FOUNDATIONS OF SATISFIABILITY MODULO THEORIES

SC^2 Summer School

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The University of Iowa

Many thanks to

- Clark Barrett
- Dejan Jovanovic
- Albert Oliveras

for contributing some of the material used in these slides.

Disclamer: The literature on SMT and its applications is vast. The bibliographic references provided here are just a sample. Apologies to all authors whose work is not cited.

INTRODUCTION

Historically, automated reasoning $\equiv$ uniform proof-search procedures for First Order Logic

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Limited success: is FOL the best compromise between expressivity and efficiency?

Most recently R&D has focused on:

- addressing mostly (expressive enough) decidable fragments of a certain logic
- incorporating domain-specific reasoning, e.g on:
  - arithmetic reasoning
  - equality
  - data structures (arrays, lists, stacks, ...)

Examples of this trend:

SAT: propositional formalization, Boolean reasoning

- \+ high degree of efficiency
- – expressive (all NP-complete problems) but involved encodings

SMT: first-order formalization, Boolean + domain-specific reasoning

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These lectures: an overview of SMT and its formal foundations

Some problems are more naturally expressed in logics other than propositional logic, e.g:

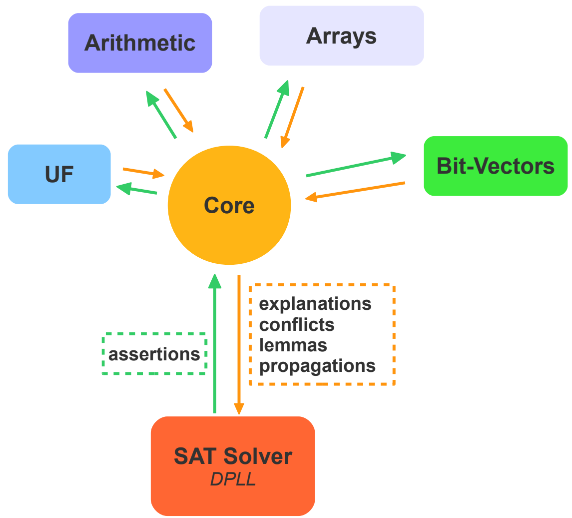
- Software verification needs reasoning about equality, arithmetic, data structures, ...

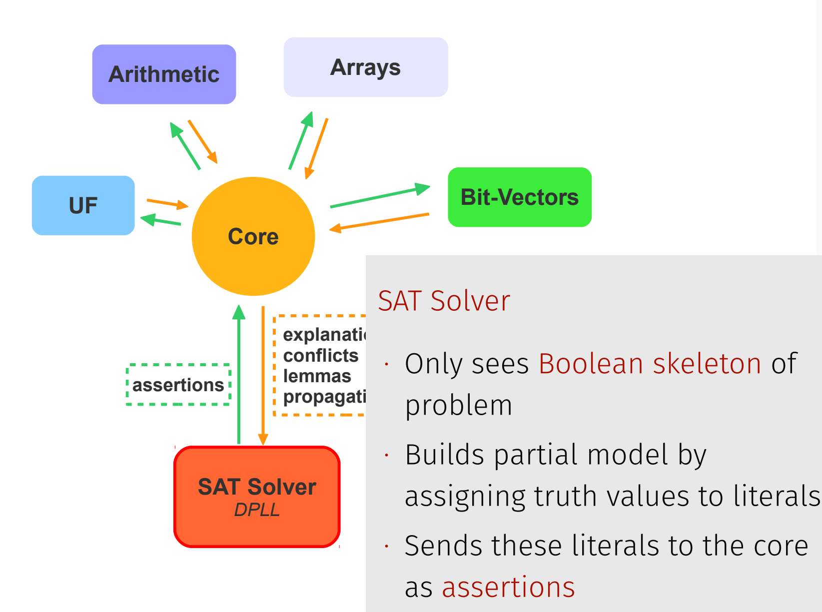
SMT is about deciding the satisfiability of a (usually quantifier-free) FOL formula with respect to some *background theory*

- Example (Equality with Uninterpreted Functions):

$$g(a) = c \quad \wedge \quad (f(g(a)) \neq f(c) \ \vee \ g(a) = d) \quad \wedge \quad c \neq d$$

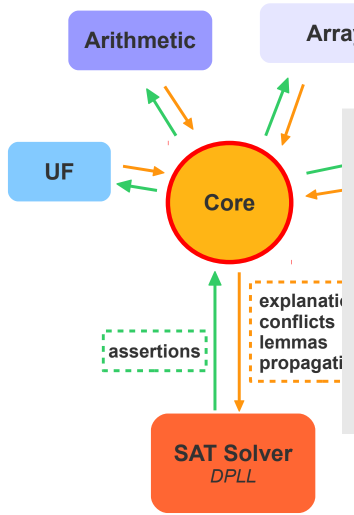
Wide range of applications: Extended Static Checking [FLL+02], Predicate abstraction [LNO06], Model checking [AMP06, HT08], Scheduling [BNO+08b], Test generation [TdH08], ...


Arithmetic
Arrays
UF
Core
Bit-Vectors
assertions
explanations
conflicts
lemmas
propagations
SAT Solver
DPLL



SAT Solver

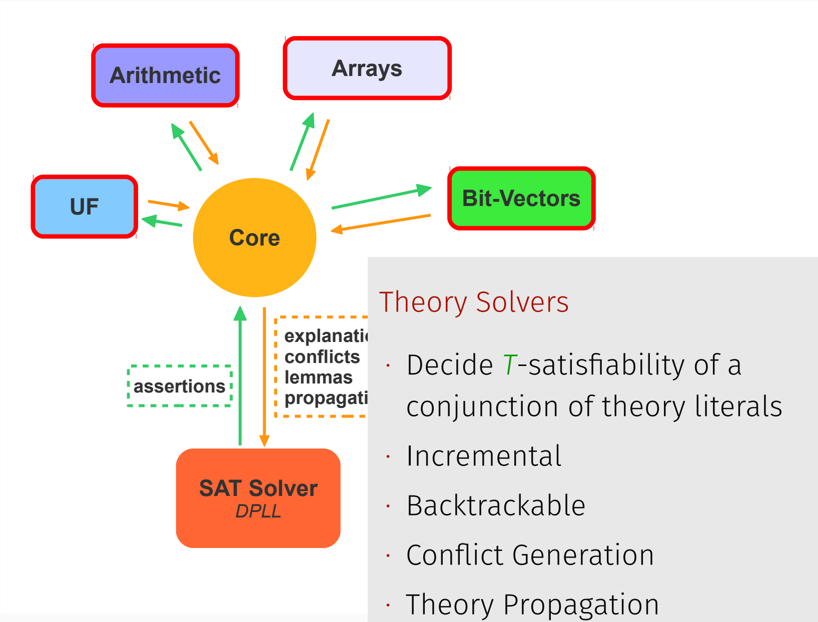
- Only sees Boolean skeleton of problem
- Builds partial model by assigning truth values to literals
- Sends these literals to the core as assertions



Core

- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles theory combination

SMT SOLVERS



Theory Solvers

- Decide T-satisfiability of a conjunction of theory literals
- Incremental
- Backtrackable
- Conflict Generation
- Theory Propagation

THEORIES

Equality ($=$) with Uninterpreted Functions [NO80, BD94, NO07a]

Typically used to abstract unsupported constructs, e.g.:

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If we abstract it to

$$mul(a, add(abs(b), c)) = d \quad \wedge \quad mul(b, add(abs(a), c)) \neq d \quad \wedge \quad a = b$$

it is still unsatisfiable

Very useful, for obvious reasons

Restricted fragments (over the reals or the integers) support more efficient methods:

- Bounds: $x \bowtie k$ with $\bowtie \in \{<, >, \leq, \geq, =\}$ [BBC+05a]
- Difference logic: $x - y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [NO05, WIGG05, CM06]
- UTVPI: $\pm x \pm y \bowtie k$, with $\bowtie \in \{<, >, \leq, \geq, =\}$ [LM05]
- Linear arithmetic, e.g: $2x - 3y + 4z \leq 5$ [DdM06a]
- Non-linear arithmetic, e.g: $2xy + 4xz^2 - 5y \leq 10$ [BLNM+09, ZM10, JdM12]

Used in software verification and hardware verification (for memories) [SBDL01, BNO+08a, dMB09a]

Two interpreted function symbols _[_] and store

Axiomatized by:

- $\forall a\, \forall i\, \forall v.\ \mathrm{store}(a, i, v)[i] = v$
- $\forall a\, \forall i\, \forall j\, \forall v.\ i \neq j \Rightarrow \mathrm{store}(a, i, v)[j] = a[j])$

Sometimes also with *extensionality*:

- $\forall a\, \forall b.\ (\forall i.\ a[i] = b[i]\ \Rightarrow a = b)$

Is the following set of literals satisfiable in this theory?

$$\mathrm{store}(a, i, x) \neq b,\ \ b[i] = y,\ \ \mathrm{store}(b, i, x)[j] = y,\ \ a = b,\ \ i = j$$

Useful both in hardware and software verification [BCF+07, BB09a, HBJ+14b]

Universe consists of (fixed-sized) vectors of bits

Different types of operations:

- *String-like*: concat, extract, ...
- *Logical*: bit-wise not, or, and, ...
- *Arithmetic*: add, subtract, multiply, ...
- *Comparison*: $<, >$, ...

Is this formula satisfiable over bit vectors of size 3?

$$a[1:0] \neq b[1:0] \ \wedge \ (a \mid b) = c \ \wedge \ c[0:0] = 0 \ \wedge \ a[1:0] + b[1:0] = 0$$

OTHER INTERESTING THEORIES

- Floating point arithmetic [BDG+14, ZWR14]
- Ordinary differential equations [GKC13]
- (Co)Algebraic data-types [BST07, RB16]
- Strings and regular expressions [LRT+14, KGG+09]
- Finite sets with cardinality [BRBT16]
- Finite relations [MRTB17]
- ...

THEORY SOLVERS

Given a theory T, a *Theory Solver* for T takes as input a set Φ of literals and determines whether Φ is T-satisfiable.

Φ is *T-satisfiable* iff there is some model M of T such that each formula in Φ holds in M.

