Nonlinear Polynomials, Interpolants and Invariant Generation for System Analysis

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with Rodriguez-Carbonell, Zhihai Zhang, Hengjun Zhao, Stephan Falke, Naijun Zhan, Ting Gan, Bican Xia and others
(work in progress)
Outline

- Ideal-theoretic approach for generating nonlinear polynomial equalities as invariants.
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- Quantifier Elimination Approach for Generating (Loop) Invariants - Review with examples.
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- Challenges for symbolic computation community.
Building reliable and safe software is critical because of its use everywhere, especially in critical applications.
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- Automation and scalability are critical for success.
Example

\( x := 1, y := 1, z := 0; \)
\[\text{while } (x \leq N) \{\]
  \( x := x + y + 2; \)
  \( y := y + 2; \)
  \( z := z + 1 \)
\[\} \]
\( \text{return } z \)
Invariants: Integer Square Root

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Explore methods that can generate (strong) loop invariants (useful program properties) automatically for a large class of programs
Two Approaches for Generating Loop Invariants Automatically

1. Ideal-Theoretic Methods

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Interplay of Computational Logic and Algebra
Generating Loop Invariant: Approach

- Guess/fix the shape of invariants of interest at various program locations with some parameters which need to be determined.

\[ Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Jz + K = 0. \]

- Generate verification conditions using the hypothesized invariants from the code.

\[ VC1: \text{At first possible entry of the loop (from initialization): } A + B + D + G + H + K = 0. \]

\[ VC2: \text{For every iteration of the loop body: } (I(x, y, z) \land x \leq N) \Rightarrow I(x + y + 2, y + 2, z + 1). \]

- Using quantifier elimination, find constraints on parameters \( A, B, C, D, E, F, G, H, J, K \) which ensure that the verification conditions are valid for all possible program variables.
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Quantifier Elimination from Verification Conditions

Considering VC2:

\[( A x^2 + B y^2 + C z^2 + D xy + E xz + F yz + G x + H y + J z + K = 0) \implies ( A (x+y+2)^2 + B (y+2)^2 + C (z+1)^2 + D (x+y+2)(y+2) + E (x+y+2)(z+1) + F (y+2)(z+1) + G (x+y+2) + H (y+2) + J (z+1) + K = 0) \]
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Expanding the conclusion gives:

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Ax^2 + (A + B + D)y^2 + cz^2 + (D + 2A)xy + Exz + (E + F)yz + (G + 4A + 2D + E)x + (H + 4A + 4B + 4D + E + F + G)y + (J + 2C + 2E + 2F)z + (4A + 4B + C + 4D + 2E + 2F + 2G + 2H + J + K) = 0
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- Since this should be 0 for all values of \(x, y, z\): we have:
  \[A + D = 0; A = 0; E = 0\] which implies \(D = 0\); using these gives:
  \[2C + 2F = 0\] which implies \(C = -F\); using all these:
  \[G = -4B - F, H = -G - K - B\] and \(J = -2B - F + 2K.\]
Generating the Strongest Invariant

- Constraints on parameters are:

\[ C = -F, \quad J = -2B - F + 2K, \quad G = -4B - F, \quad H = 3B + F - K. \]
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- Every value of parameters satisfying the above constraints leads to an invariant (including the trivial invariant true when all parameter values are 0).

  - \[ K = 1, \quad H = -1, \quad J = -1 \] gives \[ -y + 2z + 1 = 0. \]

  - \[ F = 1, \quad C = -1, \quad J = -1, \quad G = -1, \quad H = 1 \] gives \[ -z^2 + yz - x + y - z - 0. \]

  - \[ B = 1, \quad J = -2, \quad G = -4, \quad H = 3 \] gives \[ y^2 - 4x + 3y - 2z = 0. \]

The most general invariant describing all invariants of the above form is a conjunction of:

\[ y = 2z + 1; \quad z^2 - yz + z + x - y = 0 \]

\[ y^2 - 2z - 4x + 3y = 0. \]
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- 7 parameters and 4 equations, so 3 independent parameters, say \( B, F, K \). Making every independent parameter 1 separately with other independent parameters being 0, derive values of dependent parameters.
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  - \( F = 1, C = -1, J = -1, G = -1, H = 1 \) gives \(-z^2 + yz - x + y - z - 0\).
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- The most general invariant describing all invariants of the above form is a conjunction of:
  \[ y = 2z + 1; \quad z^2 - yz + z + x - y = 0 \quad y^2 - 2z - 4x + 3y = 0, \]
  from which \( x = (z + 1)^2 \) follows.
Method for Automatically Generating Invariants by Quantifier Elimination

