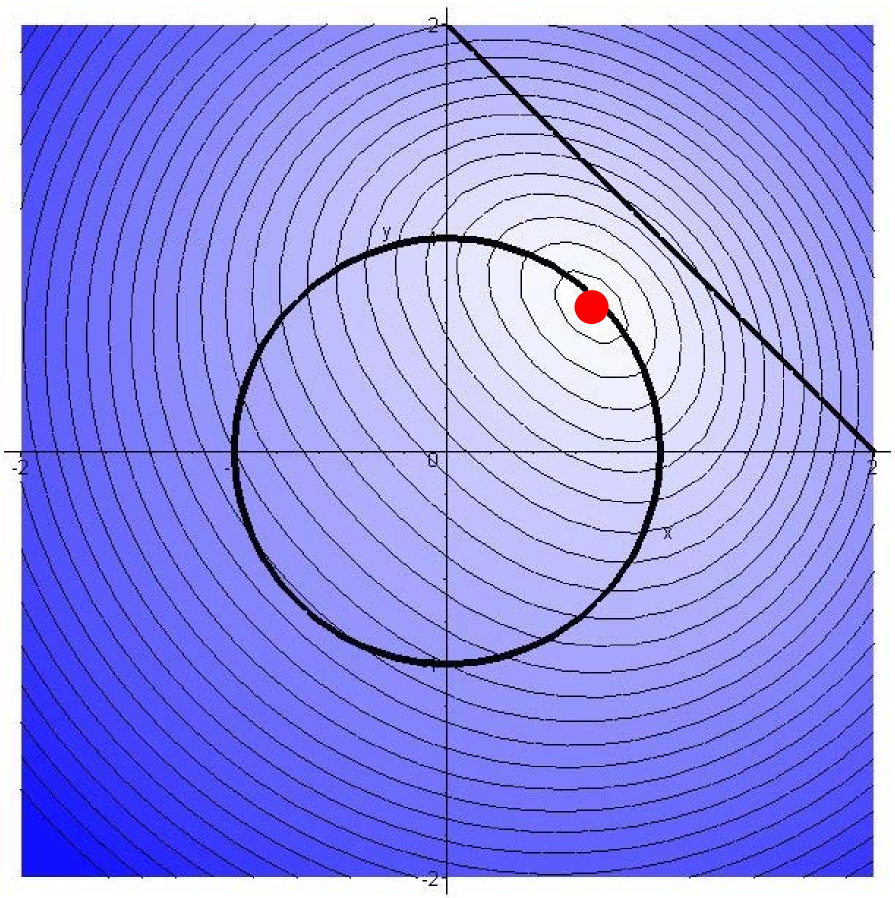
Sample and Test Ex. 2

$$f = \left(-x_1^2 \overset{+}{-} x_2^2 + 1,\; x_1 + x_2 - 2\right)$$



Sample and Test Ex. 2

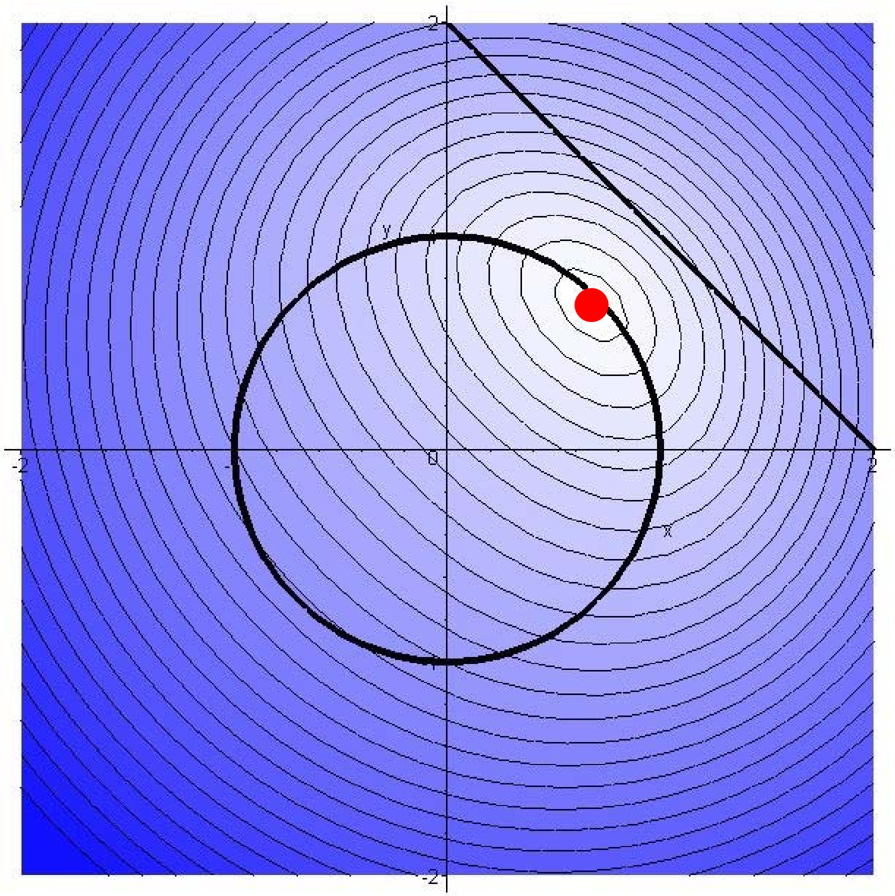
$$f = (\overset{+}{-x_1^2 - x_2^2 + 1},\ \overset{-}{x_1 + x_2 - 2})$$



Sample and Test Ex. 2

$+$ $-$

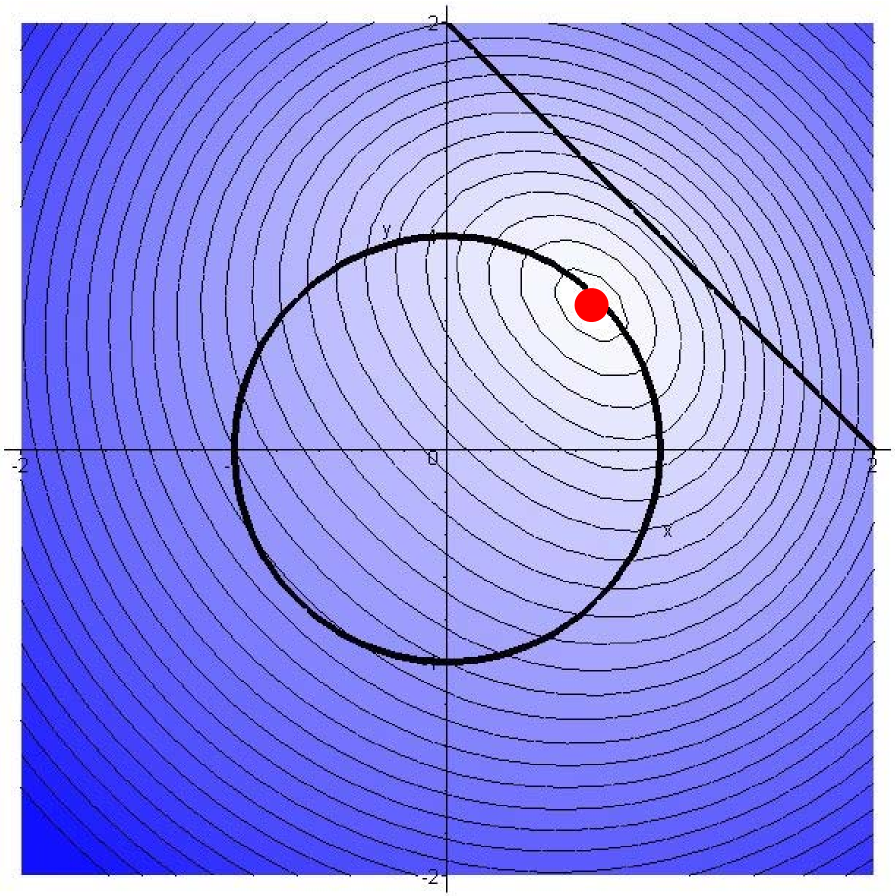
$$f = (-x_1^2 - x_2^2 + 1,\ x_1 + x_2 - 2)$$



e

Sample and Test Ex. 2

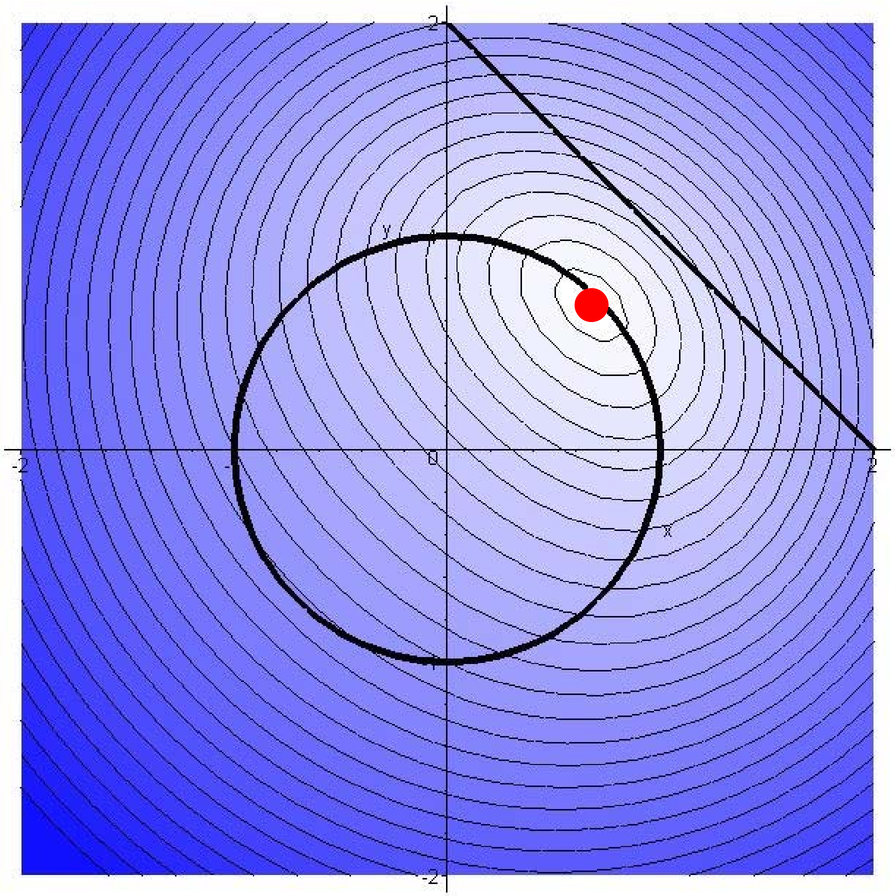
$$f = (\overset{+}{-x_1^2 - x_2^2 + 1},\ \overset{-}{x_1 + x_2 - 2})$$



$e = 2.3$

Sample and Test Ex. 2

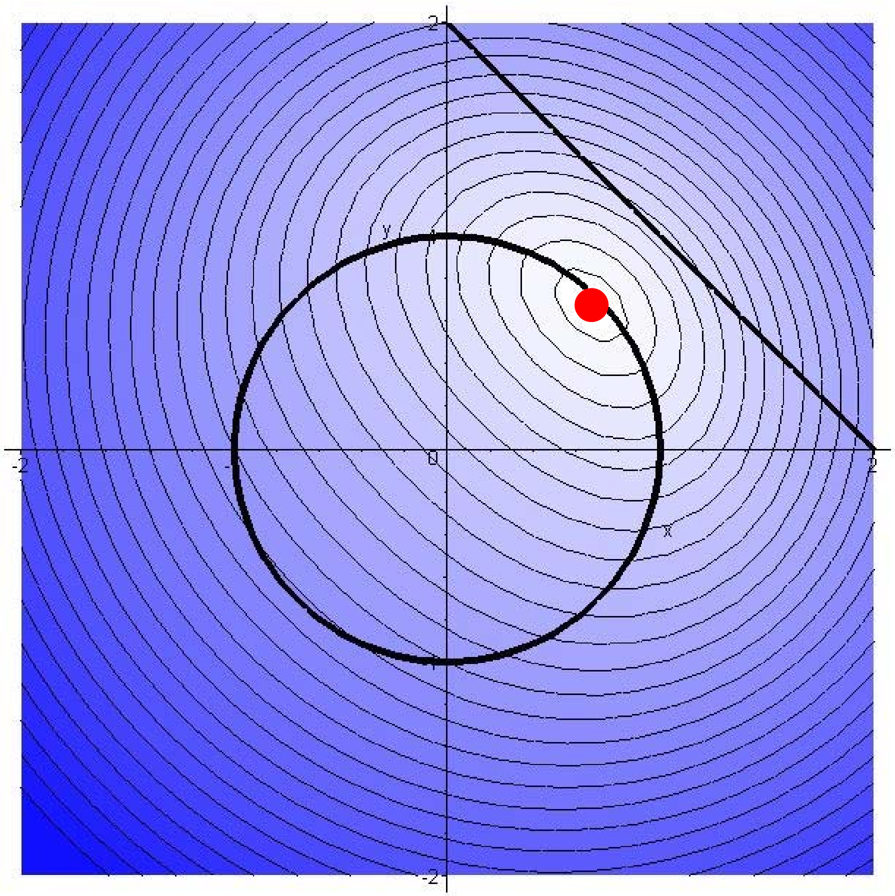
$$f = (\overset{+}{-x_1^2 - x_2^2 + 1},\ \overset{-}{x_1 + x_2 - 2})$$



$e = 2.3 > 2$

Sample and Test Ex. 2

$$f = \left(\overset{+}{-x_1^2 - x_2^2 + 1},\ \overset{-}{x_1 + x_2 - 2}\right)$$



$e = 2.3 > 2$

False!