- Literals are of the form $t_1 = t_2$ and $t_1 \neq t_2$
- Can be decided in $O(n \log(n))$ based on congruence closure
- Efficient theory propagation for equalities
- Can generate:
  - small explanations [DNS05]
  - minimal (i.e., non-redundant) explanations [NO07b]
  - smallest explanations (NP-hard) [FFHP]
- Typically the core of the SMT solver and used in other theories

Main idea: apply congruence axiom:

$$x_1 = y_1 \wedge \cdots \wedge x_n = y_n \Rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$$

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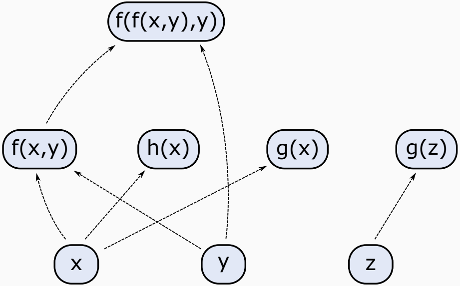
$$[\ f(x,y) = x,\ h(y) = g(y),\ f(f(x,y),y) = z,\ g(x) \neq g(z)\ ]$$



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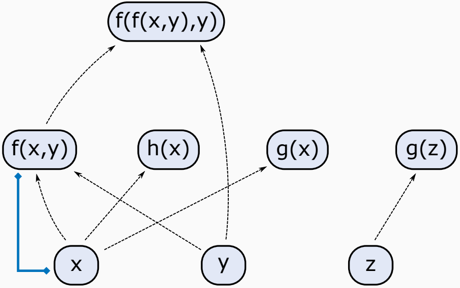

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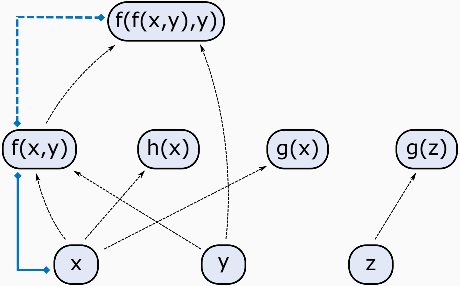

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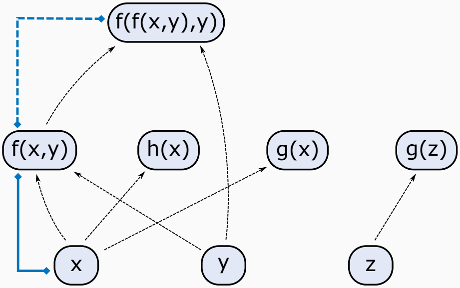

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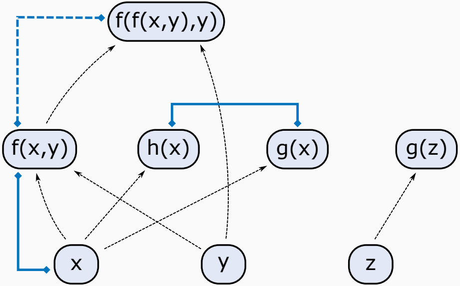

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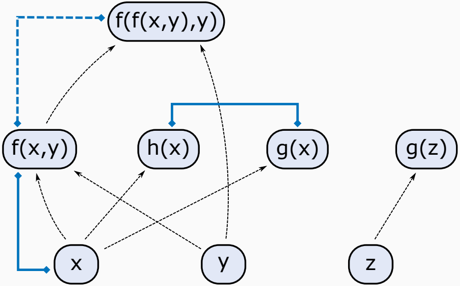

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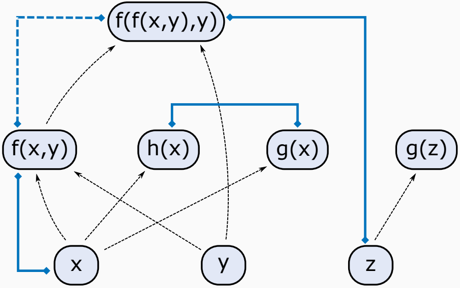

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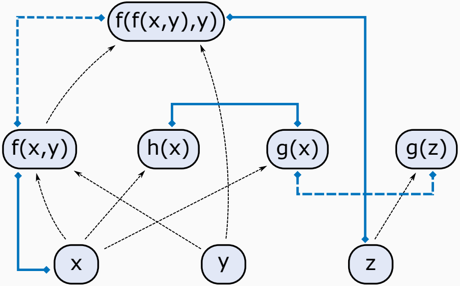

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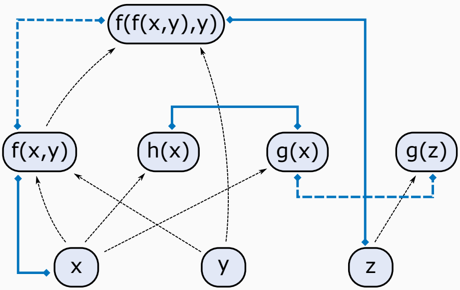

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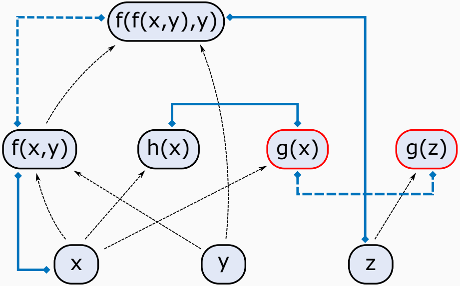

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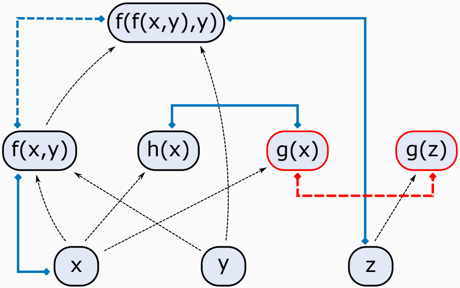

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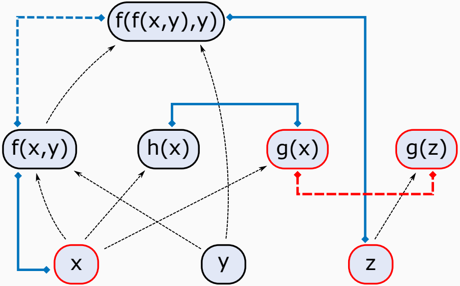

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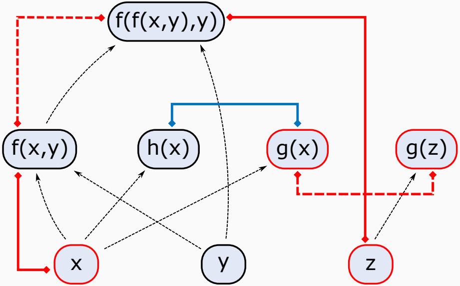

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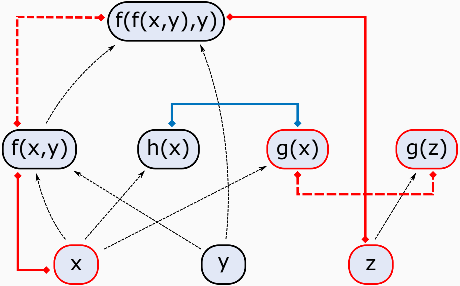
2. $f(f(x,y),y) = z$



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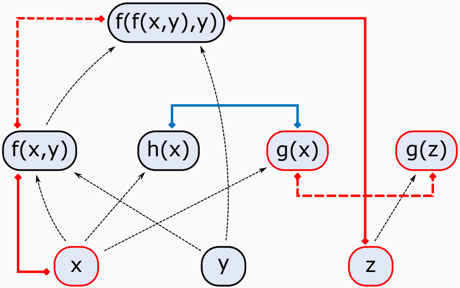
3. $f(x,y) = x$



$$\begin{aligned}
\forall a, i, e &: \ \mathrm{store}(a, i, e)[i] = e \\
\forall a, i, j, e &: \ i \neq j \Rightarrow \mathrm{store}(a, i, e)[j] = a[j] \\
\forall a, b &: \ a \neq b \Rightarrow \exists i : a[i] \neq b[i]
\end{aligned}$$

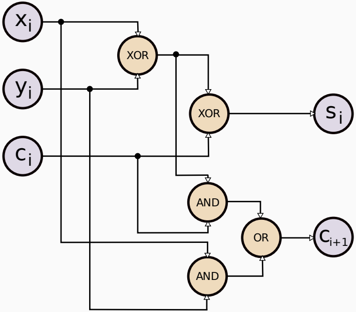
Common approach:

- UF + lemmas on demand [BB09b, DMB09b]
- Use EUF as if store and _[_] were uninterpreted
- If UNSAT in EUF, then UNSAT in arrays too
- If SAT and solution satisfies array axioms, then SAT (lucky case)
- If not, then refine by instantiating violated axioms

Common approach:

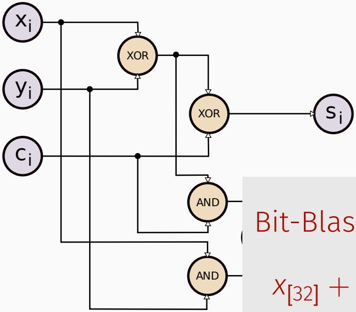
1. Simplify/preprocess (heavily)
2. Encode to SAT (aka, bit blasting)
3. Send to a SAT solver

Alternatives [HBJ+14a, ZWR16] not yet mature



Translation to CNF

- Each node a new variables
- XOR introduces 4 clauses
- AND introduces 3 clauses
- OR introduces 3 clauses
- 17 new clauses
- 5 new variables



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Bit-Blasting Addition/Multiplication

$x_{[32]} + y_{[32]}$ 544 new clauses, 160 new variables

$x_{[32]} \times y_{[32]}$ 10016 new clauses, 3008 new variables

Language:

- Literals of the form

$$x - y \leq k$$

  with x and y variables (integer or real) and k constant (integer or real)

- Reductions:

$$\begin{array}{lcll} x - y = k & \longrightarrow & x - y \leq k \ \wedge \ y - x \leq k & \\ x - y < k & \longrightarrow & x - y \leq k - 1 & \text{(integers)} \\ x - y < k & \longrightarrow & x - y \leq k - \delta & \text{(reals)} \end{array}$$

- Any solution to a set of literals can be shifted:
  - if v is a satisfying assignment, so is $v' = \lambda x.\ v(x) + k$
- We can use this to also process simple bounds $x \leq k$:
  - introduce fresh variable z (for zero),
  - rewrite each $x \leq k$ to $x - z \leq k$,
  - given a solution v, shift it so that $v'(z) = 0$
- If we allow (dis)equalities as literals,
  - in reals, satisfiability is polynomial
  - in integers, satisfiability is NP-hard

Common approach: Cycle detection

1. Construct a graph from literals
2. Check if there is a negative path

Theorem *Literals unsatisfiable* $\Leftrightarrow \exists$ *negative path*

DIFFERENCE LOGIC: FROM LITERALS TO GRAPH

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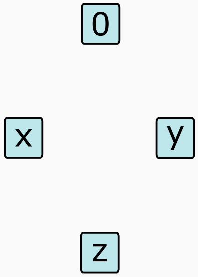
Theorem *Literals unsatisfiable* $\Leftrightarrow \exists$ *negative path*

Example

$$[\, x \leq 1,\ x - y \leq 2,\ y - z \leq 3,\ z - x \leq -6 \,]$$



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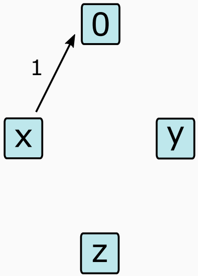

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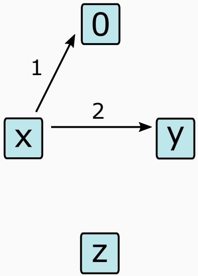

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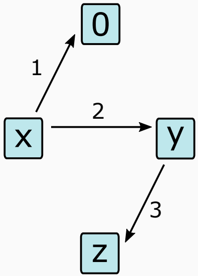

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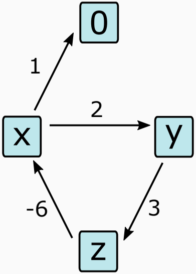

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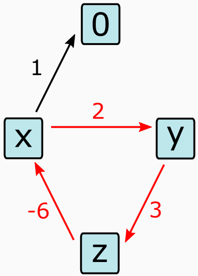

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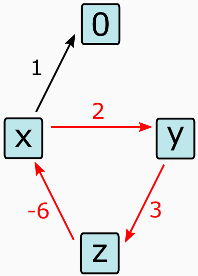
Conflict Set:

$x - y \leq 2,\ y - z \leq 3,\ z - x \leq -6$



Language:

- Literals of the form $a_1x_1 + \cdots + a_nx_n \bowtie b$ with a_1 positive and $\bowtie \in \{\leq, \geq\}$
- Reductions:

$$\begin{array}{lcll} t = b & \longrightarrow & t \leq b \ \wedge \ t \geq b & \\ t < b & \longrightarrow & t \leq b - 1 & \text{(integer arith.)} \\ t < b & \longrightarrow & t \leq b - \delta & \text{(real arith.)} \end{array}$$

Common approach: variant of Simplex designed for SMT [DDM06b]

- Incremental
- Cheap backtracking
- Can do theory propagation
- Can generate minimal explanations
- Worst case exponential but fast in practice

Rewrite each $\sum a_i x_i \bowtie b$ as $s \bowtie b$ with $s = \sum a_i x_i$

We get tableau of equations + simple bounds on variables

- Tableau is fixed (modulo pivoting and substitutions)
- Bounds can be asserted and retracted

Tableau

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

$$
\begin{gathered}
\vdots \\
l_i \le s_i \le u_i \\
\vdots \\
l_j \le x_j \le u_j \\
\vdots
\end{gathered}
$$

Tableau

$$s_1 = a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n$$

$$\vdots$$

$$s_i = a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n$$

$$\vdots$$

$$s_m = a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n$$

Bounds

$$\vdots$$

$$l_i \leq s_i \leq u_i$$

$$\vdots$$

$$l_j \leq x_j \leq u_j$$

$$\vdots$$

Tableau

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,j} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,j} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,j} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

- lhs variables are basic, rhs variables are non-basic

Tableau | **Bounds**

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

- lhs variables are basic, rhs variables are non-basic

Tableau

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

- lhs variables are basic, rhs variables are non-basic

Tableau

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

- lhs variables are basic, rhs variables are non-basic
- Keep an assignment v of all variables:
  - v satisfies the tableau,
  - v satisfies bounds on the non-basic variables

Tableau

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

- lhs variables are basic, rhs variables are non-basic
- Keep an assignment v of all variables:
  - v satisfies the tableau,
  - v satisfies bounds on the non-basic variables
- Initially $v(x) = 0$ and $-\infty \leq x \leq +\infty$

Tableau | **Bounds**

$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

Case 1:

- v satisfies bound on the basic variables too
- Satisfiable, v is the model!

