

Polynomial Invariant Generation for Multi-Path Loops

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joint work with

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logics



Logical Methods in
Computer Science

Roadmap

- Why invariant generation?
- Single-path loops
- Multi-path loops

Verifying algorithms

Division of a by b where $b \neq 0$:

```
quo := 0;  
rem := a;  
while  $b \leq \textit{rem}$  do  
    rem := rem - b;  
    quo := quo + 1;  
end while
```

Verifying algorithms

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```

Question

- Is this algorithm correct?

Verifying algorithms

Division of a by b where $b \neq 0$:

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rem := a;  
while  $b \leq \textit{rem}$  do  
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end while
```

Question

- Is this algorithm correct?
- Does it terminate?

Loop invariants

Definition

A **loop invariant** \mathcal{I} for the loop **while** \mathcal{S} **do** B is an assertion that satisfies

$$\{\mathcal{I} \wedge \mathcal{S}\} B \{\mathcal{I}\}$$

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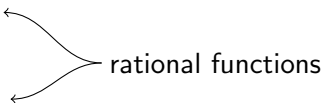
```
while  $b \leq rem$  do  
   $rem := rem - b$ ;  
   $quo := quo + 1$ ;  
end while
```

$$\underbrace{\{(quo \cdot b + rem - a = 0)\}}_{\mathcal{I}} \wedge \underbrace{\{(b \leq rem)\}}_{\mathcal{S}} \\ \begin{array}{l} rem := rem - b; \\ quo := quo + 1; \end{array} \\ \underbrace{\{(quo \cdot b + rem - a = 0)\}}_{\mathcal{I}}$$

Our programming model (for single-path loops)

```
while  $pred(v_1, \dots, v_m)$  do  
   $v_1 := f_1(v_1, \dots, v_m)$   
   $\vdots$   
   $v_m := f_m(v_1, \dots, v_m)$   
end while
```

rational functions



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end while
```

rational functions

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while  $\text{pred}(v_1, \dots, v_m)$  do  
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```

rational functions

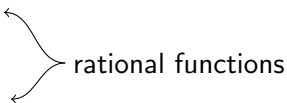
Proposition (Müller-Olm and Seidl [2004])

*The set of all equality invariants is **not computable** when considering affine equality tests.*

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$$\{I \wedge S\}B\{I\}$$

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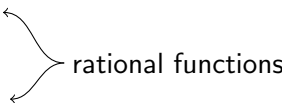
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rational functions

Proposition (Müller-Olm and Seidl [2004])

*The set of all equality invariants is **not computable** when considering affine equality tests.*

$$\{\mathcal{I}\} B^n \{\mathcal{I}\} \text{ for all } n \in \mathbb{N}$$

Polynomial invariants

Definition

A polynomial $p \in \mathbb{K}[X]$ is a **polynomial invariant** of **while** S **do** B among the loop variables $V = v_1, \dots, v_m$ with initial values V_0 if

$$\{p(V) = 0 \wedge V = V_0\} B^n \{p(V) = 0\} \quad \text{for all } n \in \mathbb{N}.$$

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Observation

The set of polynomial invariants is an ideal.

Definition

A subset $\mathcal{I} \subset \mathbb{K}[X]$ is an **ideal** if it satisfies:

- (1) $0 \in \mathcal{I}$.
- (2) If $f, g \in \mathcal{I}$, then $f + g \in \mathcal{I}$.
- (3) If $f \in \mathcal{I}$ and $h \in \mathbb{K}[X]$, then $h \cdot f \in \mathcal{I}$.

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Observation

The set of polynomial invariants is an ideal.

\Rightarrow Symbolic Computation

Some facts about ideals

An ideal is generated by a set of elements:

$$\underbrace{\langle e_1, \dots, e_n \rangle}_{\text{basis}} = \{r_1 e_1 + \dots + r_n e_n \mid r_i \in \mathbb{K}[X]\}$$

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Theorem

Every ideal $\mathcal{I} \triangleleft \mathbb{K}[X]$ has a *finite basis*.

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Theorem

Every ideal $\mathcal{I} \triangleleft \mathbb{K}[X]$ has a *finite basis*.

