Computing a Complete Basis for Equalities Implied by a System of LRA Constraints

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7/30/2017

Linear Arithmetic / Linear Programming

Input: $\sum_{j=1}^{n} a_{ij} \cdot x_j \circ_i b_i \qquad \text{for } i = 1, \ldots, m$

coefficients ($\in \mathbb{Z}$)

variables ($\exists x_j$)

predicate ($\in \{<, \leq\}$)

bound ($\in \mathbb{Z}$)

Linear Arithmetic / Linear Programming

Input: $a_i^T \mathbf{x} \circ_i b_i \quad$ for $i = 1, \ldots, m$

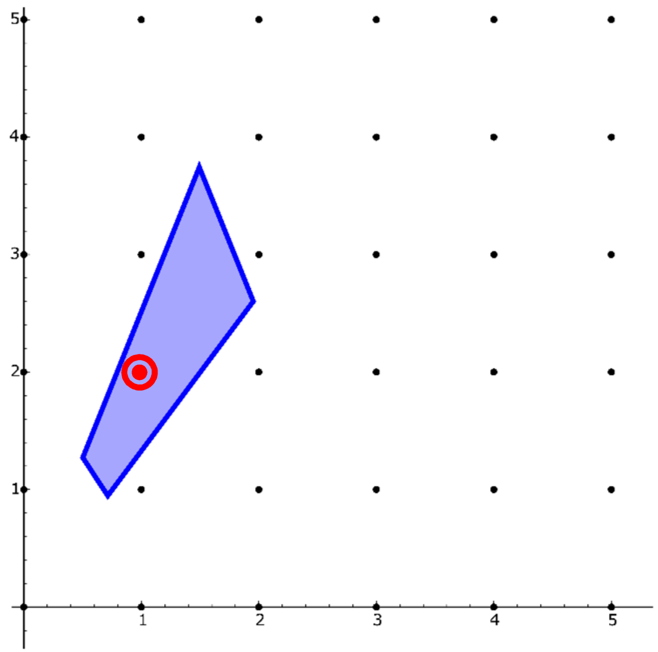
Goal: $\mathbf{x} \in \mathbb{R}^n$ (LRA/LP) or $\mathbf{x} \in \mathbb{Z}^n$ (LIA/ILP)

Complexity: P NP

Example:

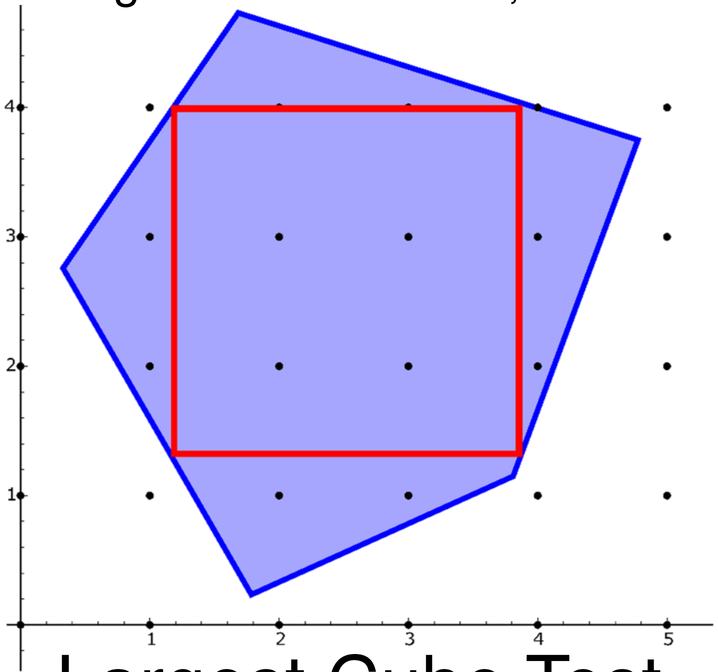
$$2y \leq 5x, \qquad 3y \geq 4x,$$
$$2y \leq -5x + 15, \qquad 2y \geq -3x + 4,$$

LIA: $(x, y) = (1, 2)$



Fast Cube Tests
for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)

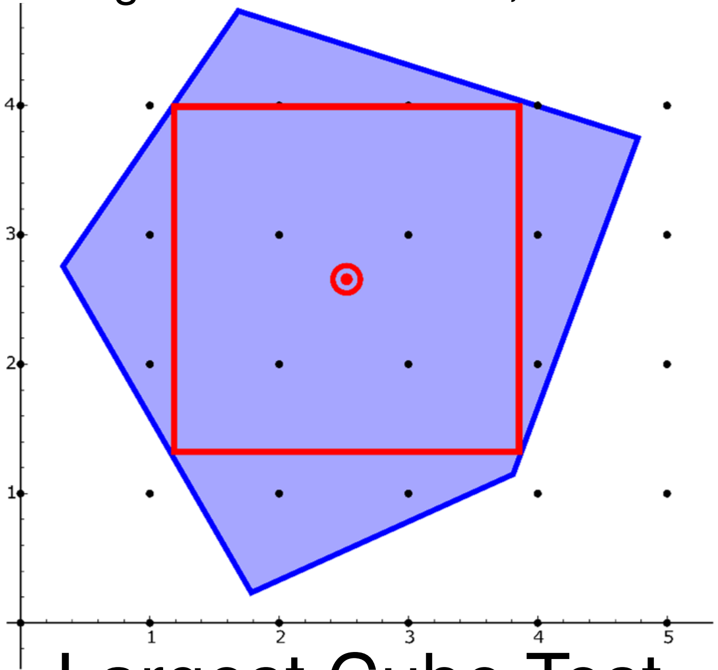


Largest Cube Test

- largest cube inside the set of real solutions

Fast Cube Tests for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)