- Hypothesize assertions, which are parametrized formulas, at various points in a program.
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- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
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- Generate verification conditions for every path in the program (a path from an assertion to another assertion including itself).
  - Depending upon the logical language chosen to write invariants, approximations of assignments and test conditions may be necessary.
- Find a formula expressed in terms of parameters eliminating all program variables (using quantifier elimination).
Quality of Invariants

Soundness and Completeness

- Every assignment of parameter values which make the formula true, gives an inductive invariant.
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- Every assignment of parameter values which make the formula true, gives an inductive invariant.
  - If no parameter values can be found, then invariants of hypothesized forms may not exist. Invariants can be guaranteed not to exist if no approximations are made, while generating verification conditions.
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▶ If all assignments making the formula true can be finitely described, invariants generated may be the strongest of the hypothesized form. Invariants generated are guaranteed to be the strongest if no approximations are made, while generating verification conditions.
How to Scale this Approach

▶ Quantifier Elimination Methods typically do not scale up due to high complexity even in this restricted case of $\exists \forall$.

▶ Even for Presburger arithmetic, complexity is doubly exponential in the number of quantifier alternations and triply exponential in the number of quantified variables.

▶ Output is huge and difficult to decipher.

▶ In practice, they often do not work (i.e., run out of memory or hang).

▶ Linear constraint solving on rationals and reals (polyhedral domain), while of polynomial complexity, has been found in practice to be inefficient and slow, especially when used repeatedly as in abstract interpretation approach [Miné].
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  - In practice, they often do not work (i.e., run out of memory or hang).

- Linear constraint solving on rationals and reals (polyhedral domain), while of polynomial complexity, has been found in practice to be inefficient and slow, especially when used repeatedly as in abstract interpretation approach [Miné]
Making QE based Method Practical

- Identify (atomic) formulas and program abstractions resulting in verification conditions with good shape and structure.

- Octagonal formulas: $l \leq x \pm y \leq h$, a highly restricted subset of linear constraints (at most two variables with coefficients from $\{-1, 0, 1\}$).

  - This fragment is the most expressive fragment of linear arithmetic over the integers with a polynomial time decision procedure.

- Max, Min formulas: $\max(\pm x - l, \pm y - h)$, expressing disjunction $(x - l \geq y - h \land x - l \geq 0) \lor (y - h \geq x - h \land y - h \geq 0)$.

  - Combination of Octagonal and Max formulas.
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- Octagonal formulas over two variables have a fixed shape. Its parameterization can be given using 8 parameters.
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- Given $n$ variables, the most general formula (after simplification) is of the following form:
  \[ \bigwedge_{i,j} \left( \text{Octa}_{i,j} : \ a_{i,j} \leq x_i - x_j \leq b_{i,j}, \ c_{i,j} \leq x_i + x_j \leq d_{i,j} \\ e_i \leq x_i \leq f_i, \ g_j \leq x_j \leq h_j \right) \]
for every pair of variables $x_i, x_j$, where $a_{i,j}, b_{i,j}, c_{i,j}, d_{i,j}, e_i, f_i, g_j, h_j$ are parameters.
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- **Goal:** Performance of QE heuristic should be at least as good.
A Simple Example

Example

\[ x := 4; \ y := 6; \]
\[ \text{while} \ (x + y \geq 0) \ \text{do} \]
\[ \quad \text{if} \ (y \geq 6) \ \text{then} \ \{ \ x := -x; \ y := y - 1 \} \]
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**VC0:** \( I(4, 6) \)

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Approach: Local QE Heuristics

- A program path is a sequence of assignment statements interspersed with tests. Its behavior may have to be approximated to generate the post condition in which both the hypothesis and the conclusion are each conjunctions of atomic octagonal formulas.

\[ \bigwedge_{i, j} (\text{Octa}_{i,j} \land \alpha(x_i, x_j)) \Rightarrow \text{Octa}'_{i,j}, \] along with additional parameter-free constraints \( \alpha(x_i, x_j) \), of the same form in which lower and upper bounds are constants.