Tableau	Bounds
$s_1 = a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n$	$\vdots$
$\vdots$	$l_i \leq s_i \leq u_i$
$s_i = a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n$	$\vdots$
$\vdots$	$l_j \leq x_j \leq u_j$
$s_m = a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n$	$\vdots$

Case 2:

- v doesn't satisfy bound on some s_i and all x_j's that s_i depends on are at their bounds (can't fix)
- Unsatisfiable, the row is the explanation

Tableau

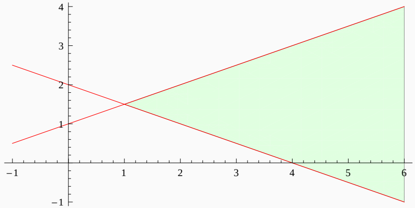
$$
\begin{aligned}
s_1 &= a_{1,1} \cdot x_1 + \cdots + a_{1,i} \cdot x_j + \cdots + a_{1,n} \cdot x_n \\
&\vdots \\
s_i &= a_{i,1} \cdot x_1 + \cdots + a_{i,i} \cdot x_j + \cdots + a_{i,n} \cdot x_n \\
&\vdots \\
s_m &= a_{m,1} \cdot x_1 + \cdots + a_{m,i} \cdot x_j + \cdots + a_{m,n} \cdot x_n
\end{aligned}
$$

Bounds

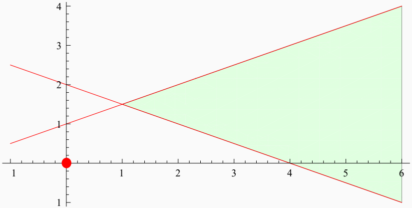
$$
\begin{gathered}
\vdots \\
l_i \leq s_i \leq u_i \\
\vdots \\
l_j \leq x_j \leq u_j \\
\vdots
\end{gathered}
$$

Case 3:

- v doesn't satisfy bound on some s_i, and some x_j's that s_i depends on has some slack
- Pivot, substitute, and continue

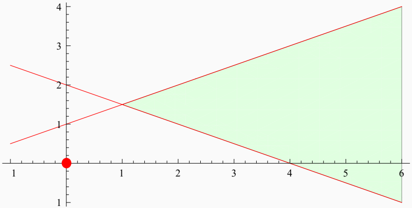


$$[\; 2y - x - 2 \leq 0,\;\; -2y - x + 4 \leq 0 \;]$$



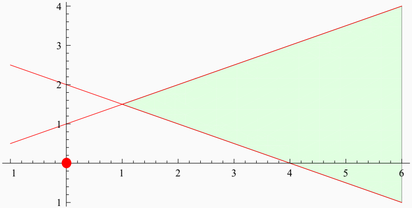
$$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$$

Tableau	Bounds	Assignment
	$-\infty \leq x \leq +\infty$	$x \mapsto 0$
$s_1 = 2y - x$	$-\infty \leq y \leq +\infty$	$y \mapsto 0$
$s_2 = -2y - x$	$-\infty \leq s_1 \leq +\infty$	$s_1 \mapsto 0$
	$-\infty \leq s_2 \leq +\infty$	$s_2 \mapsto 0$



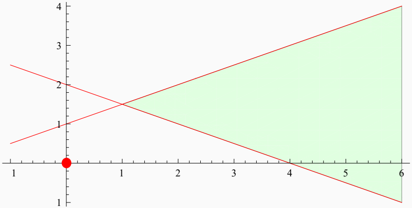
$$[\, s_1 \le 2,\ s_2 \le -4 \,]$$

Tableau	Bounds	Assignment
	$-\infty \le x \le +\infty$	$x \mapsto 0$
$s_1 = 2y - x$	$-\infty \le y \le +\infty$	$y \mapsto 0$
$s_2 = -2y - x$	$-\infty \le s_1 \le +\infty$	$s_1 \mapsto 0$
	$-\infty \le s_2 \le +\infty$	$s_2 \mapsto 0$



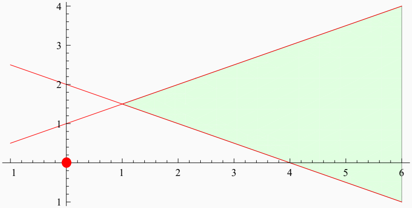
$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau	Bounds	Assignment
	$-\infty \leq x \leq +\infty$	$x \mapsto 0$
$s_1 = 2y - x$	$-\infty \leq y \leq +\infty$	$y \mapsto 0$
$s_2 = -2y - x$	$-\infty \leq s_1 \leq 2$	$s_1 \mapsto 0$
	$-\infty \leq s_2 \leq +\infty$	$s_2 \mapsto 0$



$$[\, s_1 \leq 2,\; s_2 \leq -4 \,]$$

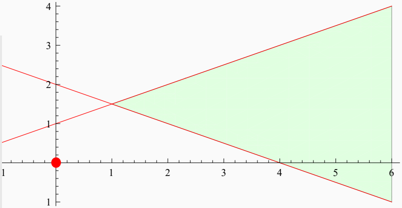
Tableau	Bounds	Assignment
	$-\infty \leq x \leq +\infty$	$x \mapsto 0$
$s_1 = 2y - x$	$-\infty \leq y \leq +\infty$	$y \mapsto 0$
$s_2 = -2y - x$	$-\infty \leq s_1 \leq 2$	$s_1 \mapsto 0$
	$-\infty \leq s_2 \leq +\infty$	$s_2 \mapsto 0$



$$[\ s_1 \leq 2,\ s_2 \leq -4\ ]$$

Tableau	Bounds	Assignment
	$-\infty \leq x \leq +\infty$	$x \mapsto 0$
$s_1 = 2y - x$	$-\infty \leq y \leq +\infty$	$y \mapsto 0$
$s_2 = -2y - x$	$-\infty \leq s_1 \leq 2$	$s_1 \mapsto 0$
	$-\infty \leq s_2 \leq -4$	$s_2 \mapsto 0$

s_2's bound violated



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = 2y - x$$
$$s_2 = -2y - x$$

Bounds

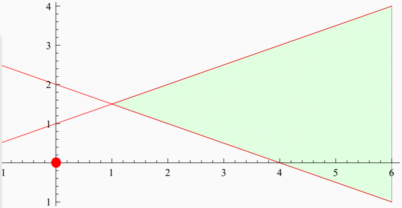
$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$
$$y \mapsto 0$$
$$s_1 \mapsto 0$$
$$s_2 \mapsto 0$$

s_2's bound violated

- There is slack in y



$[\ s_1 \leq 2,\ s_2 \leq -4\ ]$

Tableau

$$s_1 = 2y - x$$
$$s_2 = -2y - x$$

Bounds

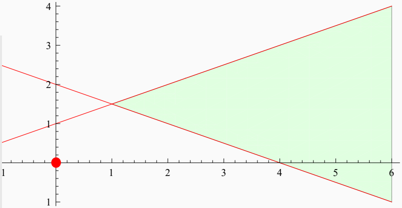
$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$
$$y \mapsto 0$$
$$s_1 \mapsto 0$$
$$s_2 \mapsto 0$$

s_2's bound violated

- There is slack in y
- Pivot s_2 and y



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Bounds

Assignment

Tableau

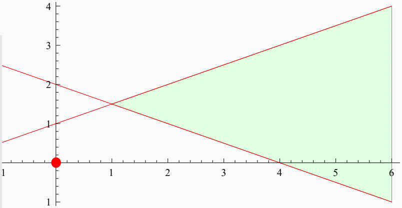
$$s_1 = 2y - x$$
$$s_2 = -2y - x$$

$-\infty \leq x \leq +\infty$	$x \mapsto 0$
$-\infty \leq y \leq +\infty$	$y \mapsto 0$
$-\infty \leq s_1 \leq 2$	$s_1 \mapsto 0$
$-\infty \leq s_2 \leq -4$	$s_2 \mapsto 0$

LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_2's bound violated

- There is slack in y
- Pivot s_2 and y



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$

$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$

$$-\infty \leq y \leq +\infty$$

$$-\infty \leq s_1 \leq 2$$

$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$

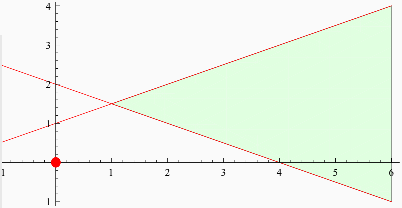
$$y \mapsto 0$$

$$s_1 \mapsto 0$$

$$s_2 \mapsto 0$$

s_2's bound violated

- There is slack in y
- Pivot s_2 and y
- Update s_2 value



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$
$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

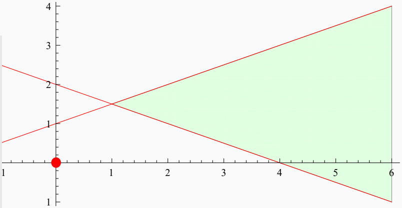
Assignment

$$x \mapsto 0$$
$$y \mapsto 0$$
$$s_1 \mapsto 0$$
$$s_2 \mapsto 0$$

LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_2's bound violated

- There is slack in y
- Pivot s_2 and y
- Update s_2 value



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$

$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$

$$-\infty \leq y \leq +\infty$$

$$-\infty \leq s_1 \leq 2$$

$$-\infty \leq s_2 \leq \ -4$$

Assignment

$$x \mapsto 0$$

$$y \mapsto 0$$

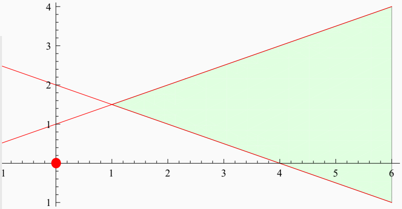
$$s_1 \mapsto 0$$

$$s_2 \mapsto -4$$

LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_2's bound violated

- There is slack in y
- Pivot s_2 and y
- Update s_2 value
- Update basic vars



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$

$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$

$$-\infty \leq y \leq +\infty$$

$$-\infty \leq s_1 \leq 2$$

$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$

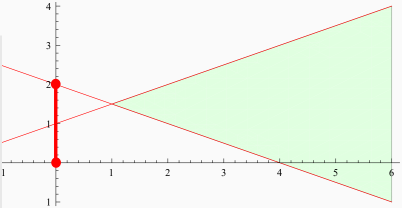
$$y \mapsto 0$$

$$s_1 \mapsto 0$$

$$s_2 \mapsto -4$$

s_2's bound violated

- There is slack in y
- Pivot s_2 and y
- Update s_2 value
- Update basic vars



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$
$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