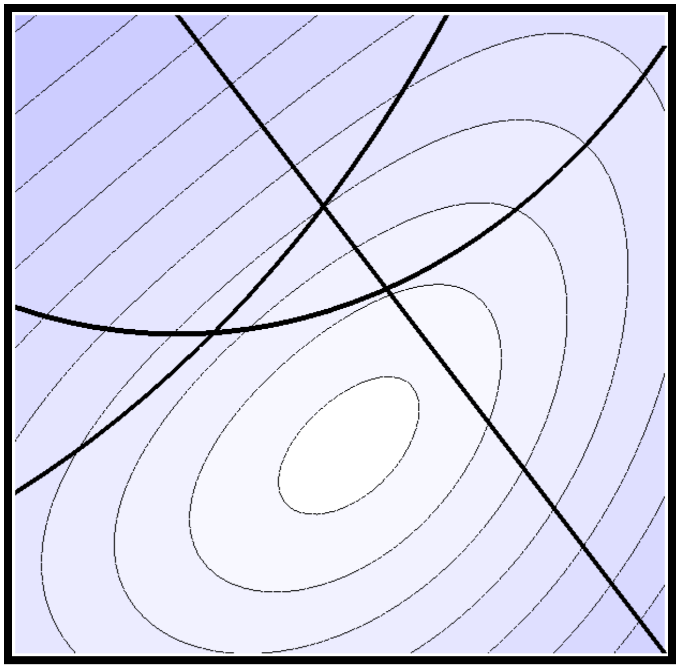
Sample and Test

Sample and Test Ex. 3

$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

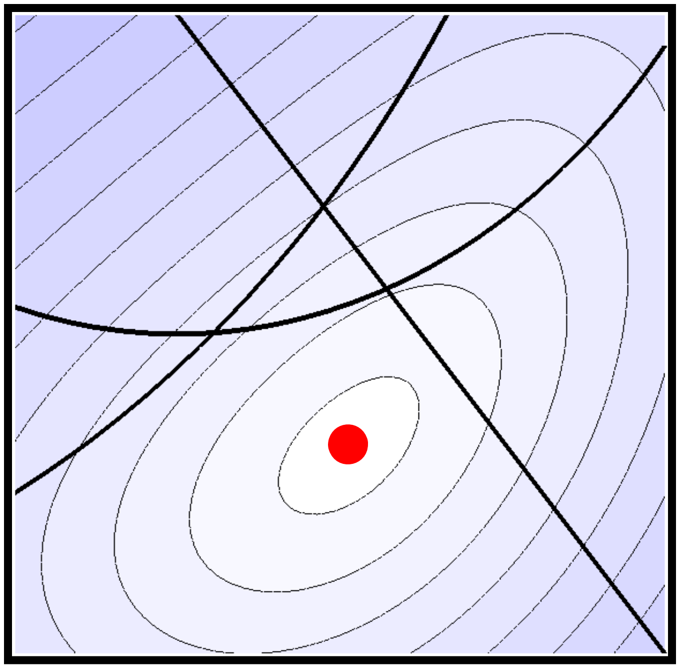
Sample and Test Ex. 3

$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$



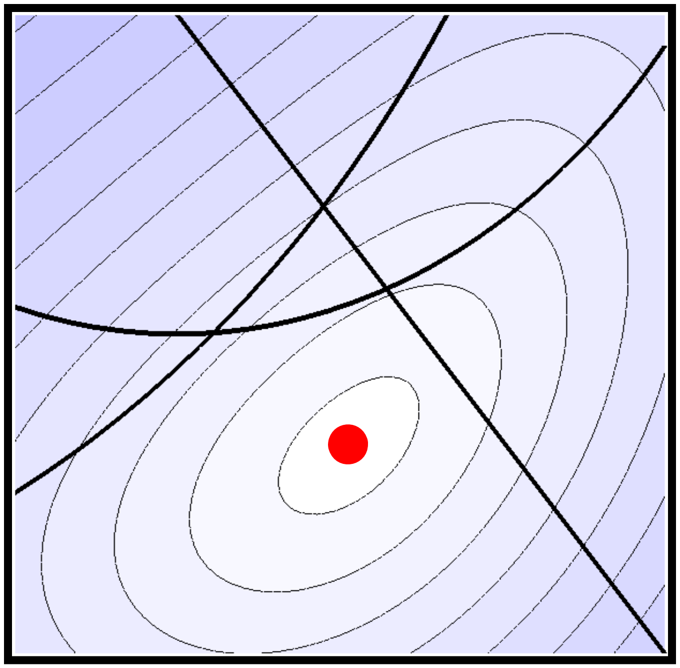
Sample and Test Ex. 3

$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$



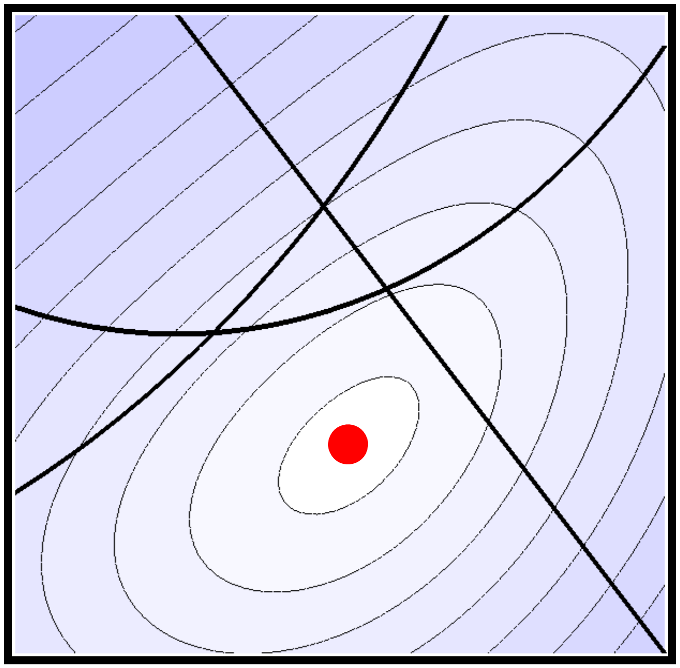
Sample and Test Ex. 3

$$f = \left(\overset{-}{-(x_1-1)^2 - (x_2-\tfrac{3}{8})^2 + 1}, \; \overset{+}{x_1^2 + x_2^2 - 3}, \; \overset{+}{-x_1 - x_2} \right)$$



Sample and Test Ex. 3

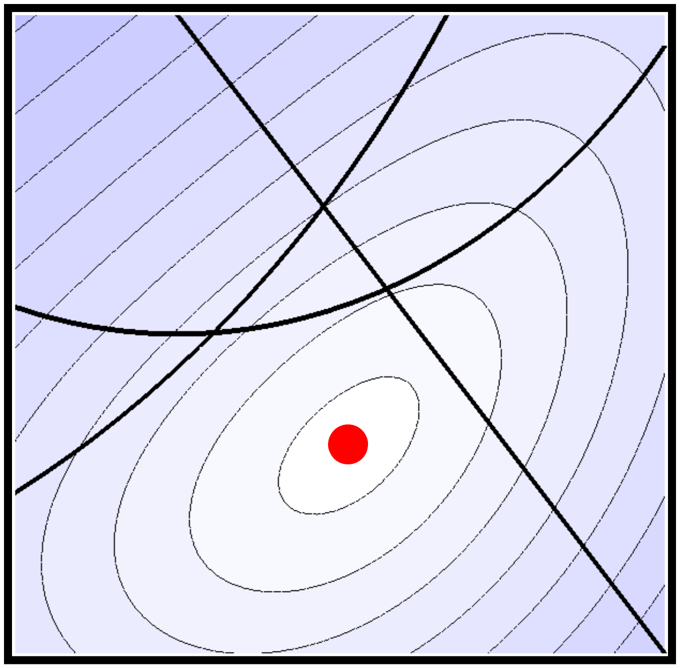
$$f = \left(\overset{-}{-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1},\ \overset{+}{x_1^2+x_2^2-3},\ \overset{+}{-x_1-x_2} \right)$$



$e = 2.7$

Sample and Test Ex. 3

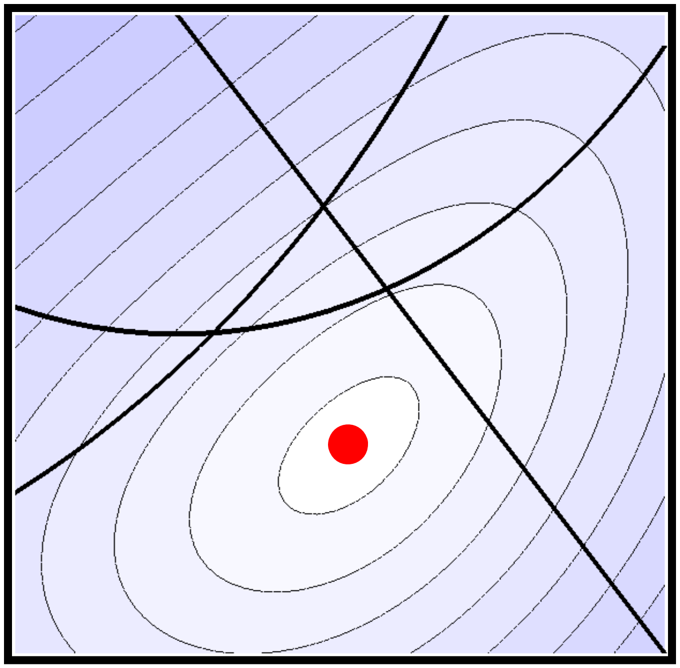
$$f = \left(\overset{-}{-(x_1-1)^2 - (x_2 - \tfrac{3}{8})^2 + 1}, \; \overset{+}{x_1^2 + x_2^2 - 3}, \; \overset{+}{-x_1 - x_2} \right)$$



$e = 2.7 < 3$

Sample and Test Ex. 3

$$f = \left(-(x_1-1)^2 \overset{-}{-} (x_2-\tfrac{3}{8})^2 + 1,\ x_1^2 \overset{+}{+} x_2^2 - 3,\ -x_1 \overset{+}{-} x_2 \right)$$



$e = 2.7 < 3$

?

Sample and Test Ex. 3

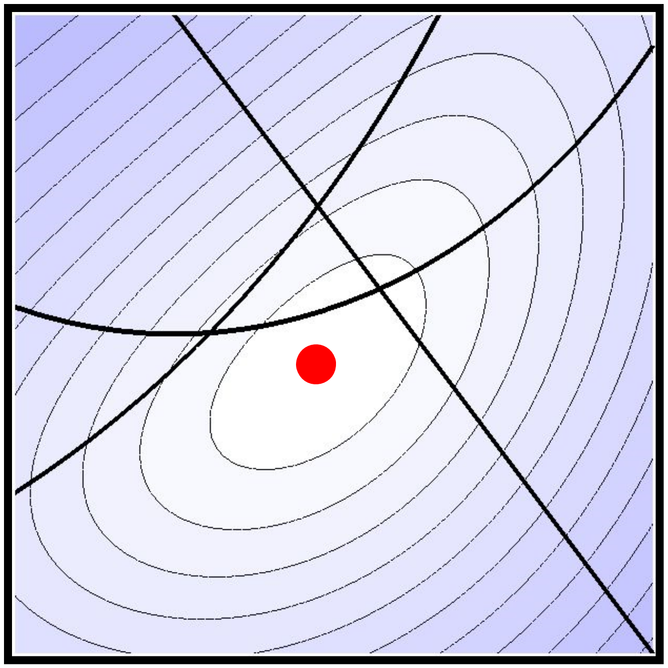
$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

Sample and Test Ex. 3

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

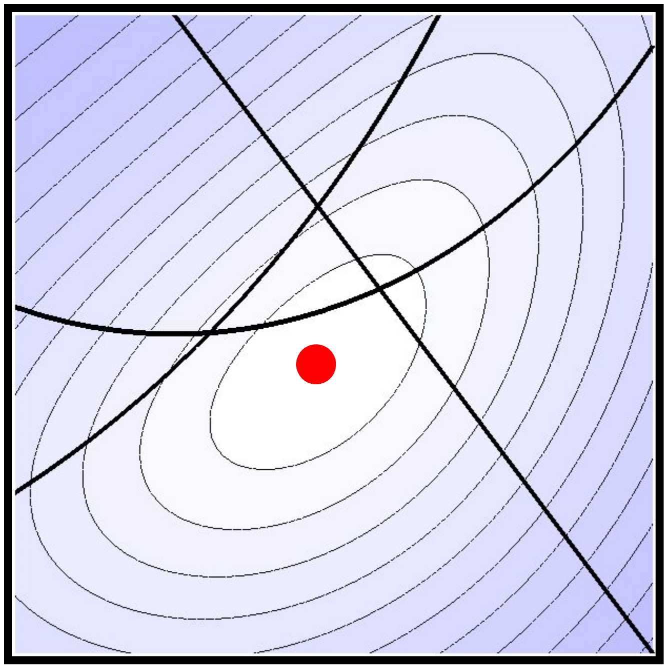
Sample and Test Ex. 3

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\; x_1^2+x_2^2-3,\; -x_1-x_2\right)$$



Sample and Test Ex. 3

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$



$e = 2.6 < 3$

?