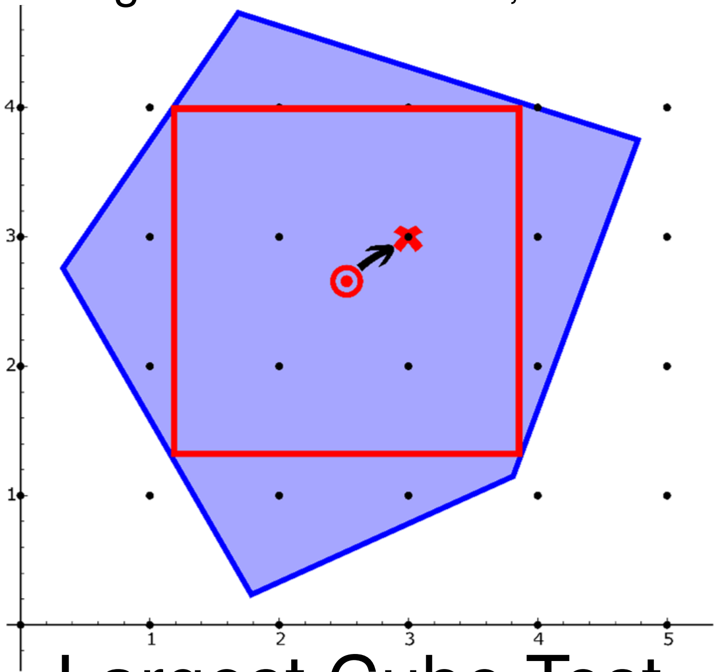


Largest Cube Test

- largest cube inside the set of real solutions
- center point

Fast Cube Tests
for LIA constraint solving

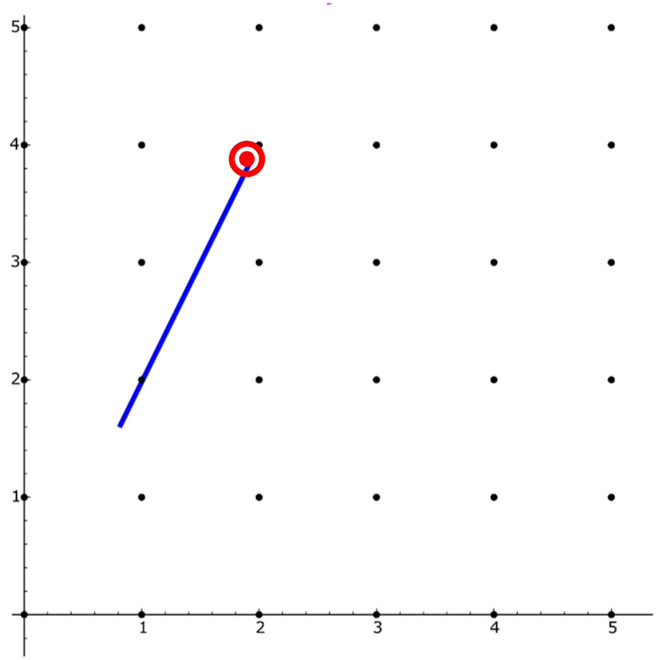
(Bromberger & Weidenbach, IJCAR 2016)



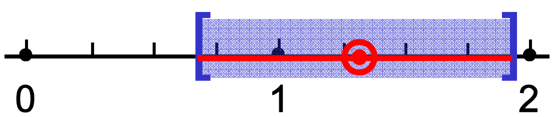
Largest Cube Test

- largest cube inside the set of real solutions
- center point → integer point
- optimization LP (LRA) + evaluation

Equalities



$\Downarrow$



$x \mapsto 1 \Rightarrow \boxed{y \mapsto 2x} \Rightarrow y \mapsto 2$

$$2y \leq 2 + 3x, \quad 2y \geq -2 + 5x,$$
$$5y \leq 25 - 3x, \quad 2y \geq 4 - 2x,$$
$$\boxed{y = 2x}$$

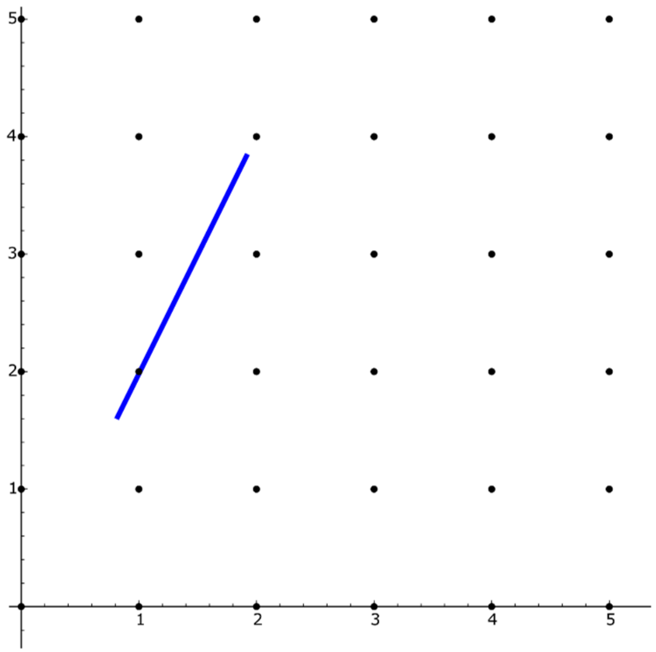
substitute $\Downarrow$ $y \mapsto 2x$

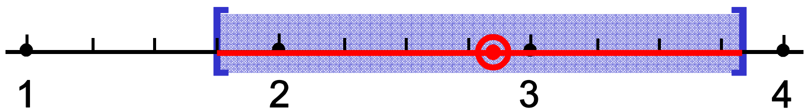
$$2 \cdot 2x \leq 2 + 3x, \quad 2 \cdot 2x \geq -2 + 5x,$$
$$5 \cdot 2x \leq 25 - 3x, \quad 2 \cdot 2x \geq 4 - 2x,$$

$\Downarrow$

$$x \leq 2, \quad 2 \geq x,$$
$$13x \leq 25, \quad 3x \geq 2,$$

Equalities





$$y \mapsto 3 \Rightarrow \boxed{x \mapsto \frac{y}{2}} \Rightarrow x \mapsto \frac{3}{2}$$

$$2y \leq 2 + 3x, \qquad 2y \geq -2 + 5x,$$
$$5y \leq 25 - 3x, \qquad 2y \geq 4 - 2x,$$
$$\boxed{y = 2x}$$

Diophantine Equation Handler

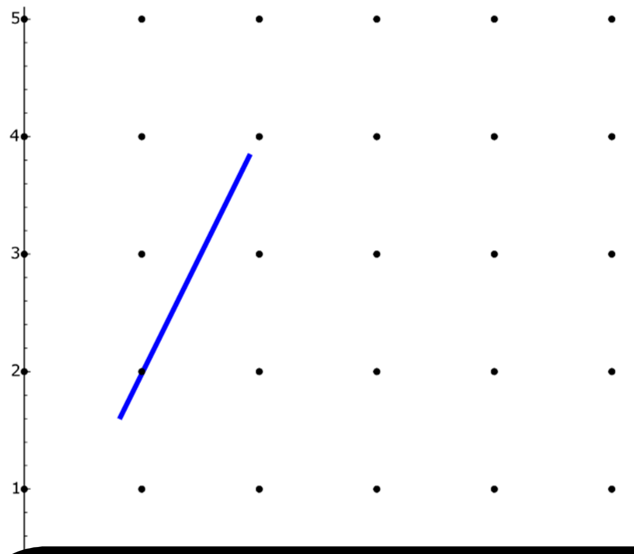
A Practical Approach to

Satisfiability Modulo Linear Integer Arithmetic

by A. Griggio. JSAT 2012

$$y \leq 4, \qquad 4 \geq y,$$
$$13y \leq 50, \qquad 3y \geq 4,$$

Equalities



$$2y \le 2 + 3x, \qquad 2y \ge -2 + 5x,$$
$$5y \le 25 - 3x, \qquad 2y \ge 4 - 2x,$$
$$\boxed{y = 2x}$$