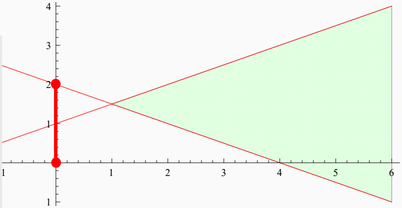
$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$
$$y \mapsto 2$$
$$s_1 \mapsto 4$$
$$s_2 \mapsto -4$$

LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_1's bound violated



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$

$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$

$$-\infty \leq y \leq +\infty$$

$$-\infty \leq s_1 \leq 2$$

$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$

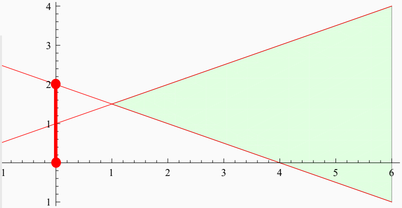
$$y \mapsto 2$$

$$s_1 \mapsto 4$$

$$s_2 \mapsto -4$$

s_1's bound violated

- There is slack in x



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$
$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

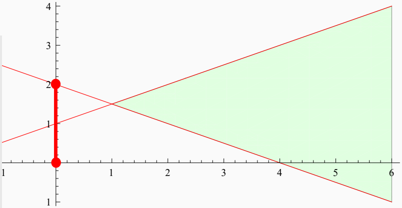
Assignment

$$x \mapsto 0$$
$$y \mapsto 2$$
$$s_1 \mapsto 4$$
$$s_2 \mapsto -4$$

LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_1's bound violated

- There is slack in x
- Pivot s_1 and x



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$s_1 = -s_2 - 2x$$
$$y = -\frac{1}{2}s_2 - \frac{1}{2}x$$

Bounds

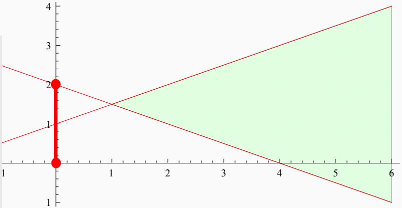
$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$
$$y \mapsto 2$$
$$s_1 \mapsto 4$$
$$s_2 \mapsto -4$$

s_1's bound violated

- There is slack in x
- Pivot s_1 and x



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$x = -\frac{1}{2}s_1 - \frac{1}{2}s_2$$
$$y = \frac{1}{4}s_1 - \frac{1}{4}s_2$$

Bounds

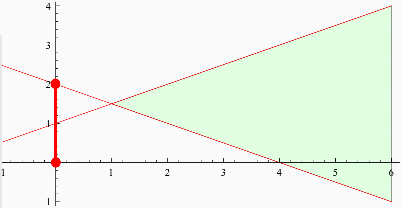
$$-\infty \leq x \leq +\infty$$
$$-\infty \leq y \leq +\infty$$
$$-\infty \leq s_1 \leq 2$$
$$-\infty \leq s_2 \leq -4$$

Assignment

$$x \mapsto 0$$
$$y \mapsto 2$$
$$s_1 \mapsto 4$$
$$s_2 \mapsto -4$$

s_1's bound violated

- There is slack in x
- Pivot s_1 and x
- Update s_1 value



$[\, s_1 \leq 2,\ s_2 \leq -4 \,]$

Tableau

$$x = -\frac{1}{2}s_1 - \frac{1}{2}s_2$$
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Bounds

$$-\infty \leq x \leq +\infty$$
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Assignment

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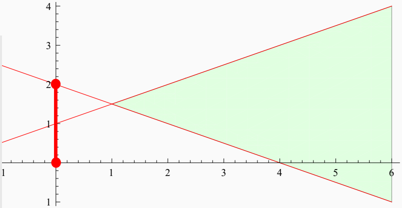
LINEAR RATIONAL ARITHMETIC: EXAMPLE

s_1's bound violated

- There is slack in x
- Pivot s_1 and x
- Update s_1 value



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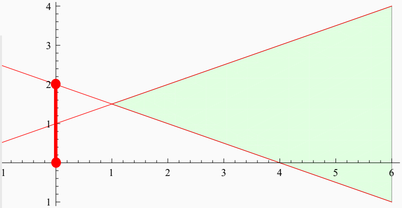

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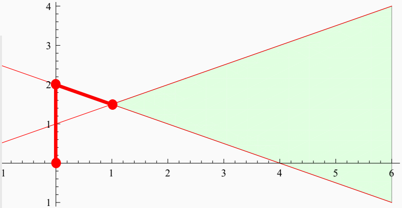

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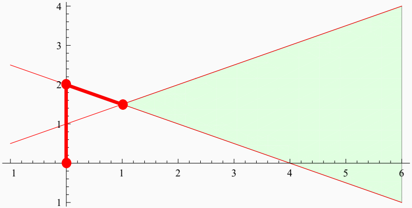

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Classic NP-complete problem [Pap81]

Admits quantifier elimination [Coo72]

Common approach:

- Simplex + Branch-And-Bound [DDM06b, Gri12, Kin14]
- Use Simplex to solve real relaxation (treat variables as real)
- If UNSAT over reals, then UNSAT over integers too
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Alternatives [JdM13, BSW15] not yet mature

$$f(\mathbf{y}, x) = a_m \cdot x^{d_m} + a_{m-1} \cdot x^{d_{m-1}} + \cdots + a_1 \cdot x^{d_1} + a_0$$

f is in $\mathbb{Z}[\mathbf{y}, x]$, a_i are in $\mathbb{Z}[\mathbf{y}]$

Examples

$$\begin{aligned} f(x, y) &= (x^2 - 1)y^2 + (x + 1)y - 1 \in \mathbb{Z}[x, y] \\ g(x) &= 16x^3 - 8x^2 + x + 16 \in \mathbb{Z}[x] \end{aligned}$$

Polynomial Constraints

$$f(x, y) > 0 \wedge g(x) < 0$$

$$p_1 > 0 \lor (p_2 = 0 \land p_3 < 0) \qquad p_1, p_2, p_3 \in \mathbb{Z}[x_1, \dots, x_n]$$

Projection (Saturation)

Project polynomials using a projection P

$$\{p_1, p_2, p_3\} \mapsto \{p_1, p_2, p_3, p_4, \dots, p_n\}$$

Lifting (Model construction)

For each variable x_k

1. Isolate roots of $p_i(\alpha, x_k)$
2. Choose a cell C and assign $x_k \mapsto \alpha_k \in C$, continue
3. If no more cells, backtrack

Model Construction

Build partial model by assigning variables to values

$$[\ldots, C_1, C_2, \ldots, x \mapsto \sqrt{2}/2, \ldots]$$

Unit Reasoning

Reason about unit constraints

$$C_1 \equiv (x^2 + y^2 < 1) \qquad C_2 \equiv (xy > 1)$$

Explain Conflicts

Explain conflicts using valid clausal reasons

$$\overline{C_1} \vee \overline{C_2} \vee x \leq 0 \vee x \geq 1$$

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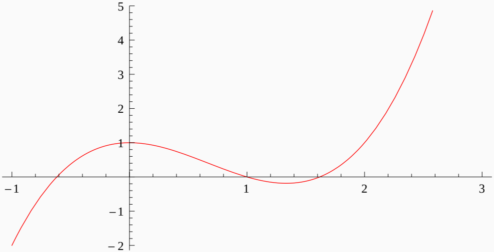
NON-LINEAR REAL ARITHMETIC



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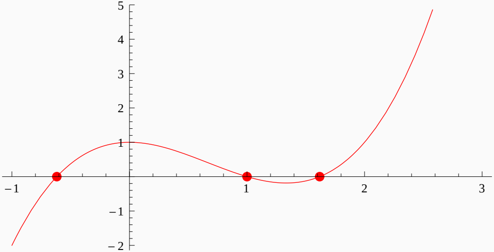

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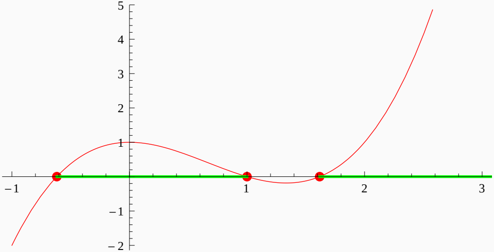

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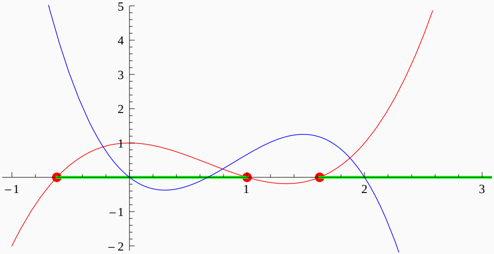

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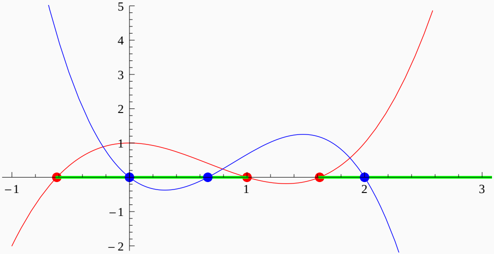

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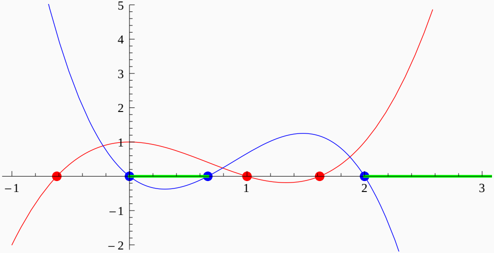

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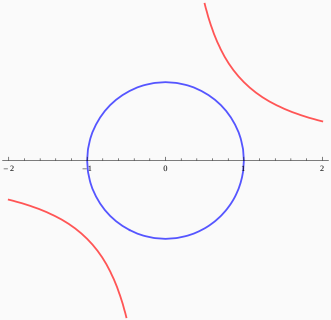

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$$[]$$

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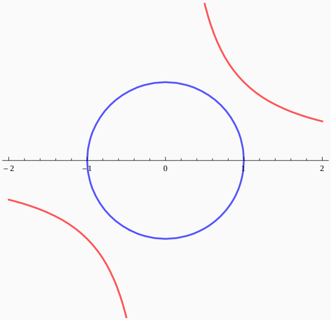

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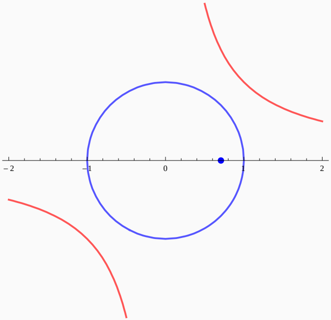

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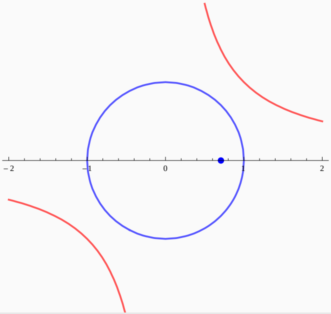
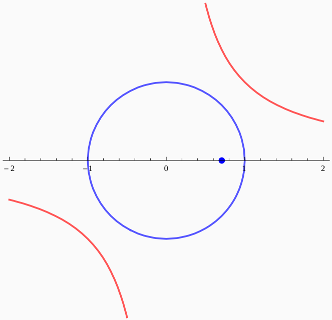