Sample and Test Ex. 3

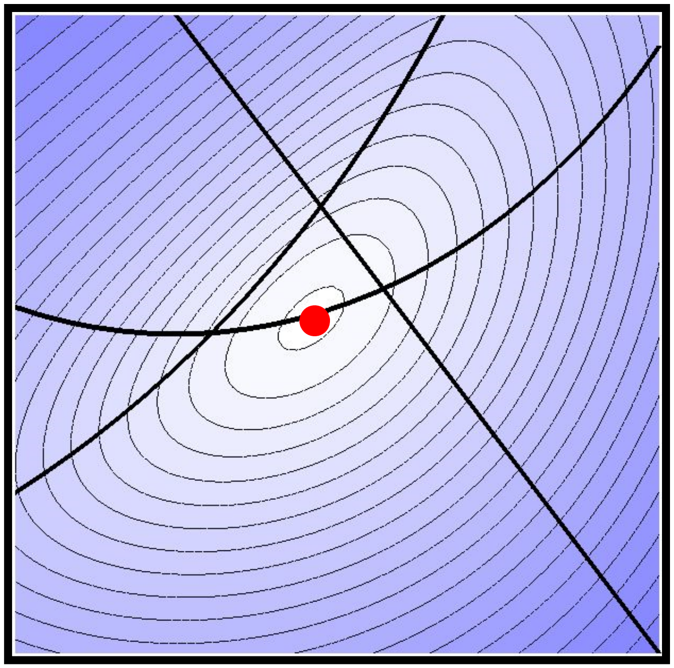
$$f = \left(-(x_1-1)^2 - (x_2 - \tfrac{3}{8})^2 + 1,\ x_1^2 + x_2^2 - 3,\ -x_1 - x_2 \right)$$

Sample and Test Ex. 3

$$f = 4\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

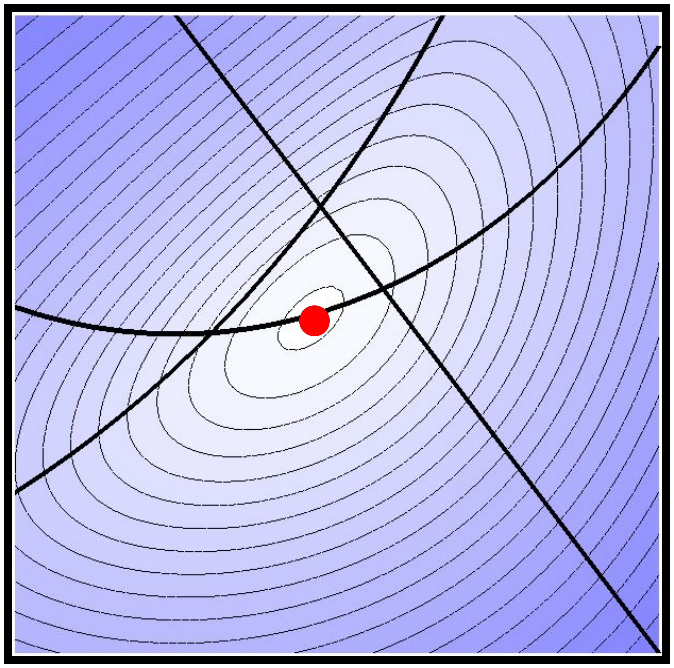
Sample and Test Ex. 3

$$f = 4\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\; x_1^2+x_2^2-3,\; -x_1-x_2\right)$$



Sample and Test Ex. 3

$$f = 4\left(\overset{-}{-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1},\ \overset{+}{x_1^2+x_2^2-3},\ \overset{+}{-x_1-x_2}\right)$$



$e = 2.3 < 3$

?

Sample and Test Ex. 3

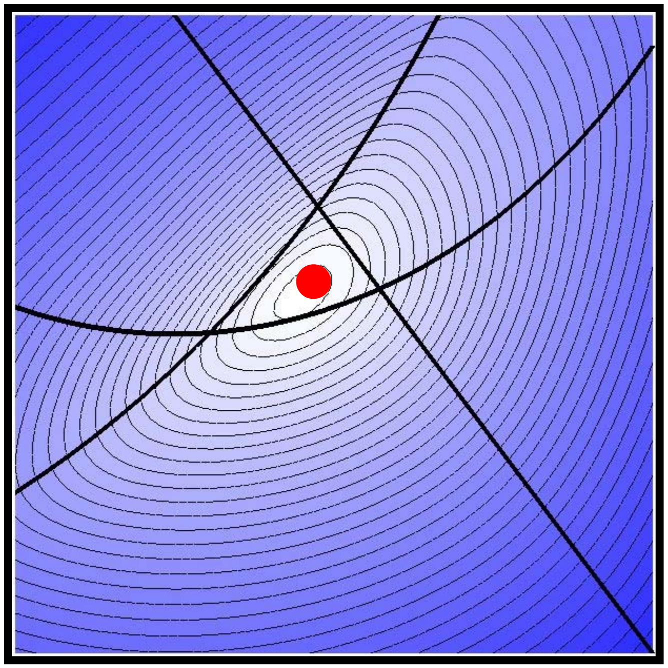
$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

Sample and Test Ex. 3

$$f = 8\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

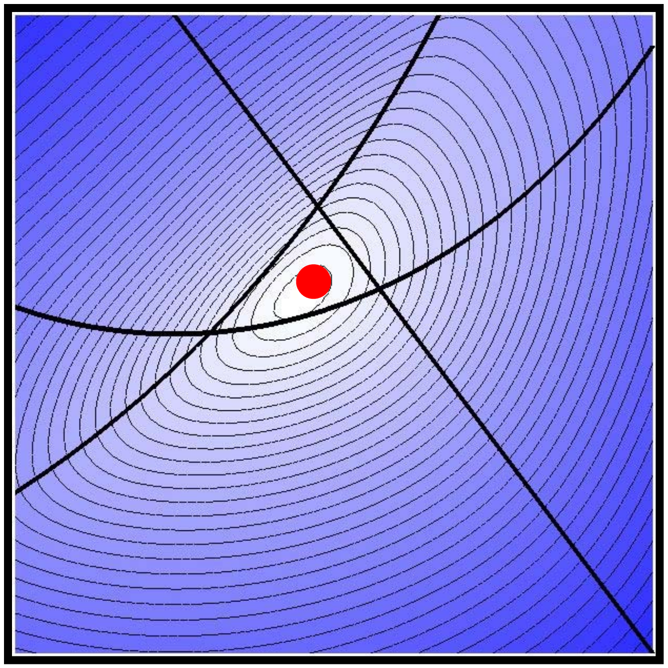
Sample and Test Ex. 3

$$f = 8\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$



Sample and Test Ex. 3

$$f = 8\left(-(x_1-1)^2 \overset{+}{-} (x_2-\tfrac{3}{8})^2+1,\; x_1^2 \overset{+}{+} x_2^2-3,\; -x_1 \overset{+}{-} x_2\right)$$

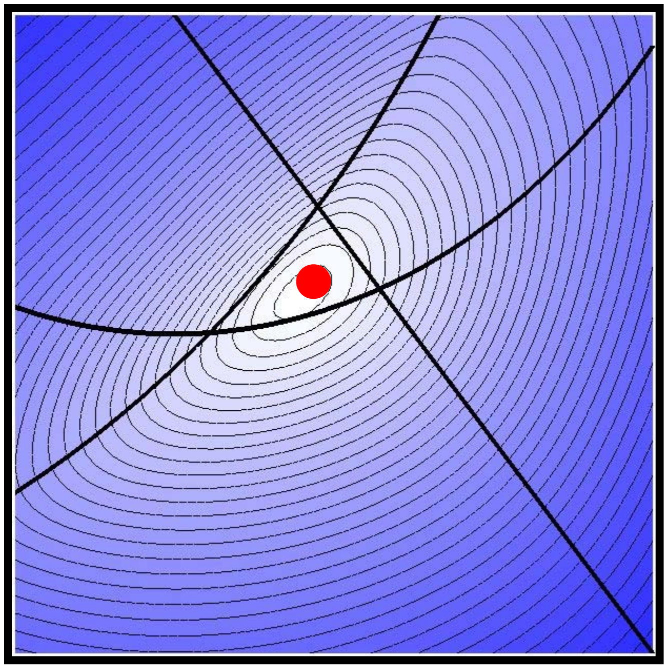


$$e = 1.9 < 3$$

Sample and Test Ex. 3

$$\qquad\qquad + \qquad\qquad\qquad + \qquad\qquad\qquad +$$

$$f = 8\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2\right)$$

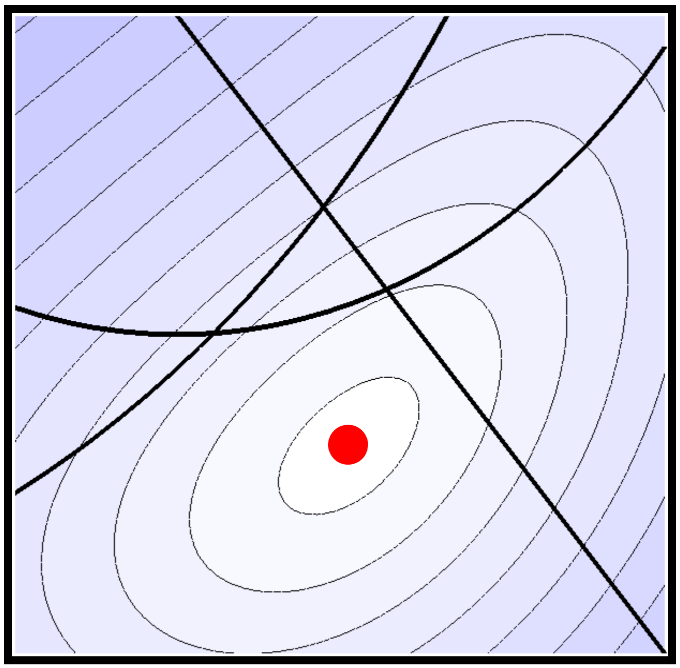


$e = 1.9 < 3$

True!

Sample and Test Ex. 3 Movie

$$f = \left(\overset{-}{-(x_1-1)^2 - (x_2 - \tfrac{3}{8})^2 + 1}, \ \overset{+}{x_1^2 + x_2^2 - 3}, \ \overset{+}{-x_1 - x_2} \right)$$

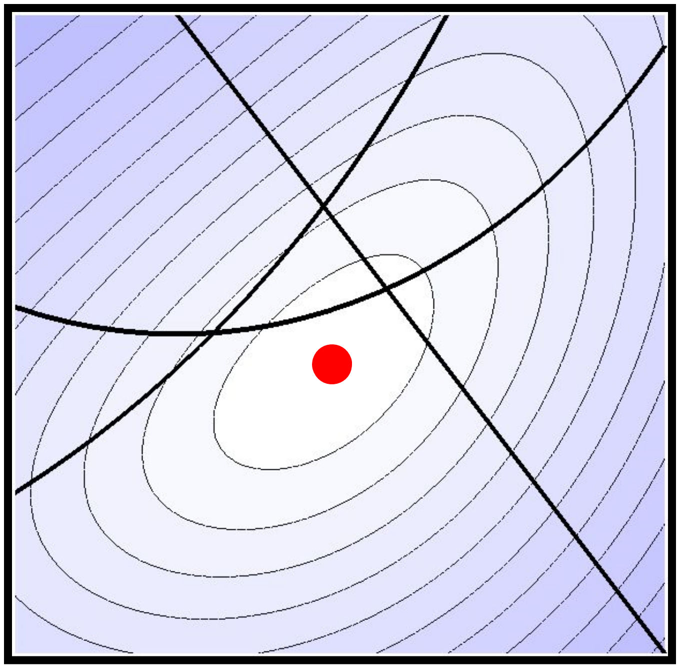


$e = 2.7 < 3$

?