- Analysis of a big conjunctive constraint on every possible pair of variables can be considered individually by considering the subformula on each distinct pair.
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- Analyze how a general octagon gets transformed due to assignments. For each assignment case, a table is built showing the effect on the parameter values.
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Quantifier elimination heuristics to generate constraints on lower and upper bounds by table lookups in $O(n^2)$ steps, where $n$ is the number of program variables.
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Table 3: Sign of exactly one variable is changed

\[ x := -x + A \]
\[ y := y + B \]

\[ \Delta_1 = A - B, \quad \Delta_2 = A + B. \]

\[ \Delta_2 - u_2 \leq x - y \]

\[ x - y \leq a \]

\[ x + y \leq u_2 \]

\[ x - y \leq \Delta_2 - l_2 \]
\[ l_2 \leq x + y \]
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\[ x := -x + A \]
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<table>
<thead>
<tr>
<th>constraint</th>
<th>present</th>
<th>absent</th>
<th>side condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x - y \leq a )</td>
<td>( a \leq \Delta_2 - l_2 )</td>
<td>( u_1 \leq \Delta_2 - l_2 )</td>
<td>-</td>
</tr>
<tr>
<td>( x - y \geq b )</td>
<td>( \Delta_2 - u_2 \leq b )</td>
<td>( \Delta_2 - u_2 \leq l_1 )</td>
<td>-</td>
</tr>
<tr>
<td>( x + y \leq c )</td>
<td>( c \leq \Delta_1 - l_1 )</td>
<td>( u_2 \leq \Delta_1 - l_1 )</td>
<td>-</td>
</tr>
<tr>
<td>( x + y \geq d )</td>
<td>( \Delta_1 - u_1 \leq d )</td>
<td>( \Delta_1 - u_1 \leq l_2 )</td>
<td>-</td>
</tr>
<tr>
<td>( x \leq e )</td>
<td>( e \leq A - l_3 )</td>
<td>( u_3 \leq A - l_3 )</td>
<td>-</td>
</tr>
<tr>
<td>( x \geq f )</td>
<td>( A - u_3 \leq f )</td>
<td>( A - u_3 \leq l_3 )</td>
<td>-</td>
</tr>
<tr>
<td>( y \leq g )</td>
<td>( u_4 \geq g + B )</td>
<td>( u_4 = +\infty )</td>
<td>( B &gt; 0 )</td>
</tr>
<tr>
<td>( y \geq h )</td>
<td>( l_4 \leq h + B )</td>
<td>( l_4 = -\infty )</td>
<td>( B &lt; 0 )</td>
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Example

\[
x := 4; \quad y := 6;
\]
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\text{while} \ (x + y \geq 0) \ \text{do}
\]
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Generating Constraints on Parameters

\textbf{VC0:}
\[ l_1 \leq -2 \leq u_1 \land l_2 \leq 10 \leq u_2 \land l_3 \leq 4 \leq u_3 \land l_4 \leq 6 \leq u_4. \]
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  \( x + y: -u_1 + 1 \leq 0 \land u_2 \leq -l_1 + 1. \)
  \( x: l_3 + u_3 = 0. \)
  \( y: l_4 \leq 5. \)

Make \( l_i \)'s as large as possible and \( u_i \)'s as small as possible:
- \( l_1 = -10, u_1 = 9, \)
- \( l_2 = -11, u_2 = 10, \)
- \( l_3 = -6, u_3 = 6, \)
- \( l_4 = -5, u_4 = 6. \)

The corresponding invariant is:
\[ -10 \leq x - y \leq 9 \land -11 \leq x + y \leq 10 \land -6 \leq x \leq 6 \land -5 \leq y \leq 6. \]
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患有 VC2:  
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\[ y: -u_4 \leq l_4 \land 5 \leq -l_4. \]
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- Parameter constraints corresponding to a specific program path are read from the corresponding entries in tables.
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- Parameter constraints corresponding to a specific program path are read from the corresponding entries in tables.
- Accumulate all such constraints on parameter values. They are also octagonal.
- Every parameter value that satisfies the parameter constraints leads to an invariant.
- Maximum values of lower bounds and minimal values of upper bounds satisfying the parameter constraints gives the strongest invariants. Maximum and minimum values can be computed using Floyd-Warshall’s algorithm.
Complexity and Parallelization

- Overall Complexity: $O(k \times n^2)$:
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  - Parametric constraints are decomposed based on parameters appearing in them: there are $O(n^2)$ such constraints on disjoint blocks of parameters of size $\leq 4$.
- Program paths can be analyzed in parallel. Parametric constraints can be processed in parallel.
Max Formulas

Pictorial representation of all possible cases of $\max(\pm x + l, \pm y + h)$. Observe that every defined region is nonconvex.