Unit Constraint Reasoning

$$x^2 + y^2 < 1 \Rightarrow -\sqrt{3/2} < y < \sqrt{3/2}$$

$$-2y - x + 4 < 0 \Rightarrow y > \sqrt{2}$$



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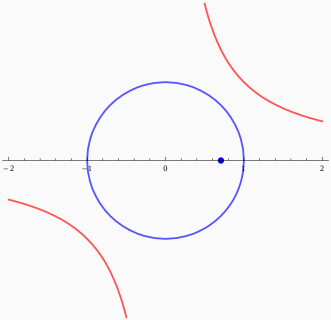

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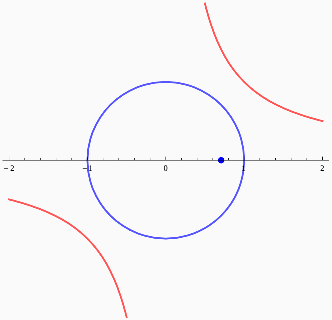

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CAD Projection

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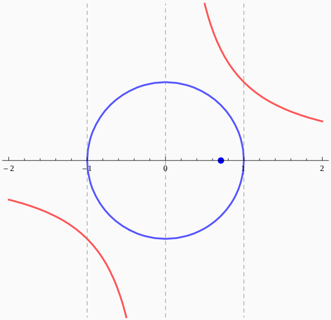

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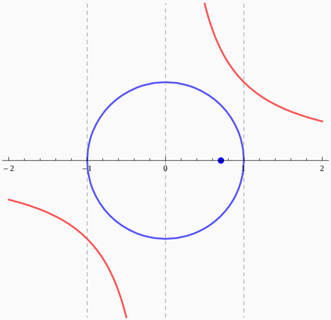

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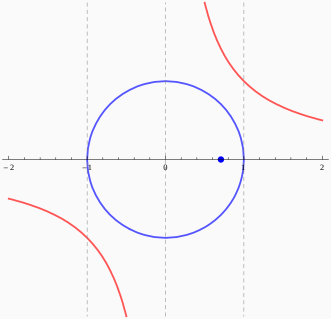

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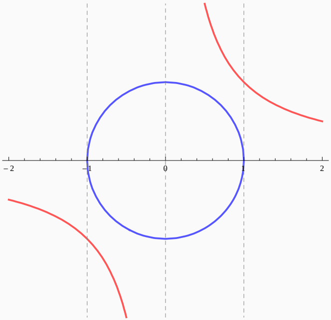

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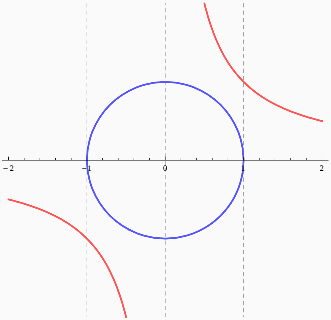

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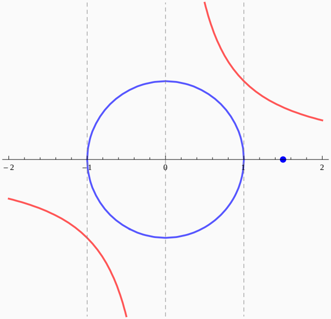

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EXTENDING THEORY SOLVERS TO QFFS

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Solution: Exploit propositional satisfiability technology

Two main approaches:

1. "Eager" [PRSS99, SSB02, SLB03, BGV01, BV02]

- translate into an equisatisfiable propositional formula
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Notable systems: UCLID

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Theory T: Equality with Uninterpreted Functions

Simplest setting:

- Off-line SAT solver
- Non-incremental *theory solver* for conjunctions of equalities and disequalities
- Theory atoms (e.g., $g(a) = c$) abstracted to propositional atoms (e.g., 1)

(VERY) LAZY APPROACH FOR SMT – EXAMPLE

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- Send $\{1,\ \overline{2} \vee 3,\ \overline{4}\}$ to SAT solver.
- SAT solver returns model $\{1,\ \overline{2},\ \overline{4}\}$.
  Theory solver finds (concretization of) $\{1,\ \overline{2},\ \overline{4}\}$ unsat.
- Send $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4\}$ to SAT solver.
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- Send $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2,\ \overline{1} \vee \overline{3} \vee 4\}$ to SAT solver.
- SAT solver finds $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4\}$ unsat.
  Done: the original formula is unsatisfiable in UF.

Several enhancements are possible to increase efficiency:

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- If M is T-unsatisfiable, backtrack to some point where the assignment was still T-satisfiable

LAZY APPROACH – MAIN BENEFITS

- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information

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- Every tool does what it is good at:
  - SAT solver takes care of Boolean information
  - Theory solver takes care of theory information
- The theory solver works only with conjunctions of literals
- Modular approach:
  - SAT and theory solvers communicate via a simple API [GHN+04]
  - SMT for a new theory only requires new theory solver
  - An off-the-shelf SAT solver can be embedded in a lazy SMT system with few new lines of code (tens)

Several variants and enhancements of lazy SMT solvers exist

They can be modeled abstractly and declaratively as *transition systems*

A transition system is a binary relation over states, induced by a set of conditional transition rules

The framework can be first developed for SAT and then extended to lazy SMT [NOT06, KG07]

An abstract framework helps one:

- skip over implementation details and unimportant control aspects
- reason formally about solvers for SAT and SMT
- model advanced features such as non-chronological bactracking, lemma learning, theory propagation, ...
- describe different strategies and prove their correctness
- compare different systems at a higher level
- get new insights for further enhancements

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The one described next is a re-elaboration of those in [NOT06, KG07]

- Modern SAT solvers are based on the DPLL procedure [DP60, DLL62]
- DPLL tries to build incrementally a satisfying truth assignment M for a CNF formula F
- M is grown by
  - deducing the truth value of a literal from M and F, or
  - guessing a truth value
- If a wrong guess for a literal leads to an inconsistency, the procedure backtracks and tries the opposite value

States:

$$\text{fail} \quad \text{or} \quad \langle M, F \rangle$$

where

- M is a sequence of literals and *decision points* $\bullet$ denoting a partial truth *assignment*
- F is a set of clauses denoting a CNF *formula*

Def. If $M = M_0 \bullet M_1 \bullet \cdots \bullet M_n$ where each M_i contains no decision points

- M_i is *decision level i* of M
- $M^{[i]} \stackrel{\text{def}}{=} M_0 \bullet \cdots \bullet M_i$

States:

$$\text{fail} \quad \text{or} \quad \langle M, F \rangle$$

Initial state:

- $\langle (), F_0 \rangle$, where F_0 is to be checked for satisfiability

Expected final states:

- fail if F_0 is unsatisfiable
- $\langle M, G \rangle$ otherwise, where
  - G is equivalent to F_0 and
  - M satisfies G

States treated like records:

- M denotes the truth assignment component of current state
- F denotes the formula component of current state

Transition rules in *guarded assignment form* [KG07]

$$\frac{p_1 \quad \cdots \quad p_n}{[\mathsf{M} := e_1] \quad [\mathsf{F} := e_2]}$$

updating M, F or both when premises $p_1, \ldots, p_n$ all hold

Extending the assignment

Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Note: When convenient, treat M as a set

Note: Clauses are treated modulo ACI of $\vee$

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Propagate $$\frac{l_1 \vee \cdots \vee l_n \vee l \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

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Note: Clauses are treated modulo ACI of $\vee$

Decide $$\frac{l \in \mathrm{Lit}(\mathsf{F}) \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M} \bullet l}$$

Note: $\mathrm{Lit}(F) \stackrel{\mathrm{def}}{=} \{l \mid l \text{ literal of } F\} \cup \{\bar{l} \mid l \text{ literal of } F\}$

Repairing the assignment

$$\textbf{Fail}\quad \frac{l_1 \vee \cdots \vee l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M \quad \bullet \notin M}{\text{fail}}$$

Repairing the assignment

Fail $$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \bullet \notin \mathsf{M}}{\mathsf{fail}}$$

Backtrack

$$\frac{l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M} \quad \mathsf{M} = M \bullet l\, N \quad \bullet \notin N}{\mathsf{M} := M\, \bar{l}}$$

Note: Last premise of **Backtrack** enforces chronological backtracking

To model conflict-driven backjumping and learning, add to states a third component C whose value is either no or a *conflict clause*

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States: fail or $\langle M, F, C\rangle$

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Expected final states:

- fail if F_0 is unsatisfiable
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Replace Backtrack with

Replace Backtrack with

Conflict $$\frac{C = \text{no} \quad l_1 \vee \cdots \vee l_n \in F \quad \bar{l}_1, \ldots, \bar{l}_n \in M}{C := l_1 \vee \cdots \vee l_n}$$

Explain $$\frac{C = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in F \quad \bar{l}_1, \ldots, \bar{l}_n \prec_M \bar{l}}{C := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump $$\frac{C = l_1 \vee \cdots \vee l_n \vee l \quad \text{lev}\,\bar{l}_1, \ldots, \text{lev}\,\bar{l}_n \leq \ i < \text{lev}\,\bar{l}}{C := \text{no} \quad M := M^{[i]}\, l}$$

Note: $l \prec_M l'$ if l occurs before l' in M
$\text{lev}\, l = i$ iff l occurs in decision level i of M

Replace **Backtrack** with

Conflict $$\frac{\mathsf{C} = \mathsf{no} \quad l_1 \vee \cdots \vee l_n \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \in \mathsf{M}}{\mathsf{C} := l_1 \vee \cdots \vee l_n}$$

Explain $$\frac{\mathsf{C} = l \vee D \quad l_1 \vee \cdots \vee l_n \vee \bar{l} \in \mathsf{F} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Backjump $$\frac{\mathsf{C} = l_1 \vee \cdots \vee l_n \vee l \quad \mathsf{lev}\,\bar{l}_1, \ldots, \mathsf{lev}\,\bar{l}_n \leq \; i < \mathsf{lev}\,\bar{l}}{\mathsf{C} := \mathsf{no} \quad \mathsf{M} := \mathsf{M}^{[i]}\, l}$$

Maintain invariant: $\mathsf{F} \models_{\mathrm{p}} \mathsf{C}$ and $\mathsf{M} \models_{\mathrm{p}} \neg\mathsf{C}$ when $\mathsf{C} \neq \mathsf{no}$

Note: $\models_{\mathrm{p}}$ denotes propositional entailment

Modify Fail to

Modify **Fail** to

$$\textbf{Fail}\ \frac{C \neq \mathrm{no} \quad \bullet \notin M}{\mathrm{fail}}$$

EXECUTION EXAMPLE

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	

EXECUTION EXAMPLE

$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$

M	F	C	rule
	F	no	
1	F	no	by Propagate

EXECUTION EXAMPLE

$$F := \{1,\ \ \bar{1} \vee 2,\ \ \bar{3} \vee 4,\ \ \bar{5} \vee \bar{6},\ \ \bar{1} \vee \bar{5} \vee 7,\ \ \bar{2} \vee \bar{5} \vee 6 \vee \bar{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by Propagate
1 2	F	no	by Propagate

EXECUTION EXAMPLE

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
1 2	F	no	by **Propagate**
1 2 $\bullet$ 3	F	no	by **Decide**

EXECUTION EXAMPLE

$$F := \{1,\ \ \overline{1} \lor 2,\ \ \overline{3} \lor 4,\ \ \overline{5} \lor \overline{6},\ \ \overline{1} \lor \overline{5} \lor 7,\ \ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by Propagate
$1\ 2$	F	no	by Propagate
$1\ 2 \bullet 3$	F	no	by Decide
$1\ 2 \bullet 3\ 4$	F	no	by Propagate