Sample and Test Ex. 3 Movie

$$\begin{array}{cccccc} & - & & + & & + \\ f = 2\Big(& -(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1, & & x_1^2+x_2^2-3, & & -x_1-x_2\Big) \end{array}$$

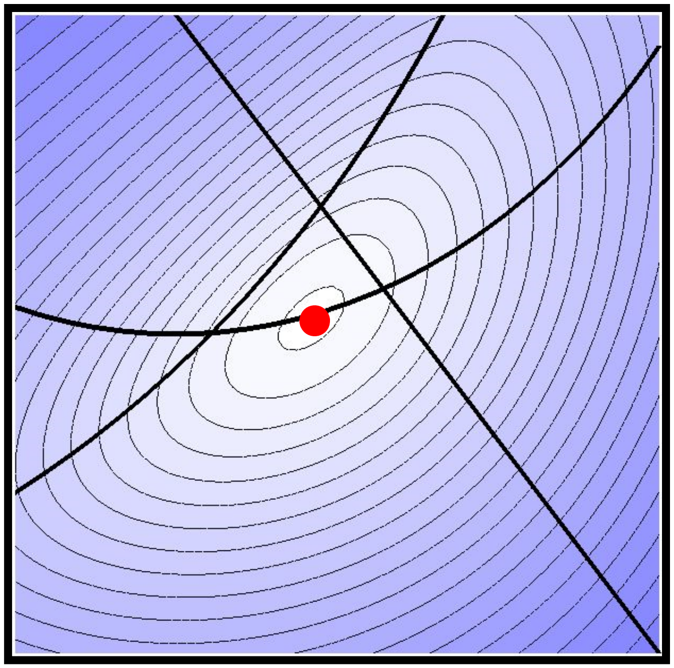


$e = 2.6 < 3$

?

Sample and Test Ex. 3 Movie

$$f = 4\left(\overset{-}{-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1},\ \overset{+}{x_1^2+x_2^2-3},\ \overset{+}{-x_1-x_2}\right)$$

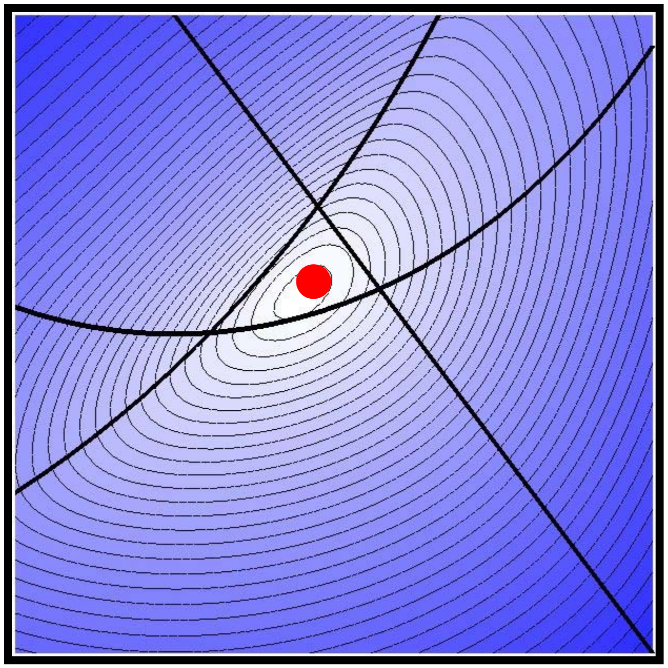


$e = 2.3 < 3$

?

Sample and Test Ex. 3 Movie

$$f = 8\left(-(x_1-1)^2 \overset{+}{-} (x_2-\tfrac{3}{8})^2+1,\ x_1^2 \overset{+}{+} x_2^2-3,\ -x_1 \overset{+}{-} x_2\right)$$

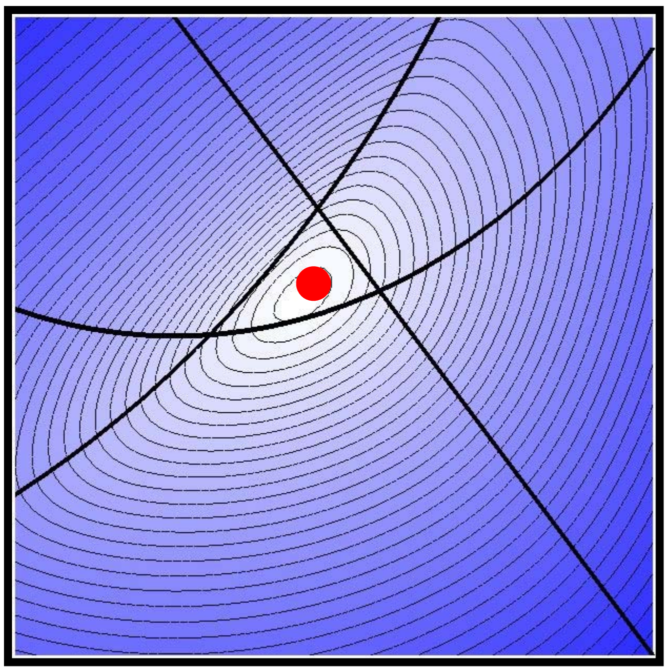


$e = 1.9 < 3$

True!

Sample and Test Ex. 3 Movie

$$f = 8\left(\overset{+}{-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1}, \overset{+}{x_1^2+x_2^2-3}, \overset{+}{-x_1-x_2}\right)$$



$e = 1.9 < 3$

True!

Theorem

$\forall$ feasible f $\forall$ suff. large C $f(x^*) > 0$.

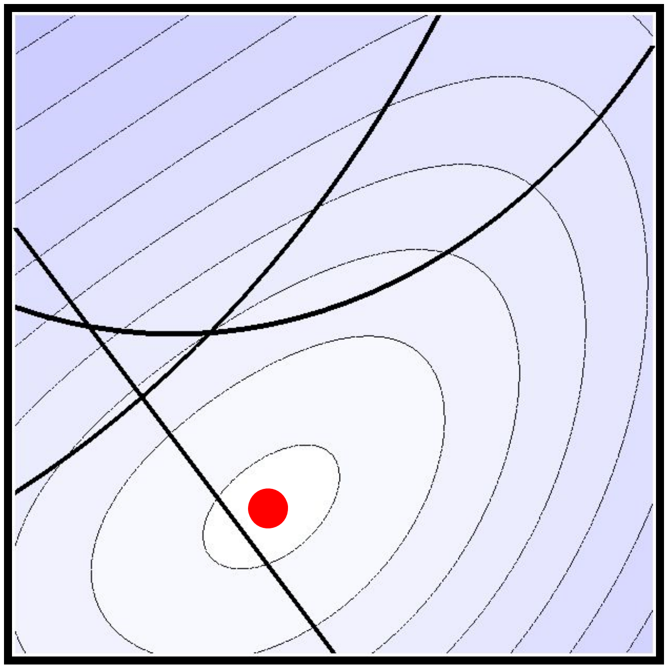
Sample and Test

Sample and Test Ex. 4

$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2-\tfrac{1}{2}\right)$$

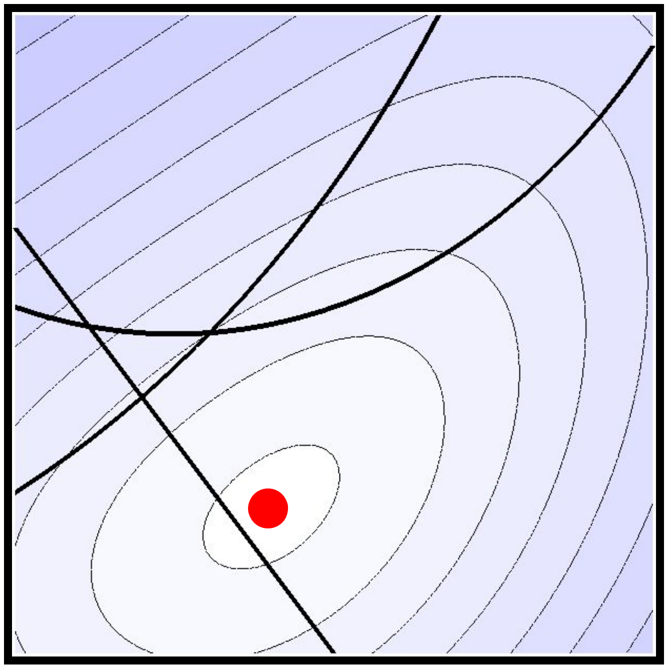
Sample and Test Ex. 4

$$f = \left(-(x_1-1)^2 - (x_2 - \tfrac{3}{8})^2 + 1,\ x_1^2 + x_2^2 - 3,\ -x_1 - x_2 - \tfrac{1}{2} \right)$$



Sample and Test Ex. 4

$$f = \left(-(x_1-1)^2 - (x_2-\tfrac{3}{8})^2 + 1,\; x_1^2 + x_2^2 - 3,\; -x_1 - x_2 - \tfrac{1}{2}\right)$$



$e = 2.9 < 3$

?

Sample and Test Ex. 4

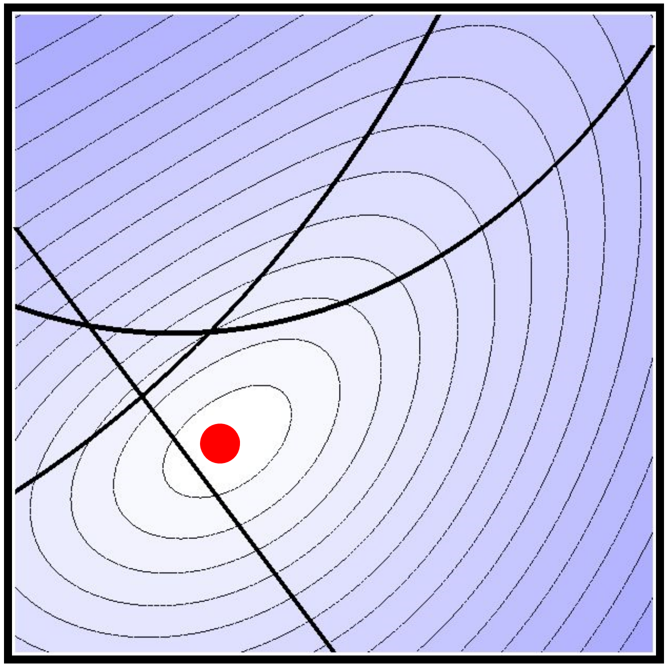
$$f = \left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2-\tfrac{1}{2}\right)$$

Sample and Test Ex. 4

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2-\tfrac{1}{2}\right)$$

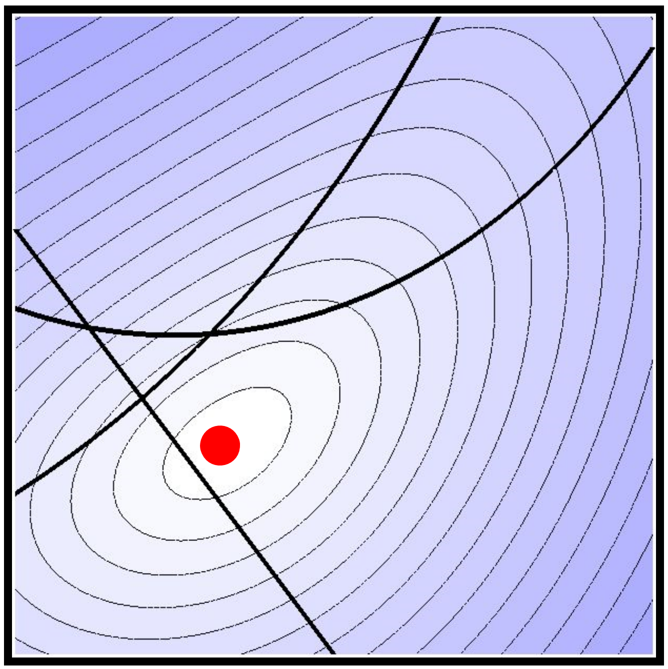
Sample and Test Ex. 4

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2-\tfrac{1}{2}\right)$$



Sample and Test Ex. 4

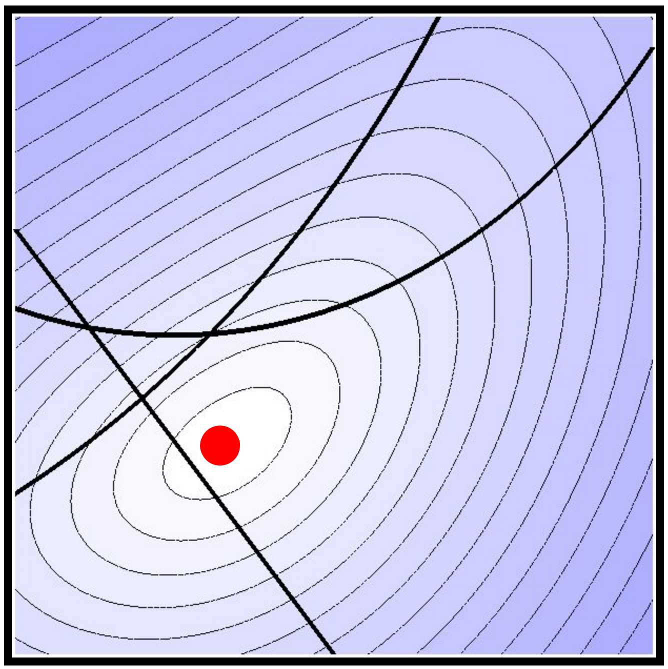
$$f = 2\left(-(x_1-1)^2 \overset{-}{-} (x_2 - \tfrac{3}{8})^2 + 1,\; x_1^2 \overset{+}{+} x_2^2 - 3,\; -x_1 \overset{-}{-} x_2 - \tfrac{1}{2}\right)$$



$$e = 3.1$$

Sample and Test Ex. 4

$$f = 2\left(\overset{-}{-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1},\ \overset{+}{x_1^2+x_2^2-3},\ \overset{-}{-x_1-x_2-\tfrac{1}{2}}\right)$$