$max(x - l_8, -y + u_8) \geq 0$
(top left corner)

$max(-x + u_5, -y + u_6) \geq 0$
(top right corner)

$max(x - l_5, y - l_6) \geq 0$
(bottom left corner)

$max(-x + u_7, y - l_7) \geq 0$
(bottom right corner)
Max Formulas

A typical template: octagonal formulas and max formulas.

\[
\begin{align*}
    l_1 & \leq x - y \leq u_1 \\
    l_2 & \leq x + y \leq u_2 \\
    l_3 & \leq x \leq u_3 \\
    l_4 & \leq y \leq u_4 \\
    \max( x - l_5, y - l_6 ) & \geq 0 \\
    \max( x - l_8, -y + u_8 ) & \geq 0 \\
    \max( -x + u_7, y - l_7 ) & \geq 0 \\
    \max( -x + u_5, -y + u_6 ) & \geq 0
\end{align*}
\]
Max Formulas – some nonconvex regions

An octagon with two corners cut out.
A square that turns into 2 disconnected components.
Table 6: Parametric Constraints for assignments with sign of one variable reversed

Assignments: $x := -x + A$, $y := y + B$

Bottom left and bottom right corners:
$max(x - l_5, y - l_6) \geq 0$ and $max(-x + u_7, y - l_7) \geq 0$

<table>
<thead>
<tr>
<th></th>
<th>$y \geq h$ absent</th>
<th>$y \geq h$ present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x \geq f$ absent</td>
<td>$(l_5 + u_7 \geq A \land l_7 - l_6 \leq B)$</td>
<td>$l_7 \leq h + B$</td>
</tr>
<tr>
<td></td>
<td>$\lor l_5 + u_7 \leq A \lor l_7 - l_4 \leq B$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lor l_2 - l_7 + u_7 \geq A - B$</td>
<td></td>
</tr>
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<td>$x \geq f$ present</td>
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<td>$u_7 \geq -f + A \lor l_6 \leq h + B$</td>
</tr>
</tbody>
</table>

The constraints for two absent tests can also be used as disjuncts in the other cases.

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<th></th>
<th>$y \geq h$ absent</th>
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<tbody>
<tr>
<td>$x \leq e$ absent</td>
<td>$(l_5 + u_7 \leq A \land l_6 - l_7 \leq B)$</td>
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</tr>
<tr>
<td></td>
<td>$\lor l_5 + u_3 \leq A \lor l_6 - l_4 \leq B$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\lor l_5 + l_6 - u_1 \geq A + B$</td>
<td></td>
</tr>
<tr>
<td>$x \leq e$ present</td>
<td>$l_5 \leq -e + A$</td>
<td>$l_5 \leq -e + A \lor l_6 \leq h + B$</td>
</tr>
</tbody>
</table>

The constraints for two absent tests can also be used as disjuncts in the other cases.
**Table 6 Contd: Parametric Constraints for assignments with sign of one variable reversed**

Assignments: \( x := -x + A, y := y + B \)

Top left and top right corners:
\[
\max(x - l_8, -y + u_8) \geq 0 \quad \text{and} \quad \max(-x + u_5, -y + u_6) \geq 0
\]

<table>
<thead>
<tr>
<th>( y \leq g ) absent</th>
<th>( y \leq g ) present</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \geq f ) absent</td>
<td>( l_8 + u_5 \geq A \land u_6 - u_8 \geq B )</td>
</tr>
<tr>
<td></td>
<td>( \lor l_3 + u_5 \geq A \lor u_6 - u_4 \geq B )</td>
</tr>
<tr>
<td></td>
<td>( \lor l_1 + u_5 + u_6 \geq A + B )</td>
</tr>
<tr>
<td>( x \geq f ) present</td>
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<tbody>
<tr>
<td>( x \leq e ) absent</td>
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</tr>
<tr>
<td></td>
<td>( \lor l_8 + u_3 \leq A \lor u_8 - u_4 \geq B )</td>
</tr>
<tr>
<td></td>
<td>( \lor l_8 + u_2 - u_8 \leq A - B )</td>
</tr>
<tr>
<td>( x \leq e ) present</td>
<td>( l_8 \leq -e + A )</td>
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</table>

The constraints for two absent tests can also be used as disjuncts in the other cases.
x := 1; y := 4;
while (y > 1) {
    if (x < 2)
        x++;
    else if (y > 3)
        y--;
    else if (x < 3)
        x++;
    else if (y > 2)
        y--;
    else if (x < 4)
        x++;
    else
        y--;
}
Example: Stairs Program