Diophantine Equation Handler

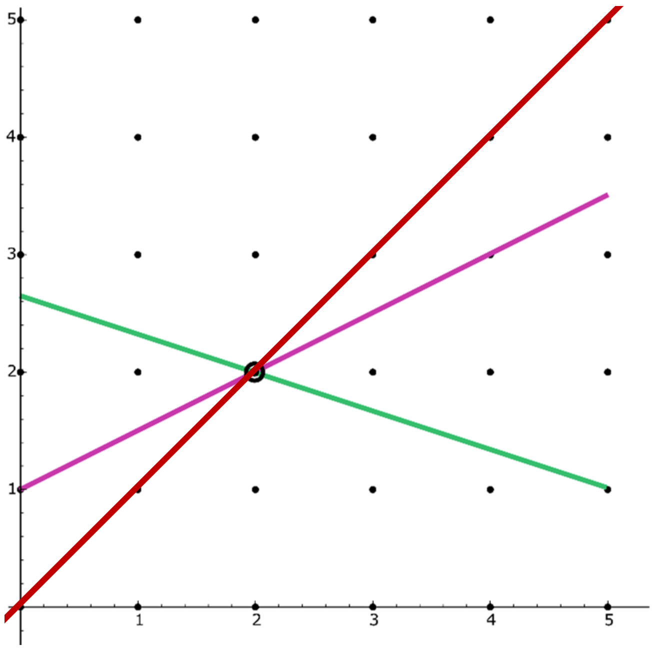
A Practical Approach to

Not Always Explicit!

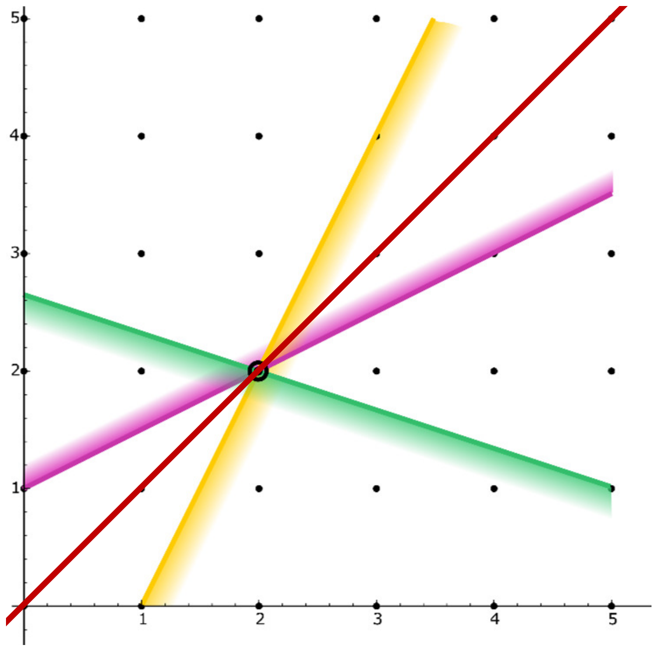
$$y \mapsto 3 \Rightarrow \boxed{x \mapsto \frac{y}{2}} \Rightarrow x \mapsto \frac{3}{2}$$

$$13y \le 50, \qquad 3y \ge 4,$$

Implied Equalities



$$\binom{x}{y} = \binom{2}{2}$$



explicit

$$\forall x \in \mathbb{R}^n . \boxed{Ax \le b} \rightarrow \boxed{d^T x = c}$$

implicit

explicit:

$$2y = x + 2,$$

$$3y \le -x + 8, \qquad 3y \ge -x + 8,$$

$$\Downarrow$$

$$2y = x + 2 \qquad y = 2x - 2,$$

$$3y = -x + 8$$

implicit:

$$2y \ge x + 2,$$

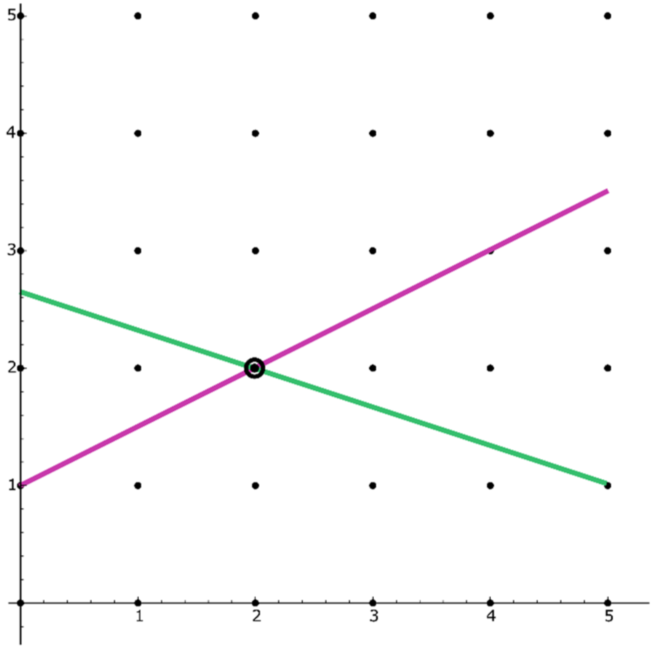
$$3y \le -x + 8, \qquad y \le 2x - 2,$$

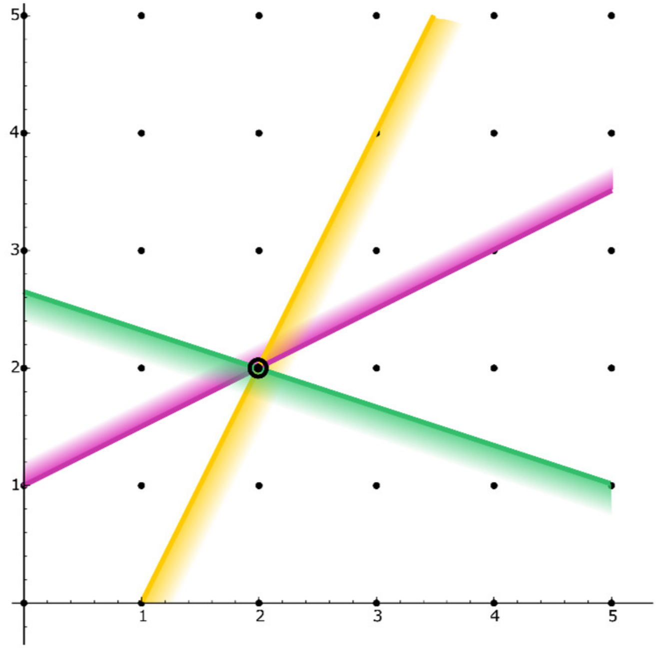
$$\Downarrow$$

$$2y = x + 2 \qquad y = 2x - 2,$$

$$3y = -x + 8$$

Equality Basis





explicit

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

implicit

Explicit:

$$\boxed{2y = x + 2,}$$
$$\boxed{3y \leq -x + 8, \qquad 3y \geq -x + 8,}$$

$$\Downarrow$$

$$\boxed{2y = x + 2} \qquad \boxed{y = 2x - 2,}$$
$$\boxed{3y = -x + 8}$$

Implicit:

$$\boxed{2y \geq x + 2,}$$
$$\boxed{3y \leq -x + 8,} \qquad \boxed{y \leq 2x - 2,}$$

$$\Downarrow$$

$$\boxed{2y = x + 2} \qquad \boxed{y = 2x - 2,}$$
$$\boxed{3y = -x + 8}$$

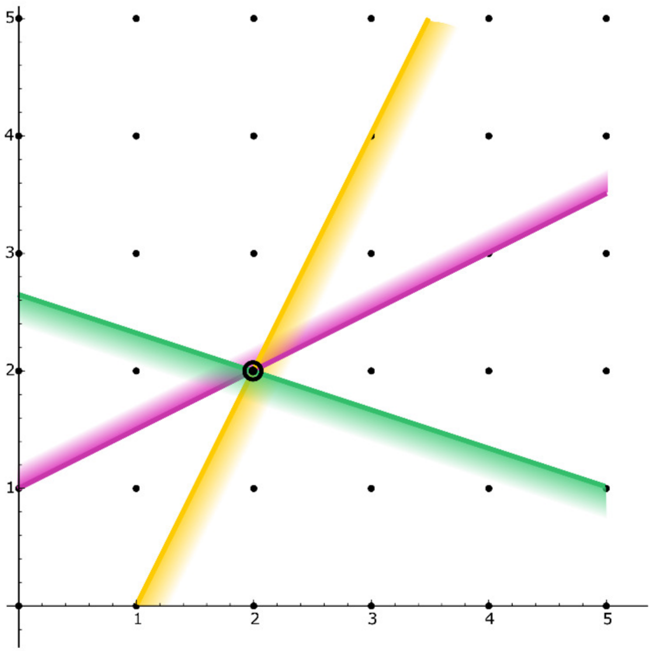
Equality Basis

Equality Basis:

1.set of

linear independent equalities

2. maximal



Implied Equalites:

All

linear combinations

of the

Equality Basis

$$\forall x \in \mathbb{R}^n . \boxed{Ax \le b} \rightarrow \boxed{d^T x = c}$$

$$\boxed{2y \ge x + 2,}$$

$$\boxed{3y \le -x + 8,} \qquad \boxed{y \le 2x - 2,}$$

$$\Downarrow$$

$$\boxed{2y = x + 2}$$

$$\boxed{3y = -x + 8} \qquad \boxed{y = 2x - 2,}$$

$$\lambda_1, \lambda_2 \in \mathbb{R}$$

$$\lambda_1 \cdot \boxed{2y = x + 2} + \lambda_2 \cdot \boxed{3y = -x + 8}$$

$$\Downarrow$$

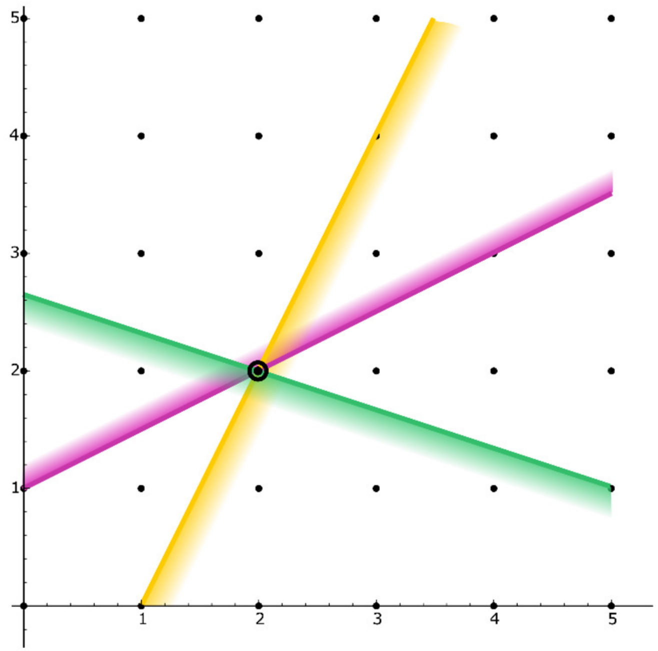
$$(2\lambda_1 + 3\lambda_2)y =$$
$$(\lambda_1 - \lambda_2)x + (2\lambda_1 - 8\lambda_2)$$

Equality Basis

Equality Basis:

1.set of
linear independent equalities

2. maximal



Implied Equalites:

All
linear combinations
of the
Equality Basis

$$\forall x \in \mathbb{R}^n.\ \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$\boxed{\boxed{2y \geq x + 2,} \quad \boxed{3y \leq -x + 8,} \quad \boxed{y \leq 2x - 2,}}$$

$$\Downarrow$$

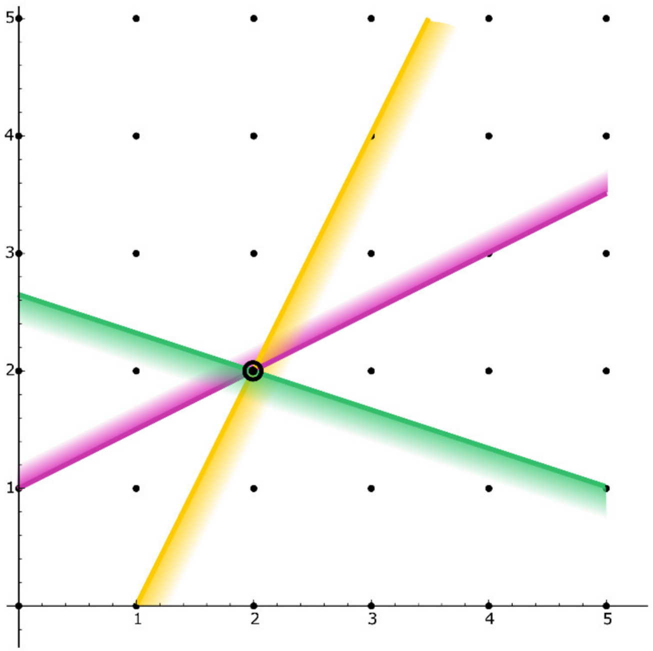
$$\boxed{\boxed{2y = x + 2} \quad \boxed{3y = -x + 8}}$$

$$\frac{7}{5} \cdot \boxed{2y = x + 2} - \frac{3}{5} \cdot \boxed{3y = -x + 8}$$

$$\Downarrow$$

$$\boxed{y = 2x - 2,}$$

Equality Basis



Simplification:

Eliminating Equalities via Substitution

$$\forall x \in \mathbb{R}^n. \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$\boxed{2y \geq x + 2,} \quad \boxed{3y \leq -x + 8,} \quad \boxed{y \leq 2x - 2,}$$

$$\Downarrow$$

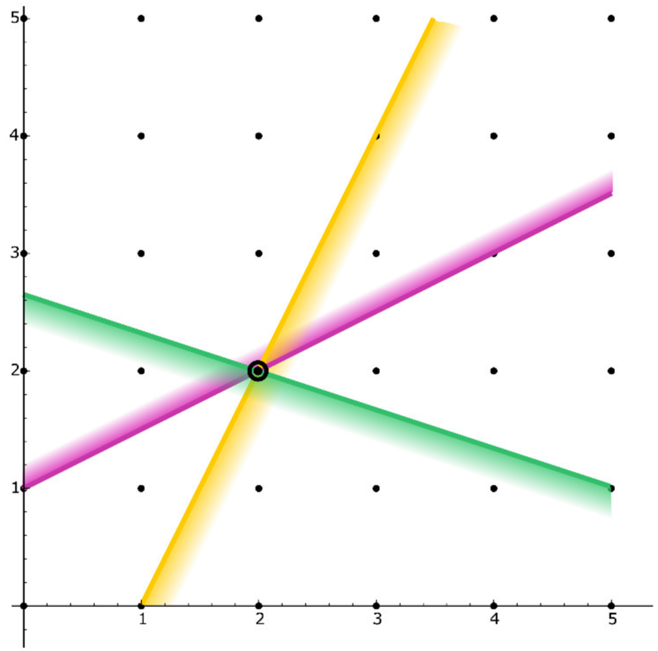
$$\boxed{2y = x + 2} \quad \boxed{3y = -x + 8}$$

$$\boxed{2y = x + 2} \Rightarrow \boxed{x \mapsto 2y - 2}$$

$$\boxed{3y = -x + 8} \Rightarrow \boxed{5y = 10} \Rightarrow \boxed{y \mapsto 2}$$

Equality Basis

<u>Simplification:</u>
Eliminating
Equalities
via
Substitution



<u>Verifying</u>
<u>Implied Equalities:</u>
via
Substitution
(Nelson-Oppen)

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$2y \geq x + 2, \quad 3y \leq -x + 8, \quad y \leq 2x - 2,$$

$$\Downarrow$$

$$2y = x + 2$$
$$3y = -x + 8$$

?

$$y = 2x - 2$$

$$x \mapsto 2y - 2$$

$$\Downarrow$$

$$y = 4y - 6$$

$$y \mapsto 2$$

$$\Downarrow$$

$$0 = 0$$

Nelson-Oppen Method

$$f(w) \neq f(z) \;\wedge\; f(x) = u \;\wedge\; f(y) = v \;\wedge$$
$$u - v = w \;\wedge\; x \leq y \;\wedge\; y + z \leq x \;\wedge\; 0 \leq z$$

$$\begin{aligned} u - v = w &\wedge \\ x \leq y &\wedge \\ y + z \leq x &\wedge \\ 0 \leq z & \end{aligned}$$

Exchange pairs of equivalent variables

$$\begin{aligned} f(w) \neq f(z) &\wedge \\ f(x) = u &\wedge \\ f(y) = v & \end{aligned}$$

LRA

True

UF

True

Nelson-Oppen Method

$$f(w) \neq f(z) \ \wedge\ f(x) = u \ \wedge\ f(y) = v \ \wedge$$
$$u - v = w \ \wedge\ x \leq y \ \wedge\ y + z \leq x \ \wedge\ 0 \leq z$$

$x = y$

$$\begin{aligned} u - v = w &\ \wedge \\ x = y &\ \wedge \\ y + z \leq x &\ \wedge \\ 0 \leq z &\ \wedge \\ u = v &\ \wedge \\ w = z & \end{aligned}$$

Exchange pairs of equivalent variables

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$x = y, w = z$

Exchange pairs of equivalent variables

$u = v$

$$\begin{aligned} u - v &= w \wedge \\ x &= y \wedge \\ y + z &\leq x \wedge \\ 0 &\leq z \wedge \\ u &= v \wedge \\ w &= z \end{aligned}$$

LRA

True

$$\begin{aligned} f(w) &\neq f(z) \wedge \\ f(x) &= u \wedge \\ f(y) &= v \wedge \\ x &= y \wedge \\ u &= v \wedge \\ w &= z \end{aligned}$$

UF

True

Nelson-Oppen Method

$$f(w) \neq f(z) \;\wedge\; f(x) = u \;\wedge\; f(y) = v \;\wedge$$
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$$\begin{aligned} u - v &= w \wedge \\ x &= y \wedge \\ y + z &\leq x \wedge \\ 0 &\leq z \wedge \\ u &= v \wedge \\ w &= z \end{aligned}$$

Exchange pairs of equivalent variables

$$\begin{aligned} f(w) &\neq f(z) \wedge \\ f(x) &= u \wedge \\ f(y) &= v \wedge \\ x &= y \wedge \\ u &= v \wedge \\ w &= z \end{aligned}$$