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by Propagate
$1\ 2$	F	no	by Propagate
$1\ 2 \bullet 3$	F	no	by Decide
$1\ 2 \bullet 3\ 4$	F	no	by Propagate
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by Decide

EXECUTION EXAMPLE

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by Propagate
$1\ 2$	F	no	by Propagate
$1\ 2 \bullet 3$	F	no	by Decide
$1\ 2 \bullet 3\ 4$	F	no	by Propagate
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by Decide
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by Propagate

EXECUTION EXAMPLE

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

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	F	no	
1	F	no	by Propagate
$1\ 2$	F	no	by Propagate
$1\ 2 \bullet 3$	F	no	by Decide
$1\ 2 \bullet 3\ 4$	F	no	by Propagate
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by Decide
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by Propagate
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by Propagate

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**

$$F := \{1,\ \bar{1} \vee 2,\ \bar{3} \vee 4,\ \bar{5} \vee \bar{6},\ \bar{1} \vee \bar{5} \vee 7,\ \bar{2} \vee \bar{5} \vee 6 \vee \bar{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \bar{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \bar{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \bar{6}\ 7$	F	$\bar{2} \vee \bar{5} \vee 6 \vee \bar{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \bar{6}\ 7$	F	$1 \vee \bar{2} \vee \bar{5} \vee 6$	by **Explain** with $\bar{1} \vee \bar{5} \vee 7$

$$F := \{1,\ \overline{1} \lor 2,\ \overline{3} \lor 4,\ \overline{5} \lor \overline{6},\ \overline{1} \lor \overline{5} \lor 7,\ \overline{2} \lor \overline{5} \lor 6 \lor \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \lor \overline{5} \lor 6 \lor \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5} \lor 6$	by **Explain** with $\overline{1} \lor \overline{5} \lor 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \lor \overline{2} \lor \overline{5}$	by **Explain** with $\overline{5} \lor \overline{6}$

$$F := \{1,\ \overline{1} \vee 2,\ \overline{3} \vee 4,\ \overline{5} \vee \overline{6},\ \overline{1} \vee \overline{5} \vee 7,\ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**

EXECUTION EXAMPLE

$$F := \{1,\ \ \overline{1} \vee 2,\ \ \overline{3} \vee 4,\ \ \overline{5} \vee \overline{6},\ \ \overline{1} \vee \overline{5} \vee 7,\ \ \overline{2} \vee \overline{5} \vee 6 \vee \overline{7}\}$$

M	F	C	rule
	F	no	
1	F	no	by **Propagate**
$1\ 2$	F	no	by **Propagate**
$1\ 2 \bullet 3$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5$	F	no	by **Decide**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	no	by **Propagate**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$\overline{2} \vee \overline{5} \vee 6 \vee \overline{7}$	by **Conflict**
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5} \vee 6$	by **Explain** with $\overline{1} \vee \overline{5} \vee 7$
$1\ 2 \bullet 3\ 4 \bullet 5\ \overline{6}\ 7$	F	$1 \vee \overline{2} \vee \overline{5}$	by **Explain** with $\overline{5} \vee \overline{6}$
$1\ 2\ \overline{5}$	F	no	by **Backjump**
$1\ 2\ \overline{5} \bullet 3$	F	no	by **Decide**
$\ldots$			

Also add

$$\textbf{Learn}\quad \frac{F \models_{\mathrm{p}} C \qquad C \notin F}{F := F \cup \{C\}}$$

$$\textbf{Forget}\quad \frac{C = \mathrm{no} \qquad F = G \cup \{C\} \qquad G \models_{\mathrm{p}} C}{F := G}$$

$$\textbf{Restart}\quad \frac{}{M := M^{[0]} \qquad C := \mathrm{no}}$$

Note: Learn can be applied to any clause stored in C when $C \neq \mathrm{no}$

At the core, current CDCL SAT solvers are implementations of the transition system with rules

Propagate, Decide,

Conflict, Explain, Backjump,

Learn, Forget, Restart

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Basic DPLL $\stackrel{\text{def}}{=}$

{ **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump** }

DPLL $\stackrel{\text{def}}{=}$ Basic DPLL + { **Learn**, **Forget**, **Restart** }

Some terminology:

Irreducible state: state for which no Basic DPLL rules apply

Execution: sequence of transitions allowed by the rules and starting with $M = \emptyset$ and $C = no$

Exhausted execution: execution ending in an irreducible state

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Note: This is not so immediate, because of **Backjump**.

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Proposition (Strong Termination) Every execution in Basic DPLL is finite.

Lemma Every exhausted execution ends with either $C = \text{no}$ or fail.

Some terminology:

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Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, the clause set F_0 is satisfied by M.

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

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  1. If $n > 0$ conflicts have been found so far, increase n and apply **Restart**
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  3. Keep applying **Explain** until **Backjump** is applicable

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  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
  4. Apply **Learn**
  5. Apply **Backjump**
  6. Apply **Propagate** to completion

- Applying
  - one Basic DPLL rule between each two **Learn** applications and
  - **Restart** less and less often

  ensures termination

- A common basic strategy applies the rules with the following priorities:
  1. If $n > 0$ conflicts have been found so far,
     increase n and apply **Restart**
  2. If a clause is falsified by M, apply **Conflict**
  3. Keep applying **Explain** until **Backjump** is applicable
  4. Apply **Learn**
  5. Apply **Backjump**
  6. Apply **Propagate** to completion
  7. Apply **Decide**

Same states and transitions but

- F contains quantifier-free clauses in some theory T
- M is a sequence of theory literals and decision points
- the DPLL system is augmented with rules

  T-**Conflict**, T-**Propagate**, T-**Explain**

- maintains invariant: $F \models_T C$ and $M \models_{\mathrm{p}} \neg C$ when $C \neq \mathrm{no}$

Def. $F \models_T G$ iff every model of T that satisfies F satisfies G as well

Fix a theory T

$$T\text{-}\mathbf{Conflict}\ \frac{C = \mathrm{no} \quad l_1, \ldots, l_n \in M \quad l_1, \ldots, l_n \models_T \bot}{C := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

Fix a theory T

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$$T\text{-Propagate}\quad \frac{l \in \mathrm{Lit}(\mathsf{F}) \quad \mathsf{M} \models_T l \quad l, \bar{l} \notin \mathsf{M}}{\mathsf{M} := \mathsf{M}\ l}$$

Fix a theory T

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$$T\text{-Propagate}\quad \frac{l \in \mathrm{Lit}(F) \quad M \models_T l \quad l, \bar{l} \notin M}{M := M\ l}$$

$$T\text{-Explain}\quad \frac{C = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_M \bar{l}}{C := l_1 \vee \cdots \vee l_n \vee D}$$

Note: $\bot$ = empty clause

Note: $\models_T$ decided by theory solver

T-**Conflict** is enough to model the naive integration of SAT solvers and theory solvers seen in the earlier UF example

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\,\overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$

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$1\,\overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**

MODELING THE VERY LAZY THEORY APPROACH

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$1\,\overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
$1\,\overline{4} \bullet \overline{2}$	$,\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**

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$1\,\overline{4} \bullet \overline{2}$	$,\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\,\overline{4}$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Restart**

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$1\ \overline{4}$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$

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$1\,\overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2 \vee 4$	by ***T*-Conflict**
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$1\,\overline{4}$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\,\overline{4}\,2\,3$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^+$
$1\,\overline{4}\,2\,3$	$,\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**

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$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
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$1\ \overline{4} \bullet \overline{2}$	$,\ \overline{1} \vee 2 \vee 4$	$\overline{1} \vee 2 \vee 4$	by **Learn**
$1\ \overline{4}$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Restart**
$1\ \overline{4}\ 2\ 3$	$,\ \overline{1} \vee 2 \vee 4$	no	by **Propagate**$^{+}$
$1\ \overline{4}\ 2\ 3$	$,\ \overline{1} \vee 2 \vee 4,\ \overline{1} \vee \overline{3} \vee 4$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**, **Learn**
fail			by **Fail**

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- An *on-line* SAT engine,
  which can accept new input clauses on the fly

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  1. check the T-satisfiability of M as it is extended and
  2. identify a small T-unsatisfiable subset of M once M becomes T-unsatisfiable

A BETTER LAZY APPROACH

$$\underbrace{g(a) = c}_{1} \quad \land \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \lor \underbrace{g(a) = d}_{3} \quad \land \quad \underbrace{c \neq d}_{\overline{4}}$$

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M	F	C	rule
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$1\,\overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**$^{+}$
$1\,\overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**

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$1\,\overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Decide**
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A BETTER LAZY APPROACH

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**

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$1\ \overline{4} \bullet \overline{2}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee 2$	by ***T*-Conflict**
$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Backjump**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by **Propagate**
$1\ \overline{4}\ 2\ 3$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{1} \vee \overline{3} \vee 4$	by ***T*-Conflict**
fail			by **Fail**

Ignoring **Restart** (for simplicity), a common strategy is to apply the rules using the following priorities:

1. If a clause is falsified by the current assignment M, apply **Conflict**
2. If M is $\mathcal{T}$-unsatisfiable, apply **$\mathcal{T}$-Conflict**
3. Apply **Fail** or **Explain**+**Learn**+**Backjump** as appropriate
4. Apply **Propagate**
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3. Apply **Fail** or **Explain**+**Learn**+**Backjump** as appropriate
4. Apply **Propagate**
5. Apply **Decide**

Note: Depending on the cost of checking the T-satisfiability of M, Step (2) can be applied with lower frequency or priority

With *T*-**Conflict** as the only theory rule, the theory solver is used just to validate the choices of the SAT engine

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With *T*-**Propagate** and *T*-**Explain**, it can also be used to guide the engine's search [Tin02]

T-**Propagate** $$\frac{l \in \mathrm{Lit}(F) \quad M \models_T l \quad l, \bar{l} \notin M}{M := M\ l}$$

T-**Explain** $$\frac{C = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_T \bar{l} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_M \bar{l}}{C := l_1 \vee \cdots \vee l_n \vee D}$$

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	

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M	F	C	rule
	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	
$1\ \overline{4}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by Propagate$^+$

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-Propagate ($1 \models_T 2$)

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M	F	C	rule
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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by ***T*-Propagate** $(1 \models_T 2)$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by ***T*-Propagate** $(1,\ \overline{4} \models_T \overline{3})$

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** ($1 \models_T 2$)
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by T-**Propagate** ($1,\ \overline{4} \models_T \overline{3}$)
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by **Conflict**

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$1\ \overline{4}\ 2$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by ***T*-Propagate** $(1 \models_T 2)$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	no	by ***T*-Propagate** $(1,\ \overline{4} \models_T \overline{3})$
$1\ \overline{4}\ 2\ \overline{3}$	$1,\ \overline{2} \vee 3,\ \overline{4}$	$\overline{2} \vee 3$	by **Conflict**
fail			by **Fail**

Note: *T*-propagation eliminates search altogether in this case
no applications of **Decide** are needed

At the core, current lazy SMT solvers are implementations of the transition system with rules

(1) **Propagate**, **Decide**, **Conflict**, **Explain**, **Backjump**, **Fail**

(2) T-**Conflict**, T-**Propagate**, T-**Explain**

(3) **Learn**, **Forget**, **Restart**

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Basic DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2)

DPLL Modulo Theories $\stackrel{\text{def}}{=}$ (1) + (2) + (3)

Updated terminology:

Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $M = \emptyset$ and $C = no$

Exhausted execution: execution ending in an irreducible state

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Proposition (Termination) Every execution in which

(a) **Learn**/**Forget** are applied only finitely many times and

(b) **Restart** is applied with increased periodicity

is finite.

Lemma Every exhausted execution ends with either $C = no$ or fail.