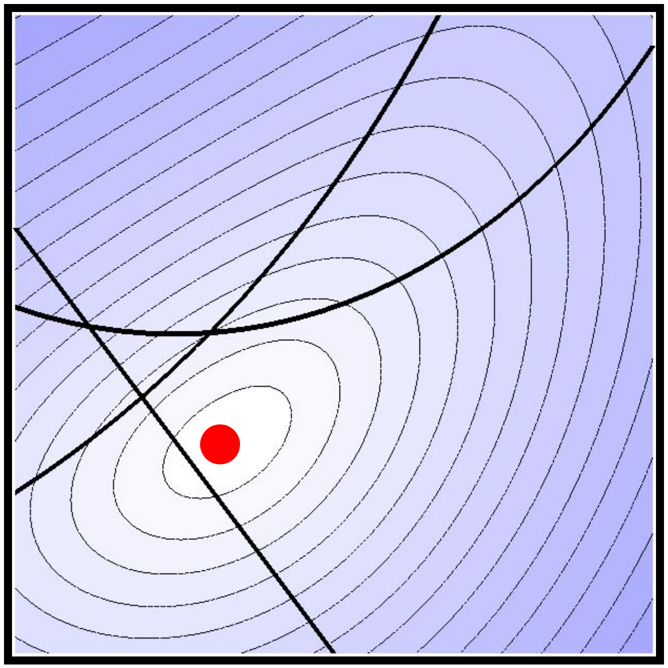


$e = 3.1 > 3$

Sample and Test Ex. 4

$$f = 2\left(-(x_1-1)^2 - (x_2-\tfrac{3}{8})^2 + 1,\; x_1^2 + x_2^2 - 3,\; -x_1 - x_2 - \tfrac{1}{2}\right)$$

$$\quad - \qquad\qquad\qquad + \qquad\qquad\qquad -$$



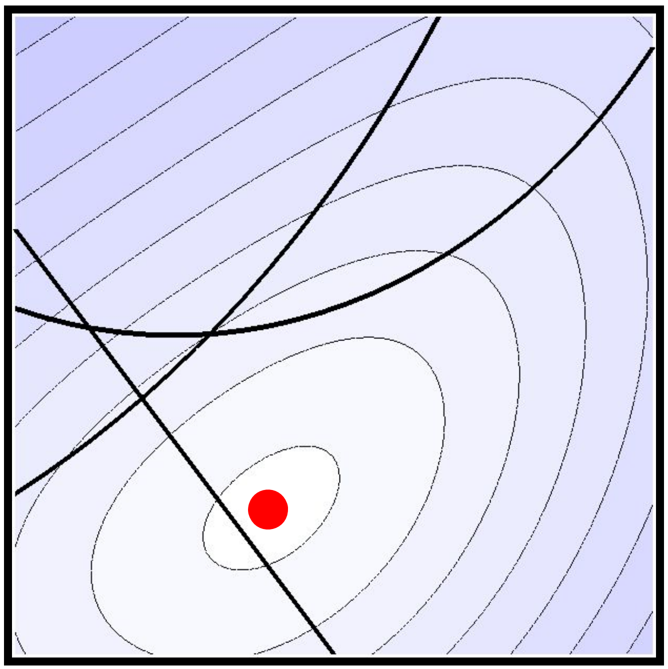
$e = 3.1 > 3$

False!

Sample and Test Ex. 4 Movie

$$f = \left(-(x_1-1)^2 - (x_2-\tfrac{3}{8})^2 + 1,\; x_1^2 + x_2^2 - 3,\; -x_1 - x_2 - \tfrac{1}{2}\right)$$

$$-\qquad\qquad + \qquad\qquad -$$

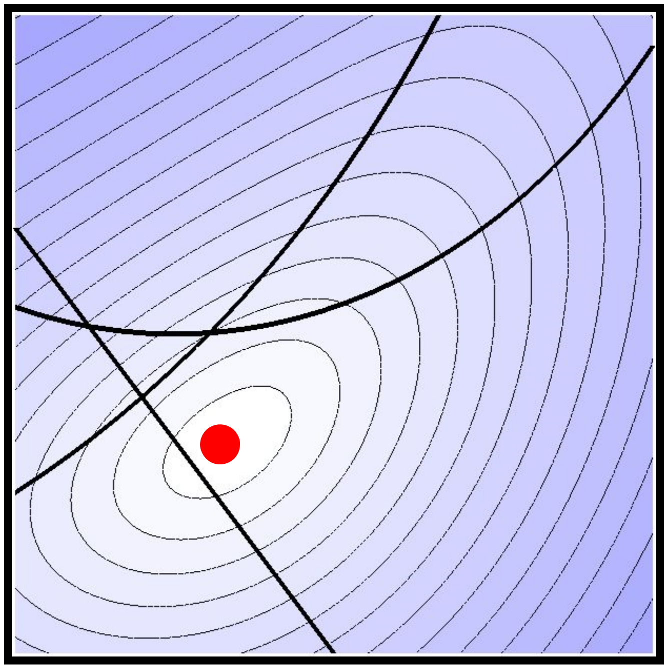


$e = 2.9 < 3$

?

Sample and Test Ex. 4 Movie

$$f = 2\left(-(x_1-1)^2-(x_2-\tfrac{3}{8})^2+1,\ x_1^2+x_2^2-3,\ -x_1-x_2-\tfrac{1}{2}\right)$$



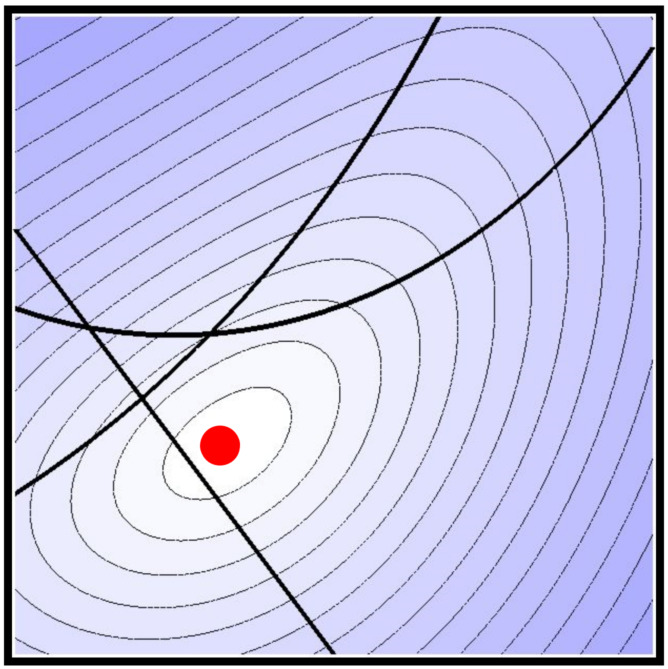
$e = 3.1 > 3$

False!

Sample and Test Ex. 4 Movie

$$f = 2\left(-(x_1-1)^2 - (x_2-\tfrac{3}{8})^2 + 1,\ x_1^2 + x_2^2 - 3,\ -x_1 - x_2 - \tfrac{1}{2}\right)$$

$-$ $+$ $-$



$e = 3.1 > 3$

False!

Theorem

$\dot{\forall}$ infeasible f $\forall$ suff. large C $e(x^*) > m$

Theorems

$\forall$ feasible f $\forall$ suff. large c $\quad f(x^*) > 0$

$\underset{\bullet}{\dot{\forall}}$ infeasible f $\forall$ suff. large c $\quad e(x^*) > m$

Algorithm

$c = 1$

loop "few" times

  $x^*, e^* = \min(e_c)$

  if $f(x^*) > 0$ return true

  if $e^* > m$ return false

  "increase" c

call symbolic

$$e_c = p(c \cdot f_1(x)) + \dots + p(c \cdot f_m(x))$$

Algorithm

$c = 1$
loop "few" times
  $x^*, e^* = \min(e_c)$ ← use intervals
  if $f(x^*) > 0$ return true
  if $e^* > m$ return false
 "increase" c
call symbolic

$$e_c = p(c \cdot f_1(x)) + \cdots + p(c \cdot f_m(x))$$

Algorithm

$C = 1$

loop "few" times

  $x^*, e^* = \min(e_c)$ ← use intervals

  if $f(x^*) > 0$ return true

  if $e^* > m$ return false

 "increase" C ← use local geom.

call symbolic

$$e_c = p(C \cdot f_1(x)) + \cdots + p(C \cdot f_m(x))$$

Algorithm

$C = 1$

loop "few" times

$x^*, e^* = \min(e_c)$ ← use intervals

if $f(x^*) > 0$ return true

if $e^* > m$ return false

"increase" C ← use local geom.

call symbolic ← only when "unstable"

$$e_c = \rho(C \cdot f_1(x)) + \cdots + \rho(C \cdot f_m(x))$$

Timing

# Var	# Poly	Deg	Previous	New
3	6	4	2 min	1 sec
4	8	5	>12 hour	5 sec
5	12	8	>72 hour	40 sec

```
degree  = 3
# polys = 3
# vars  = 1 - 20
```

degree = 3

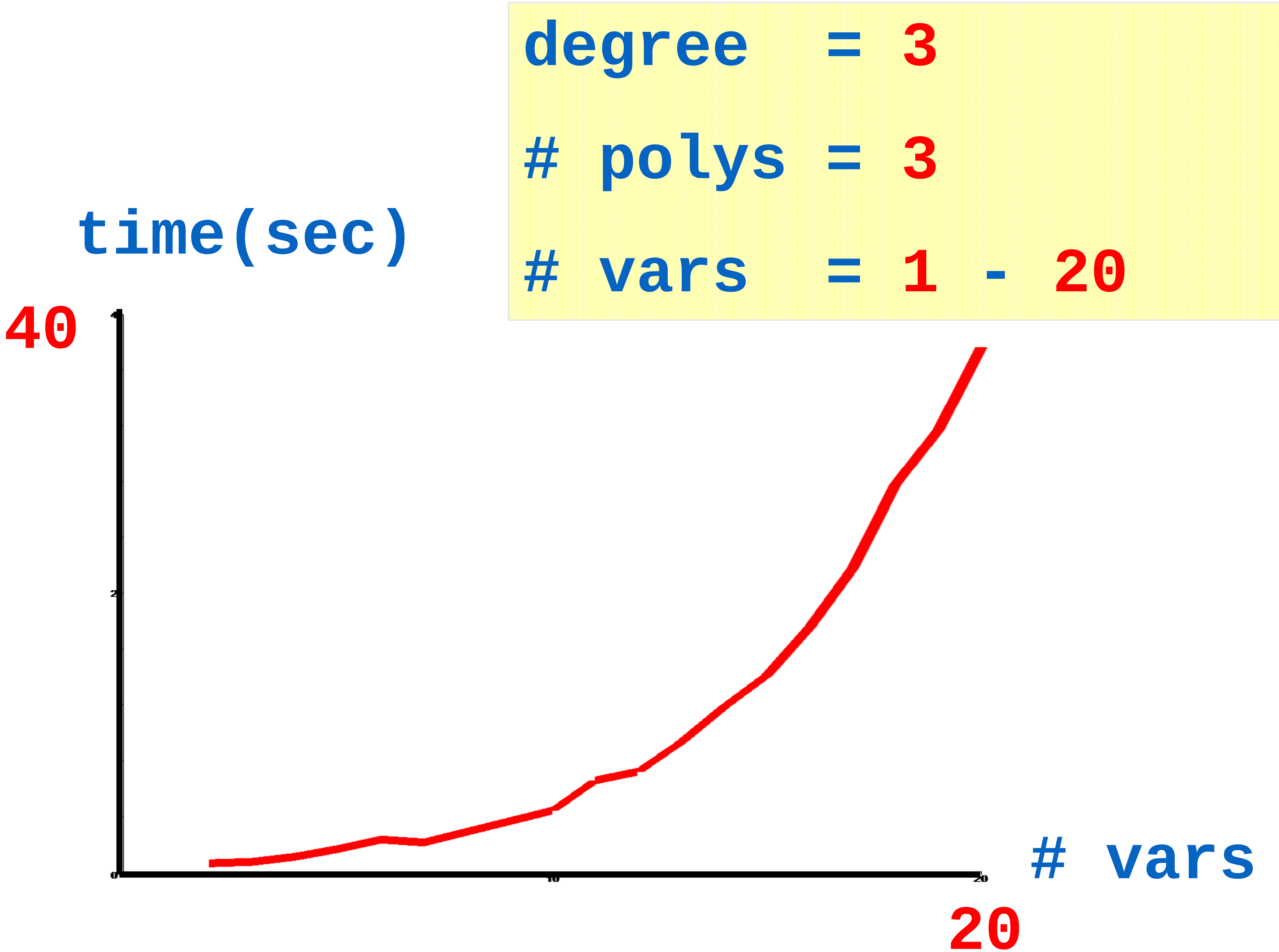
polys = 3

vars = 1 - 20

time(sec)