Parametric Constraints due to the initialization $x := 1; \ y := 4$:

\[
\begin{align*}
l_1 & \leq -3 \leq u_1 \\
l_2 & \leq 5 \leq u_2 \\
l_3 & \leq 1 \leq u_3 \\
l_4 & \leq 4 \leq u_4 \\
l_5 & \leq 1 \lor l_6 \leq 4 \\
u_5 & \geq 1 \lor u_6 \geq 4 \\
l_7 & \leq 1 \lor l_8 \geq 4 \\
u_7 & \geq 1 \lor u_8 \geq 4
\end{align*}
\]
Example: Stairs Program

Parametric Constraints from Table look up for

- Program paths in which $x$ is increasing:

  \[
  \begin{align*}
  u_1 &= +\infty \\
  u_2 &= +\infty \\
  u_3 &\geq 2 \\
  u_3 &\geq 3 \\
  u_3 &\geq 4
  \end{align*}
  \]

  \[
  \begin{align*}
  u_5 &\geq 2 \\
  u_5 &\geq 3 \lor u_6 \geq 3 \\
  u_5 &\geq 4 \lor u_6 \geq 2
  \end{align*}
  \]

- Program Paths in which $y$ is decreasing:

  \[
  \begin{align*}
  u_1 &= +\infty \\
  l_2 &= -\infty \\
  l_4 &\leq 3 \\
  l_4 &\leq 2 \\
  l_4 &\leq 1
  \end{align*}
  \]

  \[
  \begin{align*}
  l_5 &\geq 2 \lor l_6 \geq 5 \\
  l_5 &\geq 3 \lor l_6 \geq 4 \\
  l_5 &\geq 4 \lor l_6 \geq 3
  \end{align*}
  \]
Example: Stairs Program

Putting all parametric constraints together and deriving the strongest max invariant
(contrasted with the strongest octagonal invariant)

\begin{verbatim}
x := 1; y := 4;
while (y>1) {
    if (x<2) x++;
    else if (y>3) y--;
    else if (x<3) x++;
    else if (y>2) y--;
    else if (x<4) x++;
    else y--;
}
\end{verbatim}
Example: Stairs Program

Putting all parametric constraints together and deriving the strongest max invariant
(contrasted with the strongest octagonal invariant)

\[x := 1; y := 4;\]
\[\text{while } (y > 1) \{\]
  \[\text{if } (x < 2)\]
  \[x++;\]
  \[\text{else if } (y > 3)\]
  \[y--;\]
  \[\text{else if } (x < 3)\]
  \[x++;\]
  \[\text{else if } (y > 2)\]
  \[y--;\]
  \[\text{else if } (x < 4)\]
  \[x++;\]
  \[\text{else}\]
  \[y--;\]
\[\}\]
Invariants of a Program with a nested loop

```plaintext
x := 1; y := 4;
while (true) {
  if (x<2)
    x++;
  else if (y>3)
    y--;
  else if (x<3)
    x++;
  else if (y>2)
    y--;
  else if (x<4)
    x++;
  else if (y>1)
    y--;
  else
    while (x>1) {
      x--; y++;
    }
}
```
Invariants of a Program with a nested loop

\begin{verbatim}
x := 1; y := 4;
while (true) {
  if (x < 2)
    x++;
  else if (y > 3)
    y--;
  else if (x < 3)
    x++;
  else if (y > 2)
    y--;
  else if (x < 4)
    x++;
  else if (y > 1)
    y--;
  else
    while (x > 1) {
      x--; y++;
    }
}
\end{verbatim}

outer invariant
Invariants of a Program with a nested loop

\[x := 1; y := 4;\]
\[\text{while (true) } \{\]
  \[\text{if (x<2)} \]
  \[x++;\]
  \[\text{else if (y>3)} \]
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  \[x++;\]
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  \[y--;\]
  \[\text{else} \]
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    \[x--; y++;\]
  \[\}\]
\[\}\]
Max Invariants vs Octagonal Invariants

- 16 instead of 8 parameters per variable pair:
  \( l_1, u_1, \ldots, l_4, u_4, \ l_5, u_5, \ldots, l_8, u_8 \)
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  **Max**: Multiple noncomparable values for parameter tuples.
  (recall the step function invariant before)

\[
\begin{align*}
  \max(x - l_5, y - l_6) &\geq 0 \\
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  \max(-x + 2, -y + 3) &\geq 0 \\
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  \max(x - 4, y - 2) &\geq 0
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\]

Many disjunctions in Tables.

Experimentation and heuristics for determining possibilities in disjunctions that are more useful.