$u = v$

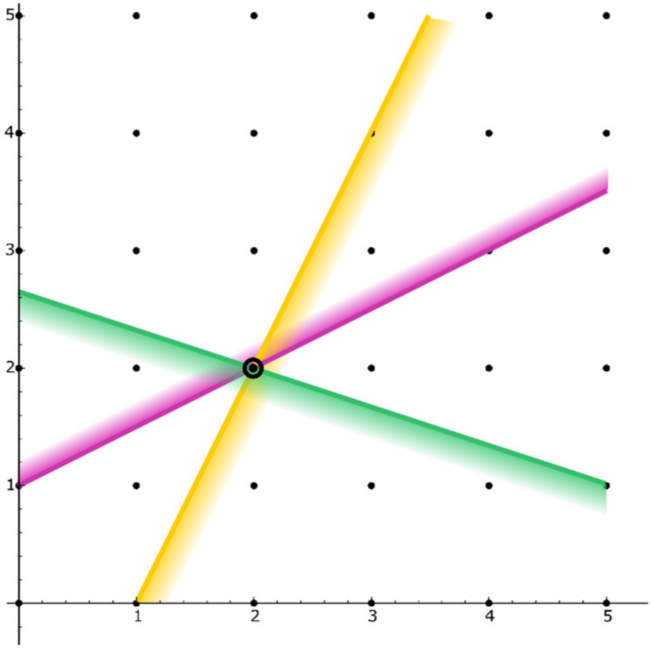
LRA

True

UF

False

Nelson-Oppen Combination



$u = v$

pairs of equivalent variables

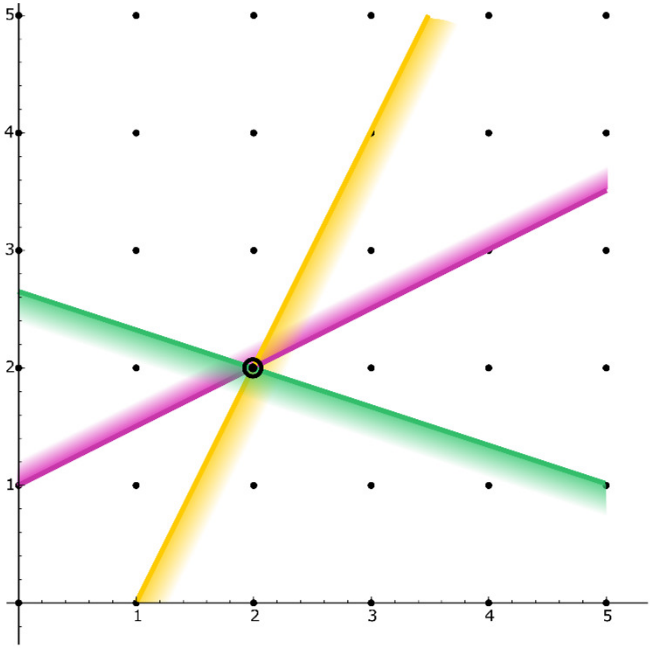
$\mathcal{T}$

$w = z$

$$\boldsymbol{\sigma} := \{\boxed{x \mapsto 2y - 2}, \boxed{y \mapsto 2}\}$$

$$x\sigma = 2$$
$$y\sigma = 2$$
$$\Longrightarrow \quad x = y$$

Nelson-Oppen Combination



$u = v$

pairs of equivalent variables

$\mathcal{T}$

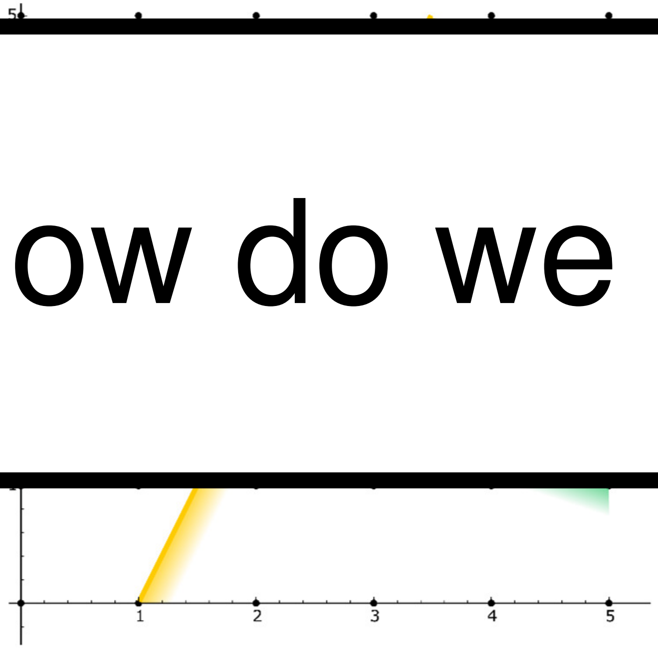
$w = z$

semantic equivalence $\xrightarrow{\sigma\ +\ \text{normalizing}}$ syntactic equivalence

Find equivalent variables with DAGs!

Nelson-Oppen Combination

$u = v$



How do we find equalities?

$w = z$

semantic equivalence $\xrightarrow{\sigma \text{ + normalizing}}$ syntactic equivalence

Find equivalent variables with DAGs!

Equality Test

subject to:

$a_i^T x \circ_i b_i$

for $i = 1, \ldots, m$

where $\circ_i \in \{\leq, <\}$

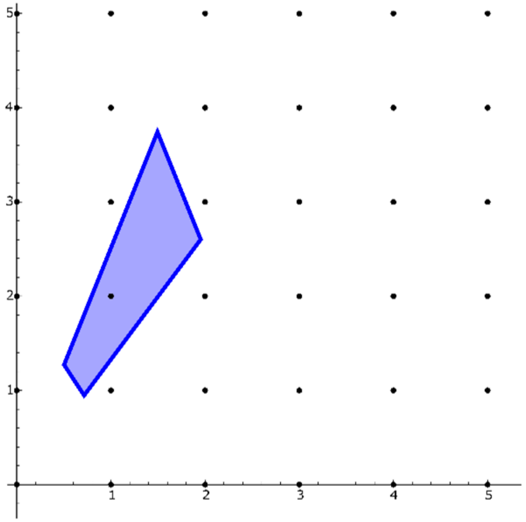
$\Rightarrow$

subject to:

$a_i^T x < b_i$

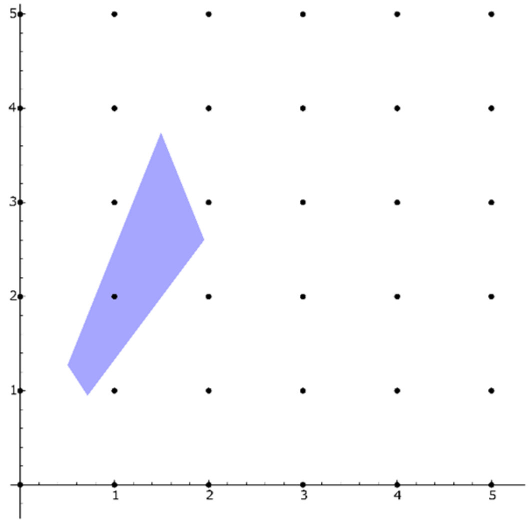
for $i = 1, \ldots, m$

Interior & Surface



$\Rightarrow$

Interior



Equality Test

subject to:

$a_i^T x \circ_i b_i$

for $i = 1, \ldots, m$

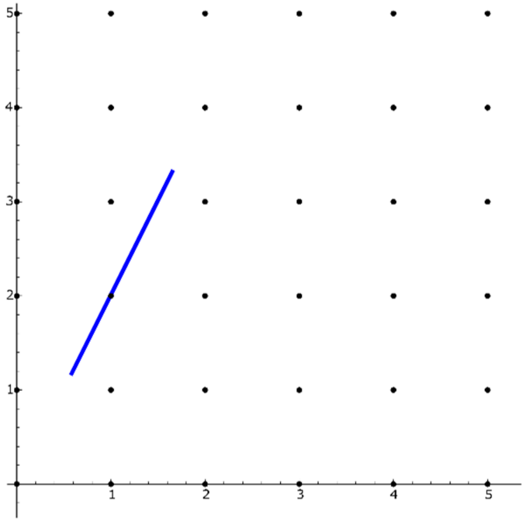
where $\circ_i \in \{\leq, <\}$

subject to:

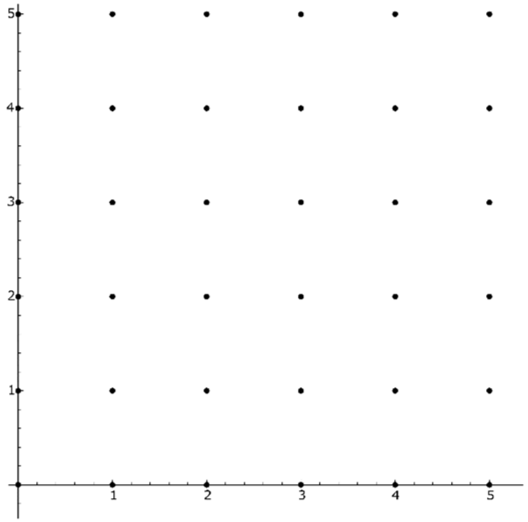
$a_i^T x < b_i$

for $i = 1, \ldots, m$

Interior & Surface



Interior



Equality Test

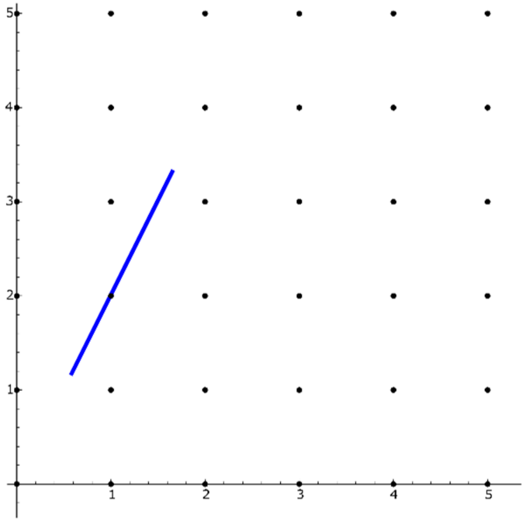
subject to:

$a_i^T x \circ_i b_i$

for $i = 1, \ldots, m$

where $\circ_i \in \{\leq, <\}$

sat

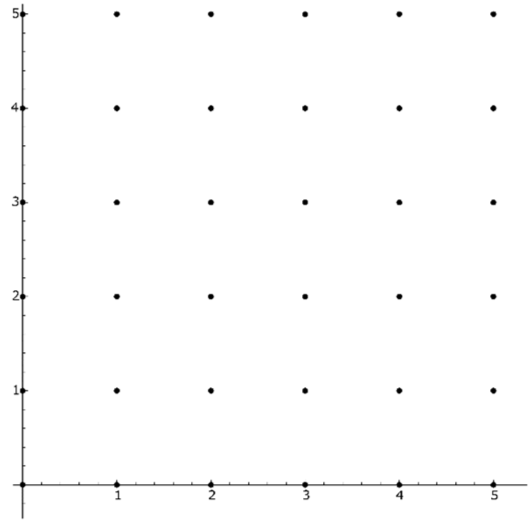


subject to:

$a_i^T x < b_i$

for $i = 1, \ldots, m$

unsat



Equality Test

subject to:

$a_i^T x \circ_i b_i$

for $i = 1, \ldots, m$

where $\circ_i \in \{\leq, <\}$

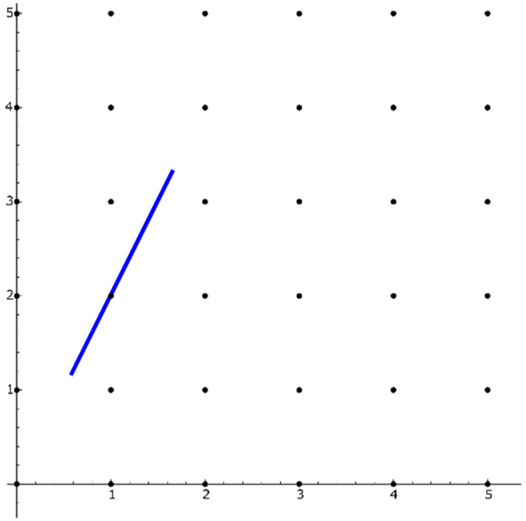
$\Rightarrow$

subject to:

$a_i^T x < b_i$

for $i = 1, \ldots, m$

sat



unsat

Conflict: $C \subseteq \{1, \ldots, m\}$

minimal subset such that

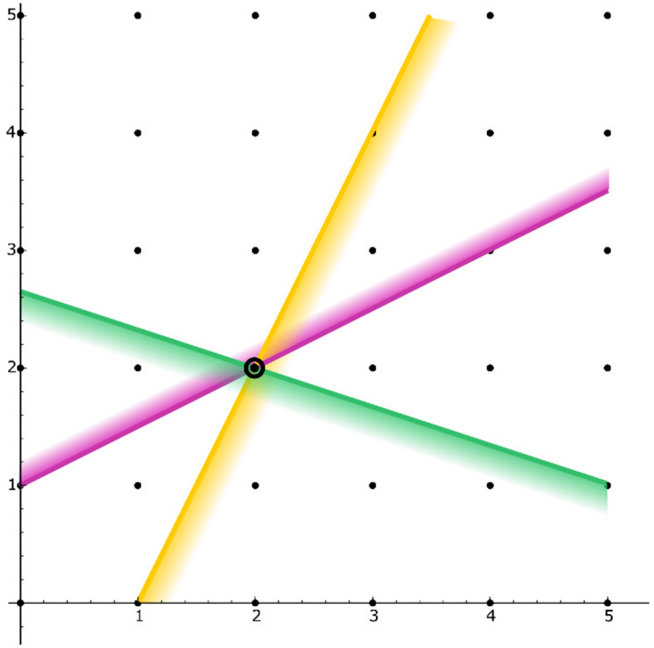
$\exists \mathbf{x} \in \mathbb{R}^n . \bigwedge_{i \in C} a_i^T \mathbf{x} < b_i$

is False

$\Downarrow$

$a_i^T x = b_i \quad \text{for } i \in C$

Equality Explanation



original

$-2x + y \leq -2,$

$x + 3y \leq 8,$

$x - 2y \leq -2,$

$\Rightarrow$

Positive Linear Combinations

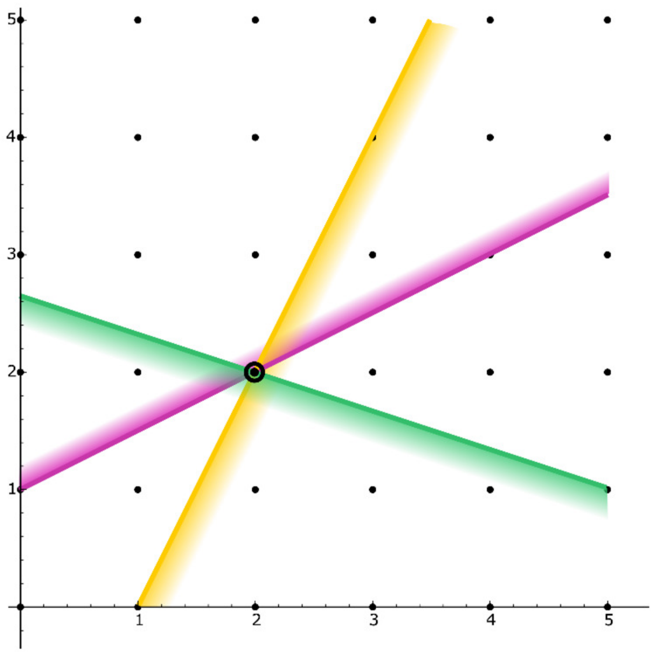
$-2x + y \leq -2,$

$2x - y \leq \quad 2,$

$\Rightarrow$

$-2x + y = -2,$

Equality Explanation



original

$-2x + y \le -2,\quad \cdot 1 \quad \rightarrow \quad -2x + y \le -2,$

$x + 3y \le 8,\quad \cdot \frac{3}{5} \quad \rightarrow \quad \frac{3}{5}x + \frac{9}{5}y \le \frac{24}{5},$