vars

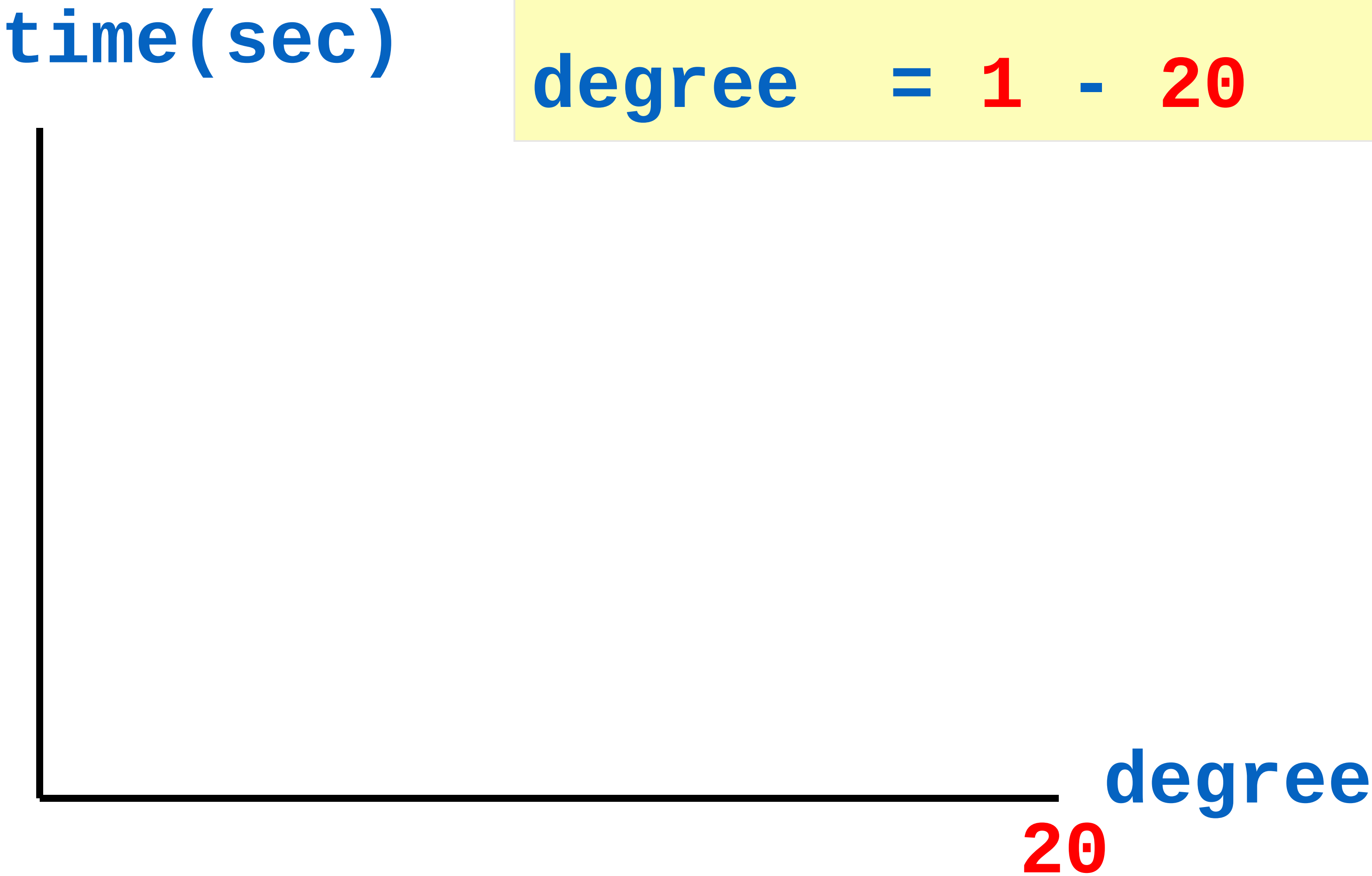
20


degree = 3
# polys = 3
# vars = 1 - 20
time(sec)
40
vars
20

polys = 3

vars = 3

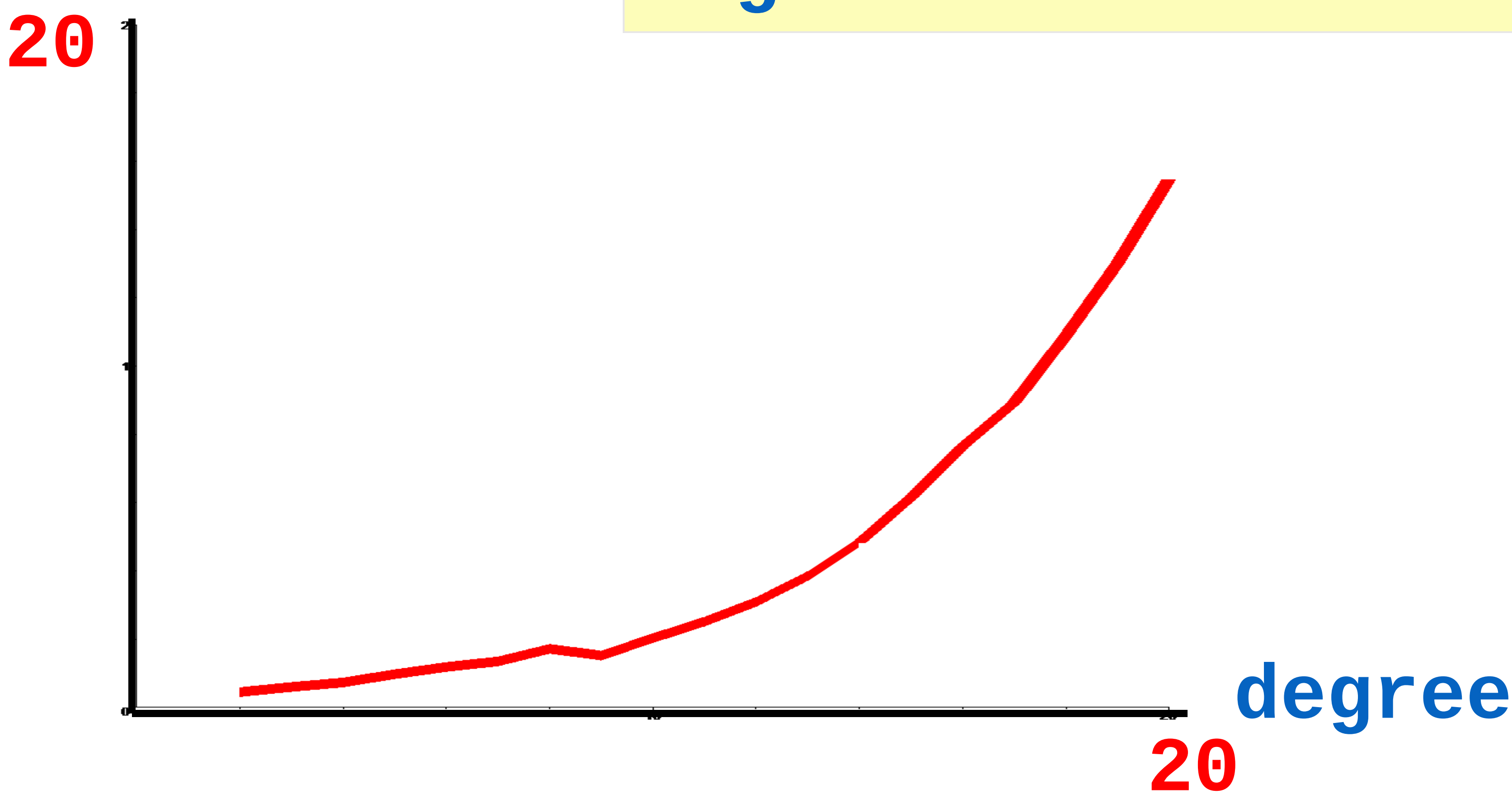
degree = 1 - 20



polys = 3

vars = 3

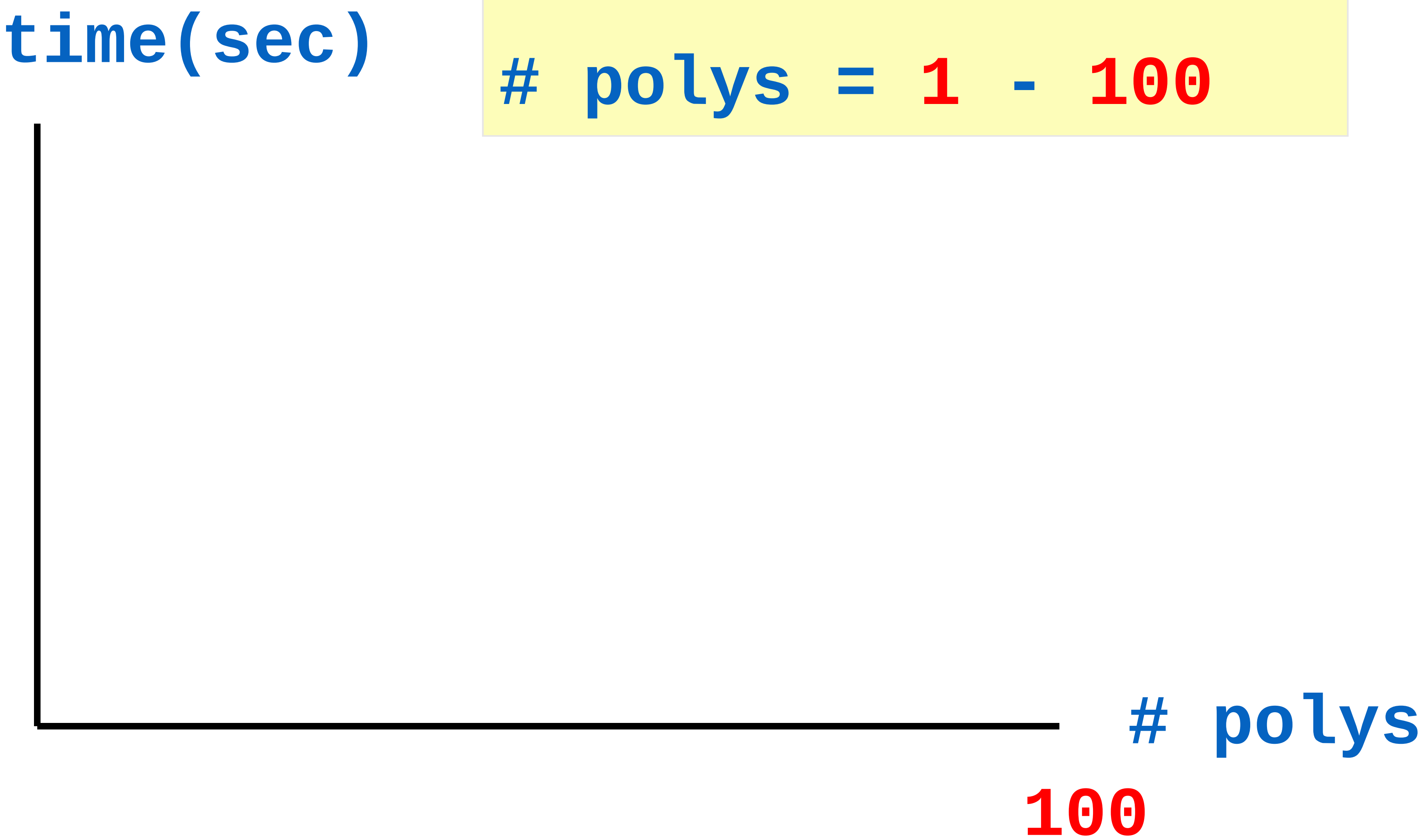
degree = 1 - 20



degree = 3

vars = 3

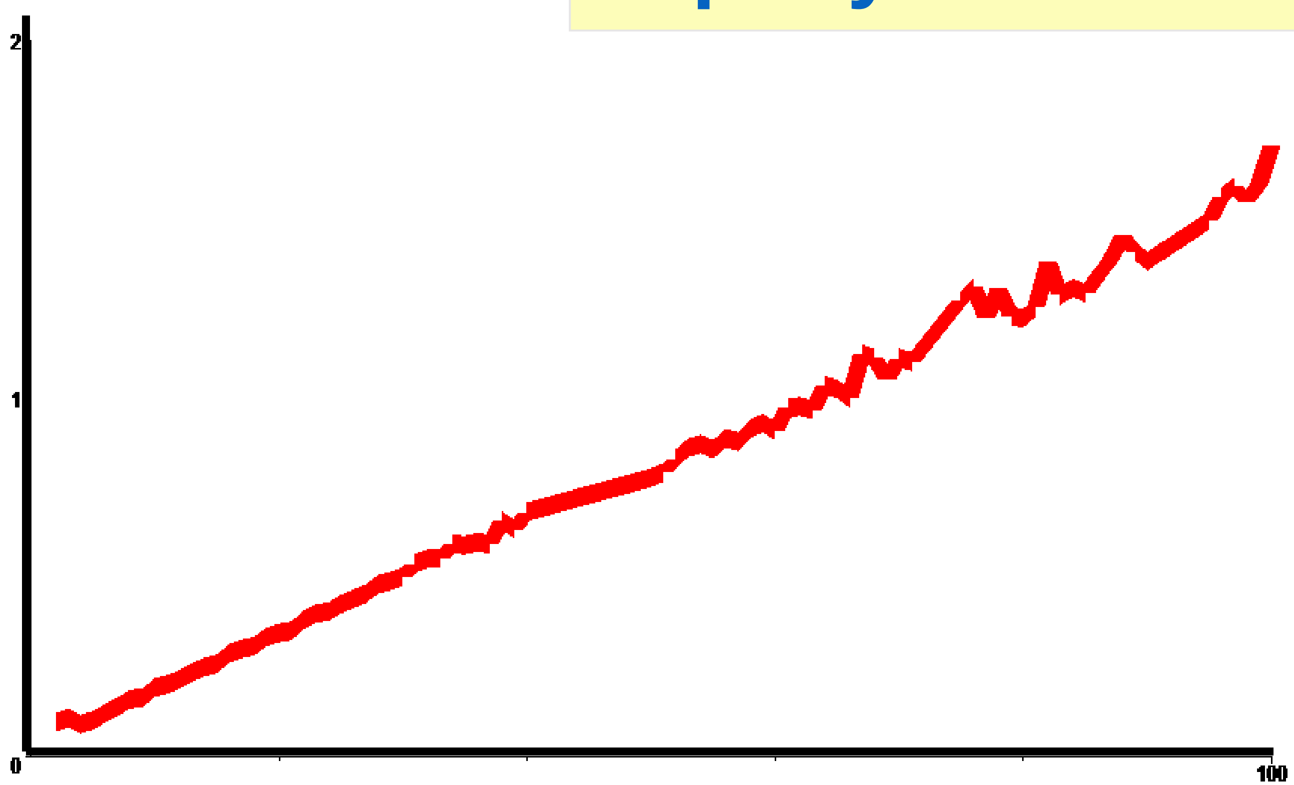
polys = 1 - 100



degree = 3

vars = 3

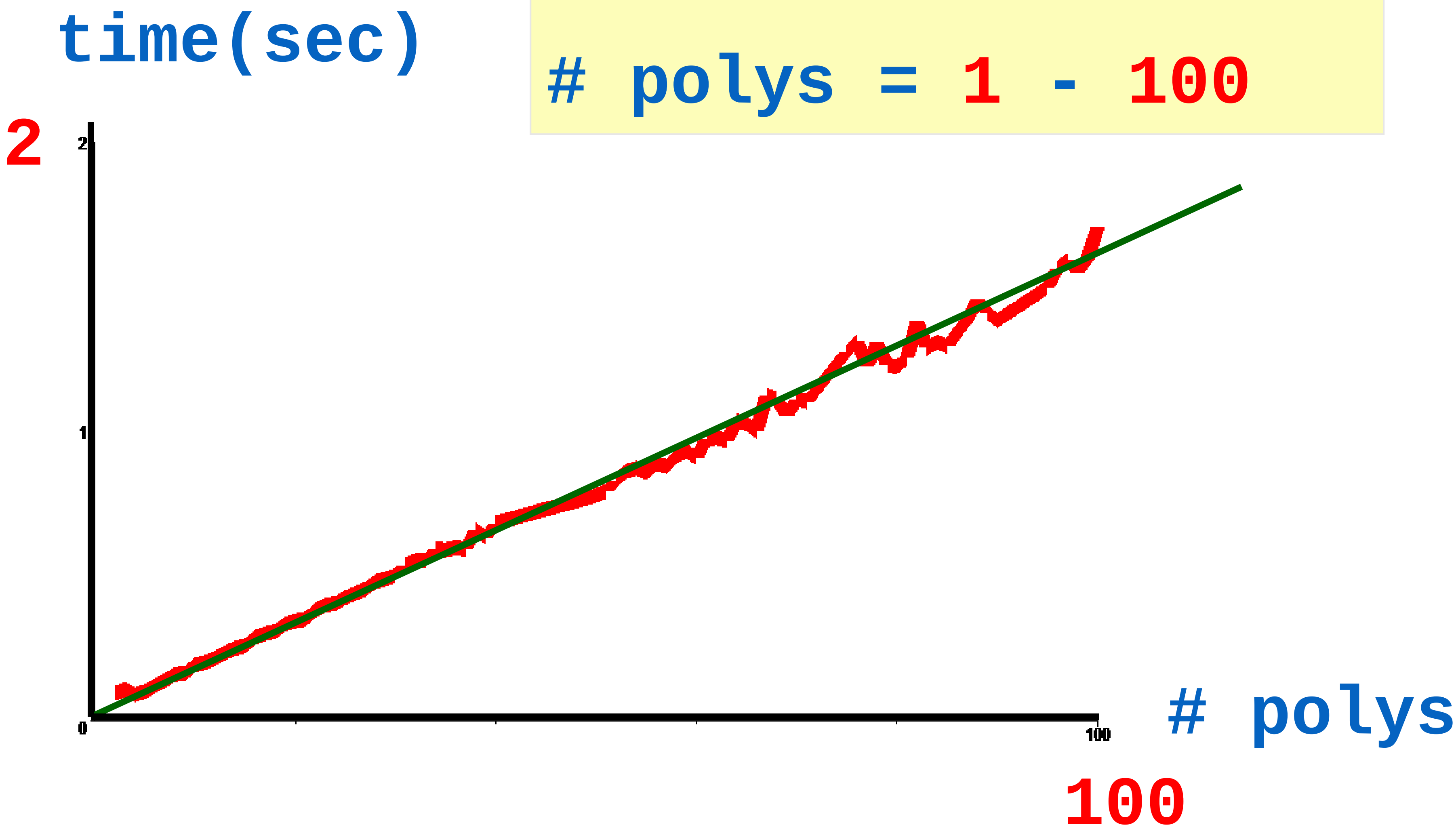
polys = 1 - 100



degree = 3

vars = 3

polys = 1 - 100



Numerical Projection

Challenging Problem

In : $\forall s,t \quad f(a,b,s,t) < 0$

Out : Quantifier-free formula in a, b

Challenging Problem

In : $\forall s,t \quad f(a,b,s,t) < 0$

Out : Quantifier-free formula in a, b

$f =$

```
8*b^6*a^2*s^4*t^8+140*b^4*a^4*s^4*t^4-4*b^4*a^2*s^8*t^4
+4*b^4*a^2*s^8*t^8+12*b^4*a^4*s^8*t^4+12*b^4*a^4*s^8*t^8
+12*b^4*a^4*s^4*t^8-12*b^4*a^2*s^4*t^8+16*b^4*a^2*s^8*t^6-
32*b^4*a^2*s^4*t^6-4*b^4*a^2*s^4*t^4+120*b^6*a^2*s^4*t^6
+8*b^6*a^2*s^8*t^6-4*a^4*b^2*s^4*t^4-12*a^4*b^2*s^8*t^4
+4*a^4*b^2*s^8*t^8-4*a^4*b^2*s^4*t^8+120*a^6*b^2*s^6*t^4
+8*a^6*b^2*s^6*t^8+8*a^6*b^2*s^8*t^4-8*a^6*b^2*s^8*t^8
+120*b^4*a^4*s^4*t^6-8*b^4*a^4*s^8*t^6-8*a^4*b^2*s^8*t^2
-24*a^4*b^2*s^6*t^2+8*a^4*b^2*s^6*t^6+16*a^4*b^2*s^6*t^8
+48*b^4*a^4*s^6*t^6-8*b^4*a^4*s^6*t^8+2*a^2*b^2*s^8*t^8
+6*a^2*b^2*s^4*t^8+34*a^2*b^2*s^4*t^4+6*a^2*b^2*s^8*t^4
+72*b^6*a^2*s^6*t^6+8*b^6*a^2*s^2*t^8-8*b^6*a^2*s^6*t^8
+56*b^6*a^2*s^2*t^6+120*b^4*a^4*s^6*t^4-8*a^6*b^2*s^8*t^6
+8*a^6*b^2*s^8*t^2+56*a^6*b^2*s^6*t^2+72*a^6*b^2*s^6*t^6
```

```
-32*a^4*b^2*s^6*t^4+2*a^2*b^2*s^2*t^8+6*a^2*b^2*s^8*t^6
+2*a^2*b^2*s^8*t^2+6*a^2*b^2*s^2*t^2+6*a^2*b^2*s^6*t^8
+10*a^2*b^2*s^2*t^6+10*a^2*b^2*s^6*t^2+22*a^2*b^2*s^6*t^6
-8*a^4*b^2*s^4*t^6+8*b^4*a^2*s^6*t^6-24*b^4*a^2*s^2*t^6
-8*b^4*a^2*s^2*t^8-8*b^4*a^2*s^6*t^4-4*b^2*s^2*t^4
+2*b^4*t^6+3*b^4*t^8-4*b^6*t^8+26*a^2*b^2*s^4*t^6
+14*a^2*b^2*s^4*t^2+26*a^2*b^2*s^6*t^4+14*a^2*b^2*s^2*t^4
-2*b^2*t^6-b^2*t^8+2*b^8*t^8+2*a^8*s^8-a^2*s^4-a^2*s^8
+3*a^4*s^8+a^4*s^4-4*a^6*s^8+2*a^4*s^6+8*a^4*s^6*t^2-