Sacrificing efficiency to generate stronger invariants.

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Ranking functions can be synthesized by hypothesizing polynomials in program variables and unary predicates on program variable in a loop body.

Example

```java
while (n>1) {
    if   n mod 2 = 0 then n := n/2
    else   n := n+1
}
```
Termination Analysis based on Quantifier Elimination

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Theorem There does not exist any polynomial in \( n \) that can serve as a ranking function.
Termination Analysis based on Quantifier Elimination

The synthesis of a polynomial ranking function of arbitrary degree can be hypothesized: much like verification conditions, leading to two constraints:

1. \( n \mod 2 = 0 \): easy.
2. otherwise: \( n' = n + 1 \), so tricky.

Must use the function \( n \mod 2 \). Consider \( n + 2(n \mod 2) \) as a possible ranking function (which can be generated from \( An + B(n \mod 2) + C \)).

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Interpolant Generation using Quantifier Elimination

**Craig:** Given $\alpha \implies \beta$, an intermediate formula $\gamma$ in common symbols of $\alpha$ and $\beta$ exists and can be constructed such that

$$\alpha \implies \gamma \land \gamma \implies \beta$$

In Kapur et al (FSE06) we showed an obvious connection between interpolation and quantifier elimination.
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- Eliminate uncommon symbols from $\alpha$: interpolant.

Similarly for $\beta$.

All interpolants between $\alpha$ and $\beta$ form a lattice using implication with the interpolant generated from $\alpha$ as the top element of the lattice and the one from $\beta$ being the bottom element.

The above assertions assume complete quantifier elimination.

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Interpolants over Equality with Uninterpreted Symbols

\( \alpha \), a finite conjunction of equality and disequalities over constants and function symbols, with their subset \( UC \) being uncommon symbols with \( \beta \)'s (\( UC \) may or may not include nonconstant function symbols).

- Run Kapur’s congruence closure algorithm (RTA 1997) on equations with two differences.
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  - Define a total ordering in which all uncommon nonconstant symbols are bigger than all uncommon constant symbols, followed by all common nonconstant symbols which are made bigger than all common constant symbols, run congruence closure which is ground completion.
  - This is in contrast to Kapur’s algorithm in which all nonconstant symbols are bigger than constant symbols.
The result is a finite set of rewrite rules of the form
\[ f(c, d) \rightarrow e \quad (f, c, d \text{ are common }) \implies e \text{ is common} \]
\[ c \rightarrow e; \quad (c \text{ is common}) \implies (e \text{ is common}) \]
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- **Horn clause introduction** From

  \[ f(a, b) \rightarrow e, \quad f(c, d) \rightarrow g, \]

  \[ (a = c \land b = d) \implies e = g \]

  where \( f \) is uncommon or least one of \( a, b, c, d \) is uncommon.
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- **Normalize Horn clauses** Run congruence closure on the antecedent and normalize the consequent. If a Horn clause becomes trivially true, it is discarded. This is done every time a new Horn clause is generated.
Conditional Rewriting The consequent of a Horn clause may have a uncommon symbol on its left side, which may also appear in an antecedent. That can be replaced in all such antecedents by carrying the conditions of this antecedent,

\[(c_1 = d_1 \land \cdots \land c_k = d_k) \implies c = d\]

\[(a_1 = b_1 \land \cdots \land a_l = b_l) \implies a = b\]

If \(a\) is some \(c_i\) or \(d_i\), then

\[(a_1 = b_1 \land \cdots \land a_l = b_l) \land (c_1 = d_1 \land b = d_i \land \cdots c_k = d_k) \implies c = d\]

Disequalities do not play since at best they can do is to delete a Horn clause or identify unsatisfiability. But if \(\alpha\) is assumed to be satisfiable in the input, then the result of this includes an interpolant which is all the equations and Horn clauses which only have common symbols.
An Example of Interpolant Generation on EUF

Mutually contradictory $\alpha = \{x_1 = z_1, z_2 = x_2, z_3 = f(x_1), f(x_2) = z_4, x_3 = z_5, z_6 = x_4, z_7 = f(x_3), f(x_4) = z_8\}$ and

$\beta = \{z_1 = z_2, z_5 = f(z_3), f(z_4) = z_6, y_1 = z_7, z_8 = y_2, y_1 \neq y_2\}$

Commons symbols are $\{f, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$. 