Proposition

For $I, J \triangleleft \mathbb{K}[X, Y]$ we can compute

$I + J$ (sum)

$I \cap \mathbb{K}[X]$ (elimination ideal; via Gröbner bases)

Assignments and Recurrences

$x := 10$

$y := 10$

while $y > 0$ **do**

$x := 2 \cdot x + 3$

$y := 1/2 \cdot y - 1$

end while

Assignments and Recurrences

```
x := 10
y := 10
while y > 0 do
  x := 2 · x + 3
  y := 1/2 · y - 1
end while
```

$$\left. \begin{array}{l} x_0 = 10 \\ y_0 = 10 \end{array} \right\} \text{initial values}$$

$$\left. \begin{array}{l} x_{n+1} = 2 \cdot x_n + 3 \\ y_{n+1} = 1/2 \cdot y_n - 1 \end{array} \right\} \text{recurrences}$$

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closed forms

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closed forms

Loop counter elimination

$$x = 2^n \cdot (x_0 + 3) - 3$$

$$y = 2^{-n} \cdot (y_0 + 2) - 2$$

Loop counter elimination

$$x = a \cdot (x_0 + 3) - 3$$

$$y = b \cdot (y_0 + 2) - 2$$

Loop counter elimination

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$$0 = 2^n \cdot 2^{-n} - 1 \quad \text{algebraic dependency among } 2^n, 2^{-n}$$

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$$\mathcal{I} =$$

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$$\mathcal{I} = \langle x - a \cdot (x_0 + 3) + 3, y - b \cdot (y_0 + 2) + 2 \rangle$$

Loop counter elimination

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algebraic dependency among $2^n, 2^{-n}$

$$\mathcal{I} = \langle x - a \cdot (x_0 + 3) + 3, y - b \cdot (y_0 + 2) + 2 \rangle + \langle a \cdot b - 1 \rangle$$

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$$\mathcal{I} \triangleleft \mathbb{K}[x_0, y_0, x, y, a, b]$$

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⇒ Gröbner bases to the rescue

$$\mathcal{I} \cap \mathbb{K}[x_0, y_0, x, y]$$

Loop counter elimination

$$x = a \cdot (x_0 + 3) - 3$$

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⇒ Gröbner bases to the rescue

$$\mathcal{I} \cap \mathbb{K}[x_0, y_0, x, y] = \langle -2x + 2x_0 - 3y - xy + 3y_0 + x_0y_0 \rangle$$

(Extended) P-solvable loops

loops with solvable mappings
(Rodríguez-Carbonell and Kapur [2004])

P-solvable loops (Kovács [2007])

C-finite sequences

extended P-solvable loops (ISSAC'17)

subsumes C-finite, rational function and hypergeometric
sequences

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$$(\mathcal{C} + \mathcal{A}) \cap \mathbb{K}[X_0, X]$$

Multi-path loops

```
while  $b$  do  
  if  $b_1$  then  $A_1$   
   $\vdots$   
  if  $b_m$  then  $A_m$   
  while  $b_{m+1}$  do  $B_{m+1}$   
   $\vdots$   
  while  $b_n$  do  $B_n$   
end while
```

Multi-path loops

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   $\vdots$   
  while  $b_n$  do  $B_n$   
end while
```


Multi-path loops

```
while  $b$  do  
  while  $b_1 \wedge f_1$  do  $B_1$   
   $\vdots$   
  while  $b_m \wedge f_m$  do  $B_m$   
  while  $b_{m+1}$  do  $B_{m+1}$   
   $\vdots$   
  while  $b_n$  do  $B_n$   
end while
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Multi-path loops

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while  $b$  do
  while  $b_1 \wedge f_1$  do  $B_1$ 
  ⋮
  while  $b_m \wedge f_m$  do  $B_m$ 
  while  $b_{m+1}$  do  $B_{m+1}$ 
  ⋮
  while  $b_n$  do  $B_n$ 
end while
```

Multi-path loops

```
while ... do
  while ... do  $B_1$ 
    ⋮
  while ... do  $B_m$ 
  while ... do  $B_{m+1}$ 
    ⋮
  while ... do  $B_n$ 
end while
```

Multi-path loops

```
while ... do  
  while ... do  $B_1$   
     $\vdots$   
  while ... do  $B_m$   $B_1^*$   
  while ... do  $B_{m+1}$   
     $\vdots$   
  while ... do  $B_n$   
end while
```

Multi-path loops

```
while ... do  
  while ... do  $B_1$   
     $\vdots$   
  while ... do  $B_m$   $B_1^*; B_2^*$   
  while ... do  $B_{m+1}$   
     $\vdots$   
  while ... do  $B_n$   
end while
```

Multi-path loops

```
while ... do  
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     $\vdots$   
  while ... do  $B_m$   
  while ... do  $B_{m+1}$   
     $\vdots$   
  while ... do  $B_n$   
end while
```

$B_1^*; B_2^*; \dots; B_n^*$

Multi-path loops

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while ... do  
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$(B_1^*; B_2^*; \dots; B_n^*)^*$

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$(B_1^*; B_2^*; \dots; B_n^*)^*$

Compute invariant ideal of

$B_1^*; \dots; B_n^*$

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$B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*$

\vdots

until a fixed-point is reached

Multi-path loops