2*a^2*s^6+48*a^5*b^3*t^7*s^5-16*a^5*b^3*t^7*s^7
+160*a^5*b^3*t^5*s^5+32*a^5*b^3*t^5*s^7-8*a^2*s^6*t^2
+112*a^5*b^3*t^3*s^5+48*a^5*b^3*t^3*s^7-8*a^2*s^6*t^6
+12*b^4*s^2*t^8+2*b^4*s^8*t^6+8*b^4*s^6*t^6-b^2*t^4
+12*b^4*s^4*t^6+8*b^4*s^2*t^6+48*a*b^7*t^7*s^3
+48*a*b^7*t^7*s^5+16*a*b^7*t^7*s^7+16*a*b^7*t^7*s
-16*a^3*b^3*s^3*t^7+16*a^3*b^3*s^5*t^7+32*a^3*b^3*s^7*t^7
-16*a^3*b^3*s^7*t^3-16*a^3*b^3*s^3*t^5+16*a^3*b^3*s^7*t^5
-16*a^5*b*t^7*s^7-48*a^5*b*t^5*s^7-48*a^5*b*t^3*s^7
-16*a^5*b*t*s^7+b^4*t^4-16*b^5*a^3*t^7*s^7+48*b^5*a^3*t^5*s^7
+48*b^5*a^3*t^7*s^3+32*b^5*a^3*t^7*s^5+112*b^5*a^3*t^5*s^3
+160*b^5*a^3*t^5*s^5-16*b^6*s^2*t^8-24*b^6*s^4*t^8-
4*b^6*s^8*t^8-16*b^6*s^6*t^8-48*b^5*a*t^7*s^3
-48*b^5*a*t^7*s^5-16*b^5*a*t^7*s^7-16*b^5*a*t^7*s
+3*b^4*s^8*t^8+12*b^4*s^6*t^8+18*b^4*s^4*t^8+8*b^8*s^2*t^8
+12*b^8*s^4*t^8+2*b^8*s^8*t^8+8*b^8*s^6*t^8+24*a^3*b*t^5*s^5
```

```
+12*a^3*b*t^5*s^7+4*a^3*b*t^7*s^3+8*a^3*b*t^7*s^5
+12*a^3*b*t^3*s^3+24*a^3*b*t^3*s^5+12*a^3*b*t^3*s^7
+12*a^3*b*t^5*s^3-4*a^2*s^4*t^2+4*a^3*b*t*s^3+8*a^3*b*t*s^5
+4*a^3*b*t*s^7+8*a^8*s^8*t^2+8*a^8*s^8*t^6+2*a^8*s^8*t^8
+12*a^8*s^8*t^4-b^2*s^8*t^8-4*b^2*s^6*t^8-6*b^2*s^4*t^8
-4*b^2*s^2*t^8-2*b^2*s^8*t^6-8*b^2*s^6*t^6-12*b^2*s^4*t^6
-8*b^2*s^2*t^6+48*a^7*b*t^3*s^7+48*a^7*b*t^5*s^7
+16*a^7*b*t^7*s^7+16*a^7*b*t*s^7-16*a^3*b^3*s^5*t^3
+12*a^4*s^8*t^2+12*a^4*s^8*t^6+4*a^4*s^4*t^6+a^4*s^4*t^8
+3*a^4*s^8*t^8+18*a^4*s^8*t^4+6*a^4*s^4*t^4+4*a^4*s^4*t^2
-16*a^6*s^8*t^2-16*a^6*s^8*t^6-4*a^6*s^8*t^8-24*a^6*s^8*t^4
-4*a^2*s^8*t^2-4*a^2*s^8*t^6-4*a^2*s^4*t^6-a^2*s^4*t^8
-a^2*s^8*t^8-6*a^2*s^8*t^4-6*a^2*s^4*t^4+4*a^3*b*t^7*s^7
+12*a^4*s^6*t^4+2*a^4*s^6*t^8+8*a^4*s^6*t^6+12*a*b^3*t^3*s^3
+12*a*b^3*t^3*s^5+4*a*b^3*t^3*s^7+8*a*b^3*t^5*s
+24*a*b^3*t^5*s^3+12*a*b^3*t^7*s^3+12*a*b^3*t^7*s^5
+4*a*b^3*t^7*s^7+24*a*b^3*t^5*s^5+8*a*b^3*t^5*s^7
+4*a*b^3*t^7*s+4*a*b^3*t^3*s-8*b^6*a^2*s^8*t^8+b^4*s^8*t^4
+4*b^4*s^6*t^4+6*b^4*s^4*t^4+4*b^4*s^2*t^4-12*a^2*s^6*t^4
-2*a^2*s^6*t^8-b^2*s^8*t^4-4*b^2*s^6*t^4-6*b^2*s^4*t^4
```