Our algorithm gives:

$\{x_1 \rightarrow z_1, x_2 \rightarrow z_2, f(z_1) \rightarrow z_3, f(z_2) \rightarrow z_4, x_3 \rightarrow z_5, x_4 \rightarrow z_6, f(z_5) \rightarrow z_7, f(z_6) \rightarrow z_8\}$

The interpolant $I_\alpha$:

$\{f(z_1) = z_3, f(z_2) = z_4, f(z_5) = z_7, f(z_6) = z_8\}$

No need to generate any Horn clauses.

The interpolant reported by McMillan's algorithm is:

$(z_1 = z_2 \land (z_3 = z_4 = \Rightarrow z_5 = z_6)) = \Rightarrow (z_3 = z_4 \land z_7 = z_8)$

Tinelli et al's algorithm is:

$(z_1 = z_2 = \Rightarrow z_3 = z_4) \land (z_5 = z_6 = \Rightarrow z_7 = z_8)$.
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$\{f(z_1) = z_3, f(z_2) = z_4, f(z_5) = z_7, f(z_6) = z_8\}$. No need to generate any Horn clauses.

The interpolant reported by McMillan’s algorithm is:
$(z_1 = z_2 \land (z_3 = z_4 \implies z_5 = z_6)) \implies (z_3 = z_4 \land z_7 = z_8)$
An Example of Interpolant Generation on EUF

Mutually contradictory \( \alpha = \{x_1 = z_1, z_2 = x_2, z_3 = f(x_1), f(x_2) = z_4, x_3 = z_5, z_6 = x_4, z_7 = f(x_3), f(x_4) = z_8\} \) and
\( \beta = \{z_1 = z_2, z_5 = f(z_3), f(z_4) = z_6, y_1 = z_7, z_8 = y_2, y_1 \neq y_2\} \)
Commons symbols are \( \{f, z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\} \).

Our algorithm gives:
\( \{x_1 \rightarrow z_1, x_2 \rightarrow z_2, f(z_1) \rightarrow z_3, f(z_2) \rightarrow z_4, x_3 \rightarrow z_5, x_4 \rightarrow z_6, f(z_5) \rightarrow z_7, f(z_6) \rightarrow z_8\} \).

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Tinelli et al’s algorithm, it is:
\( (z_1 = z_2 \implies z_3 = z_4) \land (z_5 = z_6 \implies z_7 = z_8) \).
Interpolant Generation over Octagonal formulas

Let $\alpha$ to be a conjunction of $\pm x_i \leq c_i$ and $\pm x_i \pm x_j \leq c_{i,j}$, where $x_i$ and $x_j$ are distinct.

1. For each uncommon symbol $x_i$ in $\alpha$, consider two octagon formulas in which the sign of $x_i$ is positive in one and negative in the other.
   $x_i$ is eliminated by adding the two formulas. This must be done for every pair of such formulas.

The result of all uncommon symbols is an interpolant generated from $\alpha$. This is illustrated below.
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2. In case a formula of the form $2x_j \leq a$ or $-2x_j \leq a$, it is normalized in the case octagonal formulas are over the integers.

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2. In case a formula of the form $2x_j \leq a$ or $-2x_j \leq a$, it is normalized in the case octagonal formulas are over the integers.

3. If some uncommon symbol only appears positively or negatively, all octagonal formulas containing it can be eliminated as they do not occur in the interpolant.

The result of all uncommon symbols is an interpolant generated from $\alpha$. This is illustrated below.
Griggio’s example of Octagonal Formulas

Mutually contradictory
\[ \alpha = \{ x_1 - x_2 \geq -4, \quad -x_2 - x_3 \geq 5, \quad x_2 + x_6 \geq 4, \quad x_2 + x_5 \geq -3 \}, \]
\[ \beta = \{ -x_1 + x_3 \geq -2, \quad -x_4 - x_6 \geq 0, \quad -x_5 + x_4 \geq 0 \} \]

Uncommon symbols: \{x_2\}. 
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Uncommon symbols: \( \{x_2\} \).

1. Eliminate \( x_2 \):
\[ \{ -x_3 + x_5 \geq 2, x_1 + x_6 \geq 0, x_1 + x_5 \geq -7, -x_3 + x_6 \geq 9 \}. \]
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\[ I_\alpha = \{ -x_3 + x_5 \geq 2, \quad x_1 + x_6 \geq 0, \quad x_1 + x_5 \geq -7, \quad -x_3 + x_6 \geq 9 \} \].
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Griggio’s algorithm gives the conditional interpolant
\[ (-x_6 - x_5 \geq 0) \Rightarrow (x_1 - x + 3 \geq 3) \]
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Griggio’s algorithm gives the conditional interpolant
\[ (-x_6 - x_5 \geq 0) \implies (x_1 - x + 3 \geq 3) \]
The strongest interpolant is an octagonal formula and is generated by our algorithm.
Saturation based Invariant Strengthening and Abductor Generation

- Given a formula claimed to be an invariant $I$ (or a post condition a la IC3)
Saturation based Invariant Strengthening and Abductor Generation

- Given a formula claimed to be an invariant \( I \) (or a post condition a la IC3)
- attempt to prove it inductive incrementally for every path:
  \((I \land \text{cond}) \implies I'\).