```
while ... do  
  while ... do  $B_1$   
     $\vdots$   
  while ... do  $B_m$   $(B_1^*; B_2^*; \dots; B_n^*)^*$   
  while ... do  $B_{m+1}$   
     $\vdots$   
  while ... do  $B_n$   
end while
```

Compute invariant ideal of

$B_1^*; \dots; B_n^*$
 $B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*$
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 \vdots

until a fixed-point is reached

Fixed-point
guaranteed

Multi-path loops

```
while ... do  
  while ... do  $B_1$   
   $\vdots$   
  while ... do  $B_m$   
  while ... do  $B_{m+1}$   
   $\vdots$   
  while ... do  $B_n$   
end while
```

$(B_1^*; B_2^*; \dots; B_n^*)^*$

Compute invariant ideal of

$B_1^*; \dots; B_n^*$

$B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*$

$B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*; B_1^*; \dots; B_n^*$

\vdots

until a fixed-point is reached

Fixed-point
guaranteed

Bound: $k+1$
for k loop vars

Merging loops

B_1^*

B_2^*

Merging loops

initial values

B_1^*

initial values

B_2^*

Merging loops

initial values

B_1^*

final values

initial values

B_2^*

final values

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initial values

B_1^*

final values

initial values

B_2^*

final values

Merging loops

initial values

B_1^*

final values

initial values

B_2^*

final values

$$B_1^* \begin{cases} x_1 & = 2^m \cdot (x_0 + 3) - 3 \\ y_1 & = 2^{-m} \cdot (y_0 + 2) - 2 \end{cases}$$

$$B_2^* \begin{cases} x_2 & = 2^n \cdot (x_1 + 3) - 3 \\ y_2 & = 2^{-n} \cdot (y_1 + 2) - 2 \end{cases}$$

Merging loops

initial values

B_1^*

final values

initial values

B_2^*

final values

$$C_1 \begin{cases} x_1 &= a \cdot (x_0 + 3) - 3 \\ y_1 &= b \cdot (y_0 + 2) - 2 \end{cases}$$

$$C_2 \begin{cases} x_2 &= c \cdot (x_1 + 3) - 3 \\ y_2 &= d \cdot (y_1 + 2) - 2 \end{cases}$$

Merging loops

initial values

B_1^*

final values

initial values

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final values

$$C_1 \begin{cases} x_1 &= a \cdot (x_0 + 3) - 3 \\ y_1 &= b \cdot (y_0 + 2) - 2 \end{cases}$$

$$A_1 \begin{cases} 0 &= a \cdot b - 1 \end{cases}$$

$$C_2 \begin{cases} x_2 &= c \cdot (x_1 + 3) - 3 \\ y_2 &= d \cdot (y_1 + 2) - 2 \end{cases}$$

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initial values

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$$A_2 \begin{cases} 0 &= c \cdot d - 1 \end{cases}$$

$$(C_1 + A_1 + C_2 + A_2) \cap \mathbb{K}[x_0, y_0, x_2, y_2]$$

Merging loops

initial values

B_1^*

final values

initial values

B_2^*

final values

$$C_1 \begin{cases} x_1 &= a \cdot (x_0 + 3) - 3 \\ y_1 &= b \cdot (y_0 + 2) - 2 \end{cases}$$

$$A_1 \begin{cases} 0 &= a \cdot b - 1 \end{cases}$$

$$C_2 \begin{cases} x_2 &= c \cdot (x_1 + 3) - 3 \\ y_2 &= d \cdot (y_1 + 2) - 2 \end{cases}$$

$$A_2 \begin{cases} 0 &= c \cdot d - 1 \end{cases}$$

$$(C_1 + A_1 + C_2 + A_2) \cap \mathbb{K}[x_0, y_0, x_2, y_2]$$

$$= (\underbrace{\mathcal{I}_1}_{\text{invariant ideal of first loop}} + C_2 + A_2) \cap \mathbb{K}[x_0, y_0, x_2, y_2]$$

invariant ideal of first loop

Conclusion

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ALIGATOR Mathematica package

Available at <https://ahumenberger.github.io/aligator/>

(Extended) P-solvable loops

Definition (Kovács [2007])

A loop with assignments only is called **P-solvable** if the closed forms of its recursively changed variables x_1, \dots, x_m are of the form

$$x_i(n) = p_{i,1}(n)\theta_1^n + \dots + p_{i,s}(n)\theta_s^n$$

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A loop with assignments only is called **extended P-solvable** if the closed forms of its recursively changed variables x_1, \dots, x_m are of the form

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$$v_i(n) = \sum_{k \in \mathbb{Z}^\ell} p_{i,k}(n, \theta_1^n, \dots, \theta_s^n) ((n + \zeta_1)^n)^{k_1} \dots ((n + \zeta_\ell)^n)^{k_\ell}$$