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Irreducible state: state to which no Basic DPLL MT rules apply

Execution: sequence of transitions allowed by the rules and starting with $\mathsf{M} = \emptyset$ and $\mathsf{C} = \mathsf{no}$

Exhausted execution: execution ending in an irreducible state

Proposition (Soundness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with fail, the clause set F_0 is T-unsatisfiable.

Proposition (Completeness) For every exhausted execution starting with $\mathsf{F} = F_0$ and ending with $\mathsf{C} = \mathsf{no}$, F_0 is T-satisfiable; specifically, M is T-satisfiable and $\mathsf{M} \models_{\mathrm{p}} F_0$.

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DPLL(T) = DPLL(X) engine + T-solver

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$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

DPLL(X):

- Very similar to a SAT solver, enumerates Boolean models
- Not allowed: pure literal, blocked literal detection, ...
- Required: incremental addition of clauses
- Desirable: partial model detection

The approach formalized so far can be implemented with a simple architecture named DPLL(T) [GHN+04, NOT06]

$$\text{DPLL}(T) = \text{DPLL}(X) \text{ engine} + T\text{-solver}$$

T-solver:

- Checks the T-satisfiability of conjunctions of literals
- Computes theory propagations
- Produces explanations of T-unsatisfiability/propagation
- Must be incremental and backtrackable

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Conclusion: M is T-unsatisfiable

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An alternative is to lift case splitting and backtracking from the T-solver to the SAT engine

Basic idea: encode case splits as sets of clauses and send them as needed to the SAT engine for it to split on them [BNOT06]

Possible benefits:

- All case-splitting is coordinated by the SAT engine
- Only have to implement case-splitting infrastructure in one place
- Can learn a wider class of lemmas

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$$M = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

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Basic Scenario:

$$M = \{\ldots,\ s = \underbrace{r(w(a,i,t),j)}_{s'},\ \ldots\}$$

- Main SMT module: "Is M T-unsatisfiable?"
- T-solver: "I do not know yet, but it will help me if you consider these *theory lemmas*:

$$s = s' \land i = j \Rightarrow s = t, \quad s = s' \land i \neq j \Rightarrow s = r(a,j) \text{ ''}$$

To model the generation of theory lemmas for case splits, add the rule

T-Learn

$$\frac{\models_T \exists \mathbf{v}(l_1 \vee \cdots \vee l_n) \quad l_1, \ldots, l_n \in L_{\mathrm{S}} \quad \mathbf{v} \text{ vars not in F}}{\mathrm{F} := \mathrm{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

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where L_{S} is a finite set of literals dependent on the initial set of clauses (see [BNOT06] for a formal definition of L_{S})

Note: For many theories with a theory solver, there exists an appropriate finite L_{S} for every input F

The set L_{S} does not need to be computed explicitly

Now we can relax the requirement on the theory solver:

When $M \models_{\mathrm{p}} F$, it must either

- *determine whether $M \models_{T} \perp$ or*
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The T-solver is required to determine whether $M \models_T \bot$ only if all literals in L_{S} are defined in M

Note: In practice, to determine if $M \models_T \bot$, the T-solver only needs a small subset of L_{S} to be defined in M

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by Propagate$^+$

EXAMPLE — THEORY OF FINITE SETS

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

M	F	rule
$x = y \cup z$	F	by Propagate$^+$
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$x = y \cup z \bullet y = \emptyset\ x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$	by ***T*-Learn**

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Example — Theory of finite sets

$$F: \quad x = y \cup z \quad \wedge \quad y \neq \emptyset \vee x \neq z$$

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$x = y \cup z \bullet y = \emptyset$	F	by Decide
$x = y \cup z \bullet y = \emptyset \; x \neq z$	F	by Propagate
$x = y \cup z \bullet y = \emptyset \; x \neq z$	$F, (x = z \vee e \in x \vee e \in z),$ $(x = z \vee e \notin x \vee e \notin z)$	by T-Learn
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T-solver can make the following deductions at this point:

$$e \in x \;\cdots\; \Rightarrow e \in y \cup z \;\cdots\; \Rightarrow e \in y \;\cdots\; \Rightarrow e \in \emptyset \;\Rightarrow \perp$$

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This enables an application of ***T*-Conflict** with clause

$$x \neq y \cup z \ \vee \ y \neq \emptyset \ \vee \ x = z \ \vee \ e \notin x \ \vee \ e \in z$$

Correctness results can be extended to the new rule.

Soundness: The new T-**Learn** rule maintains satisfiability of the clause set.

Completeness: As long as the theory solver can decide $M \models_T \bot$ when all literals in L_S are determined, the system is still complete.

Termination: The system terminates under the same conditions as before. Roughly:

- Any lemma is (re)learned only finitely many times
- **Restart** is applied with increased periodicity

COMBINING THEORIES AND THEIR SOLVERS

Many applications give rise to mixed-theory formulas like:

$$a \approx b + 2 \wedge A = \mathrm{store}(B, a+1, 4) \wedge$$
$$A[b+3] = 2 \vee f(a-1) \neq f(b+1)$$

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Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
- the theory of arrays (T_{A})
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Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{UF}}$?

NEED FOR COMBINING THEORIES AND SOLVERS

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Solving that formula requires reasoning over

- the theory of linear arithmetic (T_{LA})
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Question: Given solvers for each theory, can we combine them modularly into one for $T_{\mathrm{LA}} \cup T_{\mathrm{A}} \cup T_{\mathrm{UF}}$?

Under certain conditions, we can do it with the Nelson-Oppen combination method [NO79, Opp80]

Consider the following set of literals over $T_{\mathrm{LRA}} \cup T_{\mathrm{UF}}$ (T_{LRA}, linear real arithmetic):

$$
\begin{aligned}
f(f(x) - f(y)) &= a \\
f(0) &> a + 2 \\
x &= y
\end{aligned}
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First step: *purify* literals so that each belongs to a single theory

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$$f(f(x) - f(y)) = a \implies \begin{array}{r} f(e_1) = a \\ e_1 = f(x) - f(y) \end{array} \implies \begin{array}{r} f(e_1) = a \\ e_1 = e_2 - e_3 \\ e_2 = f(x) \\ e_3 = f(y) \end{array}$$

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$$f(0) > a + 2 \quad \Longrightarrow \quad \begin{array}{c} f(e_4) > a + 2 \\ e_4 = 0 \end{array} \quad \Longrightarrow \quad \begin{array}{l} f(e_4) = e_5 \\ e_4 = 0 \\ e_5 > a + 2 \end{array}$$

Second step: exchange entailed *interface equalities*, equalities over shared constants $e_1, e_2, e_3, e_4, e_5, a$

L_1	L_2
$f(e_1) = a$	$e_2 - e_3 = e_1$
$f(x) = e_2$	$e_4 = 0$
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$$L_2 \models_{\mathrm{LRA}} e_1 = e_4$$

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$e_1 = e_4$	

$$L_1 \models_{\mathrm{UF}} a = e_5$$

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Third step: check for satisfiability locally

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$L_1 \not\models_{\mathrm{UF}} \perp$

$L_2 \models_{\mathrm{LRA}} \perp$

Report unsatisfiable

Consider the following unsatisfiable set of literals over $T_{\mathrm{LIA}} \cup T_{\mathrm{UF}}$ (T_{LIA}, linear integer arithmetic):

$$\begin{aligned} 1 \leq \quad x \quad & \leq 2 \\ f(1) \quad = \quad & a \\ f(2) \quad = \quad & f(1) + 3 \\ a \quad = \quad & b + 2 \end{aligned}$$

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First step: *purify* literals so that each belongs to a single theory

$$f(1) = a \quad \Longrightarrow \quad \begin{aligned} f(e_1) &= a \\ e_1 &= 1 \end{aligned}$$

MOTIVATING EXAMPLE (NON-CONVEX CASE)

Consider the following unsatisfiable set of literals over $T_{\mathrm{LIA}} \cup T_{\mathrm{UF}}$ (T_{LIA}, linear integer arithmetic):

$$\begin{array}{rcl} 1 \leq & x & \leq 2 \\ f(1) & = & a \\ f(2) & = & f(1) + 3 \\ a & = & b + 2 \end{array}$$

First step: *purify* literals so that each belongs to a single theory

$$f(2) = f(1) + 3 \quad \Longrightarrow \quad \begin{array}{r} e_2 = 2 \\ f(e_2) = e_3 \\ f(e_1) = e_4 \\ e_3 = e_4 + 3 \end{array}$$

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
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$e_2 = 2$	
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No more entailed equalities, but $L_1 \models_{\mathrm{LIA}} x = e_1 \lor x = e_2$

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$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

Consider each case of $x = e_1 \vee x = e_2$ separately

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$e_3 = e_4 + 3$	
$a = e_4$	

Case 1) $x = e_1$

MOTIVATING EXAMPLE (NON-CONVEX CASE)

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_1$	

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
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$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_1$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_1$	

$L_2 \models_{\mathrm{UF}} a = b$, which entails $\perp$ when sent to L_1

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$e_1 = 1$	$f(e_2) = e_3$
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$e_2 = 2$	
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$1 \leq x$	$f(e_1) = a$
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$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	
$e_3 = e_4 + 3$	
$a = e_4$	

Case 2) $x = e_2$

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

L_1	L_2
$1 \leq x$	$f(e_1) = a$
$x \leq 2$	$f(x) = b$
$e_1 = 1$	$f(e_2) = e_3$
$a = b + 2$	$f(e_1) = e_4$
$e_2 = 2$	$x = e_2$
$e_3 = e_4 + 3$	
$a = e_4$	
$x = e_2$	

Second step: exchange entailed *interface equalities* over shared constants $x, e_1, a, b, e_2, e_3, e_4$

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$e_2 = 2$	$x = e_2$
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$L_2 \models_{\mathrm{UF}} e_3 = b$, which entails $\perp$ when sent to L_1

- For $i = 1, 2$, let T_i be a first-order theory of *signature* Σ_i (set of function and predicate symbols in T_i other than $=$)
- Let $T = T_1 \cup T_2$
- Let $\mathcal{C}$ be a finite set of *free* constants (i.e., not in $\Sigma_1 \cup \Sigma_2$)

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We consider only input problems of the form

$$L_1 \cup L_2$$

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Note: Because of purification, there is no loss of generality in considering only ground $(\Sigma_i \cup \mathcal{C})$-literals

Bare-bones, non-deterministic, non-incremental version

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Input: $L_1 \cup L_2$ with L_i finite set of ground $(\Sigma_i \cup \mathcal{C})$-literals
Output: **sat** or **unsat**

1. Guess an *arrangement* A, i.e., a set of equalities and disequalities over $\mathcal{C}$ such that

$$c = d \in A \quad \text{or} \quad c \neq d \in A \quad \text{for all } c, d \in \mathcal{C}$$

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2. If $L_i \cup A$ is T_i-unsatisfiable for $i = 1$ or $i = 2$, return unsat
3. Otherwise, return **sat**

Proposition (Termination) The method is terminating.

(Trivially, because there is only a finite number of arrangements to guess)

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(Because satisfiability in $(T_1 \cup T_2)$ is always preserved)

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Proposition (Soundness) If the method returns **unsat** for every arrangement, the input is $(T_1 \cup T_2)$-unsatisfiable.

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Proposition (Completeness) If $\Sigma_1 \cap \Sigma_2 = \emptyset$ and T_1 and T_2 are stably infinite, when the method returns **sat** for some arrangement, the input is $(T_1 \cup T_2)$-is satisfiable.