- How to obtain \( \psi \)?
  Approximate \( \psi \) to be \((I \land \text{cond} \land \psi) \implies I'\).
  \( \psi \) is an abductor for \((I \land \text{cond}, I')\).
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- How to obtain $\psi$?
- Approximate $\psi$ to be $(I \land \text{cond} \land \psi) \implies I'$
- $\psi$ is an abductor for $(I \land \text{cond}, I')$.
Saturation based Invariant Strengthening and Abductor Generation

Example

```pascal
var x, y, z: integer end var
x := 0, y := 0, z := 9;
while x ≤ N do
  x := x + 1;
  y := y + 1;
  z := z + x − y;
end while

Goal: z ≤ 0 is a loop invariant.
```
Saturation based Invariant Strengthening and Abductor Generation

Example

```plaintext
var x, y, z: integer end var
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z ≤ 0 ⇒ z + x − y ≤ 0.
```
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Goal: z ≤ 0 is a loop invariant.

z ≤ 0 ⇒ z + x − y ≤ 0.
(z ≤ 0 ∧ z + x − y ≤ 0) ⇒ (z + x − y ≤ 0 ∧ z + 2x − 2y ≤ 0).
```
Saturation based Invariant Strengthening and Abductor Generation

Example

\texttt{var } x, y, z: \texttt{integer } \texttt{end var}
\texttt{x := 0, y := 0, z := 9;}
\texttt{while } x \leq N \texttt{ do}
\texttt{x := x + 1; y := y + 1; z := z + x - y;}
\texttt{end while}

\textbf{Goal: } z \leq 0 \textit{ is a loop invariant.}
\texttt{z \leq 0 \implies z + x - y \leq 0.}
\texttt{(z \leq 0 \land z + x - y \leq 0) \implies (z + x - y \leq 0 \land z + 2x - 2y \leq 0).}
Strengthen it to \texttt{z \leq 0 \land x - y \leq 0}
Saturation based Invariant Strengthening and Abductor Generation

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```
var x, y, z: integer end var
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Goal: z ≤ 0 is a loop invariant.
```

\[ z \leq 0 \implies z + x - y \leq 0. \]

\[ (z \leq 0 \land z + x - y \leq 0) \implies (z + x - y \leq 0 \land z + 2x - 2y \leq 0). \]

Strengthen it to \( z \leq 0 \land x - y \leq 0 \)

Quantifier elimination comes to the rescue
Salient Points

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Salient Points

- Quantifier-elimination is ubiquitous.
- Since general (complete) QE methods are very expensive and their outputs are hard to decipher, it is better to consider special cases, sacrificing completeness as well as generality.
- There is a real trade-off between resources/efficiency and precision/incompleteness.
- Let us call a spade a spade.
Challenges

- Can we develop specialized quantifier elimination algorithms/heuristics for various fragments of real and complex arithmetic?

- Outputs generated by them need not be complete but must be useful for SMT solvers and theorem provers/verification systems.

- How can propositional reasoning, first-order and equational reasoning, redundancy checks, and preprocessing be exploited in general quantifier elimination methods to make them more effective?

- Can we have better, effective interfaces between a computer algebra system and a theorem prover?

- Can certificates be generated for outputs computed by a symbolic computation algorithm so that a theorem prover/SMT solver can trust it?
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Gröbner basis computations are being widely used in many application domains, especially for equational solving.
Parametric Gröbner Computations in Quantifier Elimination over the reals

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- Given the success of Gröbner basis computations for handling many problems in algebraic geometry, polynomial equation solving and program analysis, as well our good experience in computing comprehensive Gröbner systems, we are encouraged to build a practical incomplete heuristic for the theory of real closed field:
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2. use sum of squares heuristics (using completing square strategy): \( \sum_{u=1}^{k} p_u^2 = 0 \implies p_i = 0. \)
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5. a small step: interpolant generation for concave quadratic polynomial inequalities (over EUF).