Arose from

stability analysis of

Mc Cormack scheme for

$$\frac{\partial u}{\partial t} = p \frac{\partial u}{\partial x} + q \frac{\partial u}{\partial y}$$

- No previous methods can solve it.

  $\gg 2$ months QEPCAD

$$a^{2/3} + b^{2/3} \leq 1$$

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- Human Ad-hoc solution

  ~ 6 months

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  Huge extraneous factor!

So,

We need to find ways to avoid resultant computation.

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$$\forall \vec{y} \quad f_1(\vec{x},\vec{y})<0 \ \vee \cdots \vee f_r(\vec{x},\vec{y})<0$$

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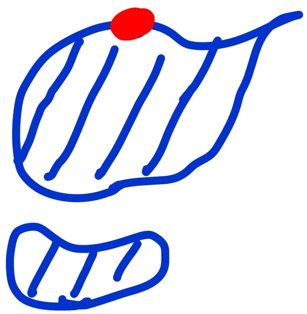

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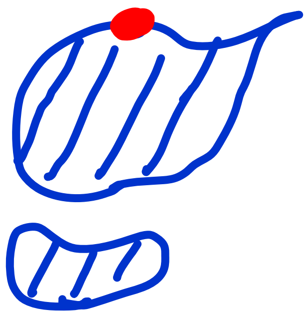

satisfies

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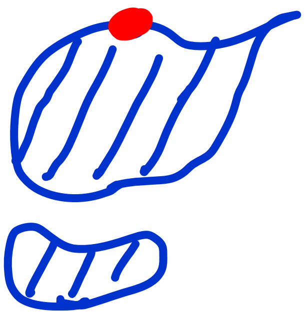

satisfies for some $j_1, \ldots, j_k$

$$f_1 = \cdots = f_r = 0$$

and

$\nabla f_{j_1}, \ldots, \nabla f_{j_k}$ lin dep.

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$f(x,y,z)=0$

$f_z(x,y,z)=0$

$\mathbb{R}^3$

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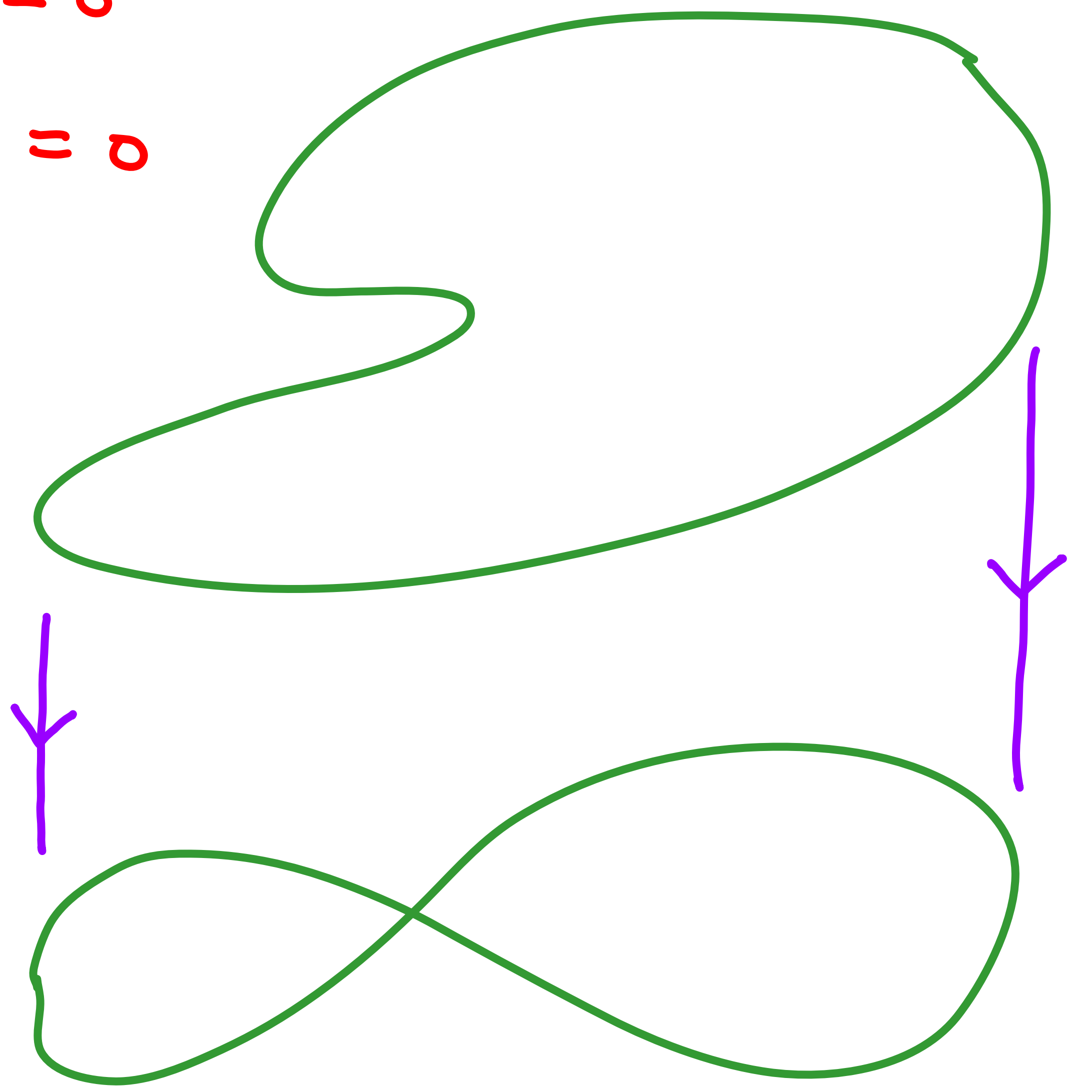

$\mathbb{R}^2$

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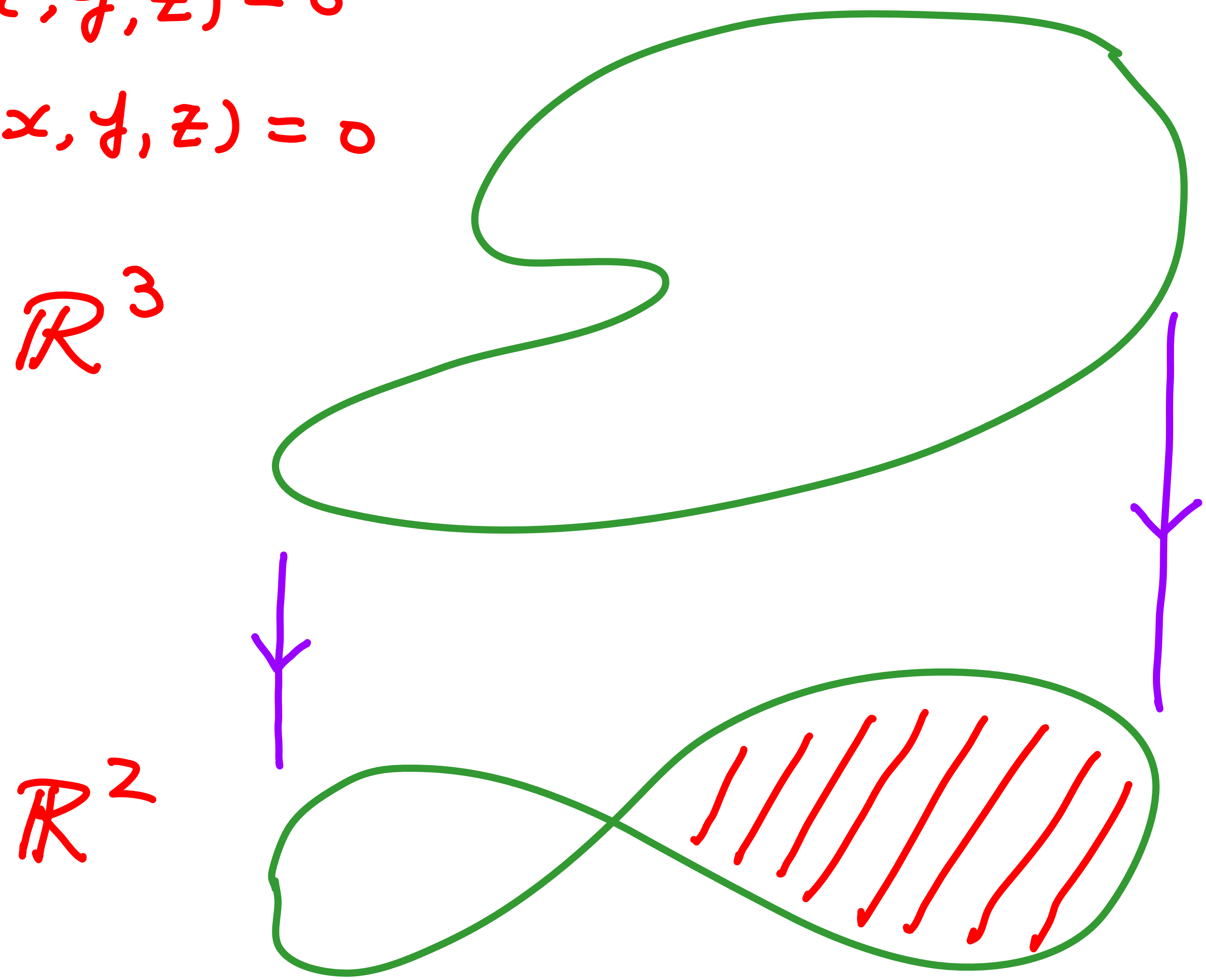

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(4.) Estimate the boundary by curve fitting

Apply to the Challenging Problem.

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$\Longrightarrow$ Curve in 4-dim space.

1. Curve Tracing
2. Project to (a,b) space

Finally . . . .

to be filled in by

to be filled in by

You!