Def. A theory T is *stably infinite* iff every quantifier-free T-satisfiable formula is satisfiable in an infinite model of T

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Many interesting theories are stably infinite:

- Theories of an infinite structure (e.g., integer arithmetic)
- Complete theories with an infinite model (e.g., theory of dense linear orders, theory of lists)
- Convex theories (e.g., EUF, linear real arithmetic)

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Def. A theory T is *convex* iff, for any set L of literals

$L \models_T s_1 = t_1 \vee \cdots \vee s_n = t_n \implies L \models_T s_i = t_i$ for some i

Note: With convex theories, arrangements do not need to be guessed, theycan be computed by (theory) propagation

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The Nelson-Oppen method has been extended to some classes of non-stably infinite theories [TZ05, RRZ05, JB10]

Let $T_1, \ldots, T_n$ be theories with respective solvers $S_1, \ldots, S_n$

How can we integrate all of them cooperatively into a single SMT solver for $T = T_1 \cup \cdots \cup T_n$?

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Quick Solution:

1. Combine $S_1, \ldots, S_n$ with Nelson-Oppen into a theory solver for T
2. Build a DPLL(T) solver as usual

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Better Solution [Bar02, BBC+05b, BNOT06]:

1. Extend DPLL(T) to DPLL($T_1, \ldots, T_n$)
2. Lift Nelson-Oppen to the DPLL($X_1, \ldots, X_n$) level
3. Build a DPLL($T_1, \ldots, T_n$) solver

- Let $n = 2$, for simplicity
- Let T_i be of signature Σ_i for $i = 1, 2$, with $\Sigma_1 \cap \Sigma_2 = \emptyset$
- Let $\mathcal{C}$ be a set of free constants
- Assume wlog that each input literal has signature $(\Sigma_1 \cup \mathcal{C})$ or $(\Sigma_2 \cup \mathcal{C})$ (no *mixed* literals)
- Let $\mathrm{M}|_i \stackrel{\text{def}}{=} \{(\Sigma_i \cup \mathcal{C})\text{-literals of M and their complement}\}$
- Let $\mathrm{I}(\mathrm{M}) \stackrel{\text{def}}{=} \{c = d \mid c, d \text{ occur in } \mathcal{C}, \mathrm{M}|_1 \text{ and } \mathrm{M}|_2\} \cup$
  $\{c \neq d \mid c, d \text{ occur in } \mathcal{C}, \mathrm{M}|_1 \text{ and } \mathrm{M}|_2\}$
  (*interface literals*)

Propagate, Conflict, Explain, Backjump, Fail (unchanged)

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

Decide $$\frac{l \in \mathrm{Lit}(F) \cup \mathrm{I}(M) \quad l, \bar{l} \notin M}{M := M \bullet l}$$

Only change: decide on interface equalities as well

Propagate, **Conflict**, **Explain**, **Backjump**, **Fail** (unchanged)

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Only change: decide on interface equalities as well

***T*-Propagate** $$\frac{l \in \mathrm{Lit}(F) \cup \mathrm{I}(M) \quad i \in \{1, 2\} \quad M \models_{T_i} l \quad l, \bar{l} \notin M}{M := M\ l}$$

Only change: propagate interface equalities as well, but reason locally in each T_i

T-Conflict

$$\frac{C = \mathsf{no} \quad l_1, \dots, l_n \in M \quad l_1, \dots, l_n \models_{T_i} \bot \quad i \in \{1,2\}}{C := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{C = l \vee D \quad \bar{l}_1, \dots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1,2\} \quad \bar{l}_1, \dots, \bar{l}_n \prec_M \bar{l}}{C := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

T-Conflict

$$\frac{\mathsf{C} = \mathsf{no} \quad l_1, \ldots, l_n \in \mathsf{M} \quad l_1, \ldots, l_n \models_{T_i} \bot \quad i \in \{1, 2\}}{\mathsf{C} := \bar{l}_1 \vee \cdots \vee \bar{l}_n}$$

T-Explain

$$\frac{\mathsf{C} = l \vee D \quad \bar{l}_1, \ldots, \bar{l}_n \models_{T_i} \bar{l} \quad i \in \{1, 2\} \quad \bar{l}_1, \ldots, \bar{l}_n \prec_{\mathsf{M}} \bar{l}}{\mathsf{C} := l_1 \vee \cdots \vee l_n \vee D}$$

Only change: reason locally in each T_i

$\mathbf{I}$-Learn

$$\frac{\models_{T_i} l_1 \vee \cdots \vee l_n \quad l_1, \ldots, l_n \in \mathsf{M}|_i \cup \mathbf{I}(\mathsf{M}) \quad i \in \{1, 2\}}{\mathsf{F} := \mathsf{F} \cup \{l_1 \vee \cdots \vee l_n\}}$$

New rule: for entailed disjunctions of interface literals

EXAMPLE — CONVEX THEORIES

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$

$$\underbrace{e_2 - e_3 = e_1}_{5} \wedge \underbrace{e_4 = 0}_{6} \wedge \underbrace{e_5 > a + 2}_{7}$$

$$\underbrace{e_2 = e_3}_{8} \quad \underbrace{e_1 = e_4}_{9} \quad \underbrace{a = e_5}_{10}$$

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M	F	C	rule
	F	no	

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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = e_2}^{1} \land \overbrace{f(y) = e_3}^{2} \land \overbrace{f(e_4) = e_5}^{3} \land \overbrace{x = y}^{4} \land$$
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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** (1, 2, 4 $\models_{\mathrm{UF}}$ 8)

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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^{+}$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** ($1, 2, 4 \models_{\text{UF}} 8$)
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** ($5, 6, 8 \models_{\text{LRA}} 9$)

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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** ($1, 2, 4 \models_{\mathrm{UF}} 8$)
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** ($5, 6, 8 \models_{\mathrm{LRA}} 9$)
0 1 2 3 4 5 6 7 8 9 10	F	no	by T-**Propagate** ($0, 3, 9 \models_{\mathrm{UF}} 10$)

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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^{+}$
0 1 2 3 4 5 6 7 8	F	no	by T-**Propagate** ($1, 2, 4 \models_{\mathrm{UF}} 8$)
0 1 2 3 4 5 6 7 8 9	F	no	by T-**Propagate** ($5, 6, 8 \models_{\mathrm{LRA}} 9$)
0 1 2 3 4 5 6 7 8 9 10	F	no	by T-**Propagate** ($0, 3, 9 \models_{\mathrm{UF}} 10$)
0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \lor \overline{10}$	by T-**Conflict** ($7, 10 \models_{\mathrm{LRA}} \bot$)

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = e_2}^{1} \wedge \overbrace{f(y) = e_3}^{2} \wedge \overbrace{f(e_4) = e_5}^{3} \wedge \overbrace{x = y}^{4} \wedge$$
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M	F	C	rule
	F	no	
0 1 2 3 4 5 6 7	F	no	by **Propagate**$^+$
0 1 2 3 4 5 6 7 8	F	no	by **T-Propagate** ($1, 2, 4 \models_{\mathrm{UF}} 8$)
0 1 2 3 4 5 6 7 8 9	F	no	by **T-Propagate** ($5, 6, 8 \models_{\mathrm{LRA}} 9$)
0 1 2 3 4 5 6 7 8 9 10	F	no	by **T-Propagate** ($0, 3, 9 \models_{\mathrm{UF}} 10$)
0 1 2 3 4 5 6 7 8 9 10	F	$\overline{7} \vee \overline{10}$	by **T-Conflict** ($7, 10 \models_{\mathrm{LRA}} \perp$)
fail			by **Fail**

EXAMPLE — NON-CONVEX THEORIES

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
$$\underbrace{1 \leq x}_{4} \wedge \underbrace{x \leq 2}_{5} \wedge \underbrace{e_1 = 1}_{6} \wedge \underbrace{a = b + 2}_{7} \wedge \underbrace{e_2 = 2}_{8} \wedge \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
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M	F	C	rule
	F	no	

EXAMPLE — NON-CONVEX THEORIES

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
$$\underbrace{1 \leq x}_{4} \land \underbrace{x \leq 2}_{5} \land \underbrace{e_1 = 1}_{6} \land \underbrace{a = b + 2}_{7} \land \underbrace{e_2 = 2}_{8} \land \underbrace{e_3 = e_4 + 3}_{9}$$

$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$

EXAMPLE — NON-CONVEX THEORIES

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
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$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9$ 10	F	no	by T-**Propagate** $(0, 3 \models_{\mathrm{UF}} 10)$

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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$$\underbrace{a = e_4}_{10} \quad \underbrace{x = e_1}_{11} \quad \underbrace{x = e_2}_{12} \quad \underbrace{a = b}_{13}$$

M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** ($0, 3 \models_{\mathrm{UF}} 10$)
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** ($\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12$)

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by T-**Propagate** $(0, 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**

$$F := \overbrace{f(e_1) = a}^{0} \land \overbrace{f(x) = b}^{1} \land \overbrace{f(e_2) = e_3}^{2} \land \overbrace{f(e_1) = e_4}^{3} \land$$
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	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0, 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **T-Propagate** $(0, 1, 11 \models_{\mathrm{UF}} 13)$

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** $(0,\ 3 \models_{\mathrm{UF}} 10)$
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** $(\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12)$
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0,\ 1,\ 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7,\ 13 \models_{\mathrm{UF}} \perp)$

EXAMPLE — NON-CONVEX THEORIES

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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M	F	C	rule
	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** ($0, 3 \models_{\mathrm{UF}} 10$)
$0 \cdots 9\ 10$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **I-Learn** ($\models_{\mathrm{LIA}} \overline{4} \vee \overline{5} \vee 11 \vee 12$)
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** ($0, 1, 11 \models_{\mathrm{UF}} 13$)
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** ($7, 13 \models_{\mathrm{UF}} \perp$)
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**

EXAMPLE — NON-CONVEX THEORIES

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	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
$0 \cdots 9\ 10$	F	no	by **T-Propagate** ($0,\ 3 \models_{\mathrm{UF}} 10$)
$0 \cdots 9\ 10$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **I-Learn** ($\models_{\mathrm{LIA}} \overline{4} \lor \overline{5} \lor 11 \lor 12$)
$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **T-Propagate** ($0,\ 1,\ 11 \models_{\mathrm{UF}} 13$)
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	$\overline{7} \lor \overline{13}$	by **T-Conflict** ($7,\ 13 \models_{\mathrm{UF}} \bot$)
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \lor \overline{5} \lor 11 \lor 12$	no	by **T-Propagate** ($0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11}$)

$$F := \overbrace{f(e_1) = a}^{0} \wedge \overbrace{f(x) = b}^{1} \wedge \overbrace{f(e_2) = e_3}^{2} \wedge \overbrace{f(e_1) = e_4}^{3} \wedge$$
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	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
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$0 \cdots 9\ 10 \bullet 11$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Decide**
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0, 1, 11 \models_{\mathrm{UF}} 13)$
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** $(7, 13 \models_{\mathrm{UF}} \bot)$
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** $(0, 1, \overline{13} \models_{\mathrm{UF}} \overline{11})$
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$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** ($7,\ 13 \models_{\mathrm{UF}} \bot$)
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** ($0,\ 1,\ \overline{13} \models_{\mathrm{UF}} \overline{11}$)
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)

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	F	no	
$0 \cdots 9$	F	no	by **Propagate**$^+$
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$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** ($0, 1, 11 \models_{\mathrm{UF}} 13$)
$0 \cdots 9\ 10 \bullet 11\ 13$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	$\overline{7} \vee \overline{13}$	by **T-Conflict** ($7, 13 \models_{\mathrm{UF}} \bot$)
$0 \cdots 9\ 10\ \overline{13}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Backjump**
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **T-Propagate** ($0, 1, \overline{13} \models_{\mathrm{UF}} \overline{11}$)
$0 \cdots 9\ 10\ \overline{13}\ \overline{11}\ 12$	$F,\ \overline{4} \vee \overline{5} \vee 11 \vee 12$	no	by **Propagate**
$\cdots$			(exercise)
fail	$\cdots$	$\cdots$	by **Fail**

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