$x - 2y \le -2,\quad \cdot \frac{7}{5} \quad \rightarrow \quad \frac{7}{5}x - \frac{14}{5}y \le -\frac{14}{5},$

$+ \quad \rightarrow \quad 2x - y \le 2,$

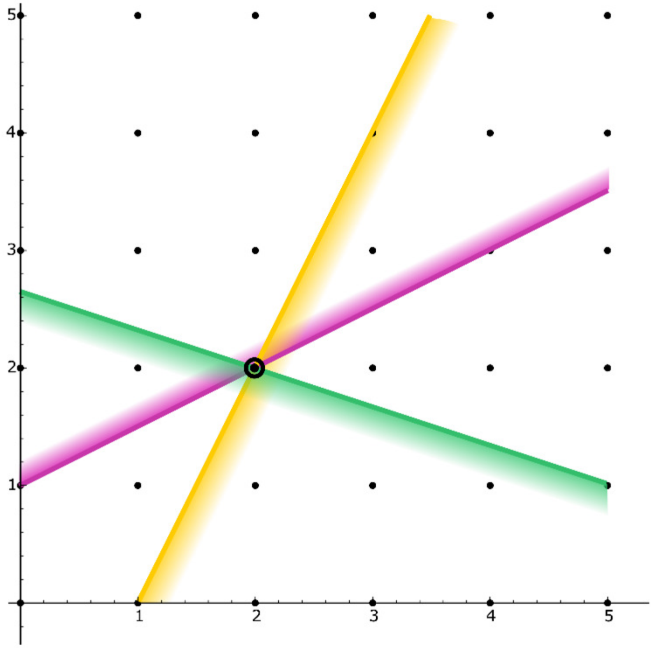
Equality Explanation

strict

$-2x + y < -2,$

$x + 3y < 8,$

$x - 2y < -2,$



original

$-2x + y \le -2,$

$x + 3y \le 8,$

$x - 2y \le -2,$

$\cdot 1$

$\cdot \frac{3}{5}$

$\cdot \frac{7}{5}$

$+$

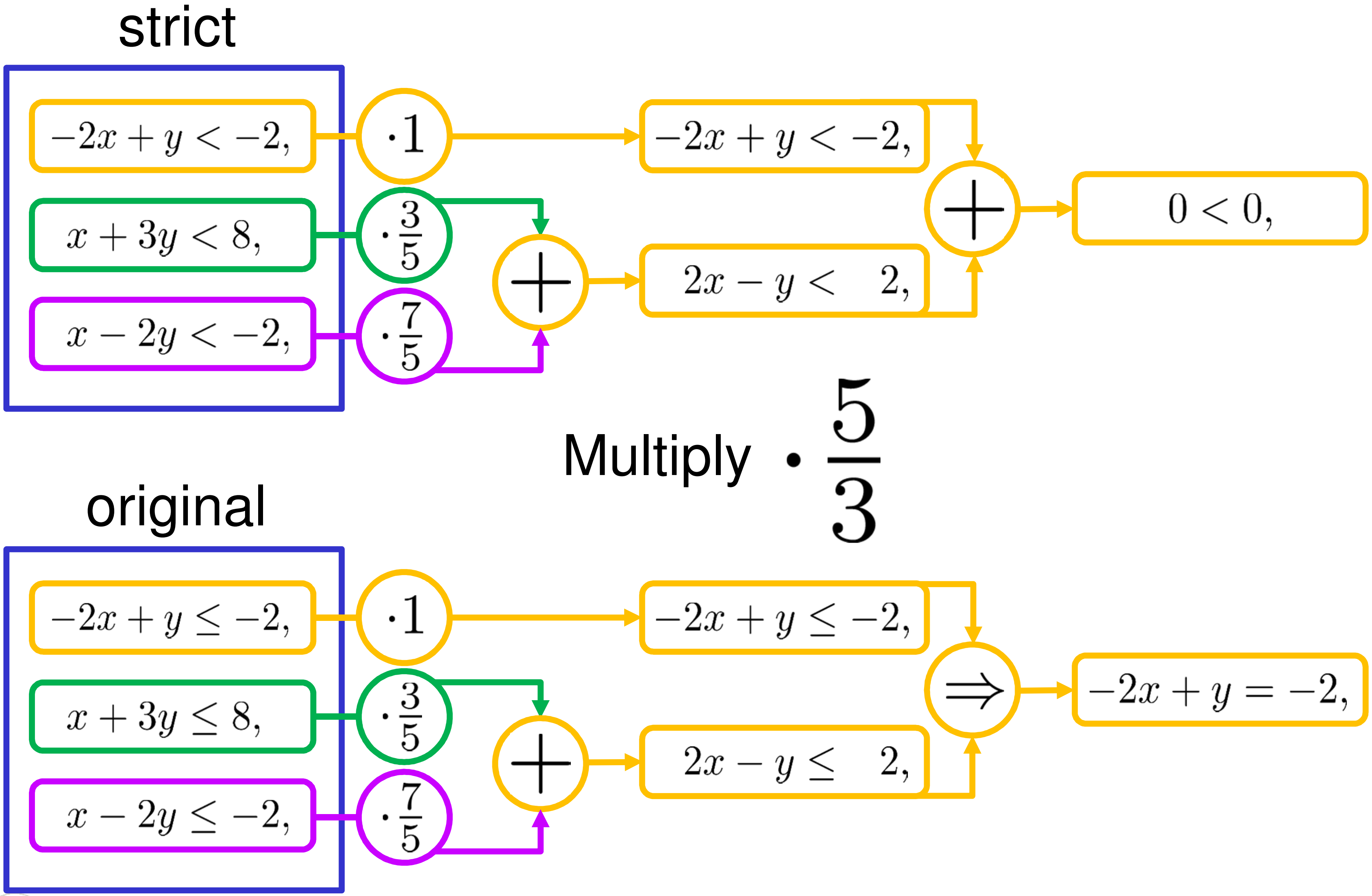
$-2x + y \le -2,$

$2x - y \le \quad 2,$

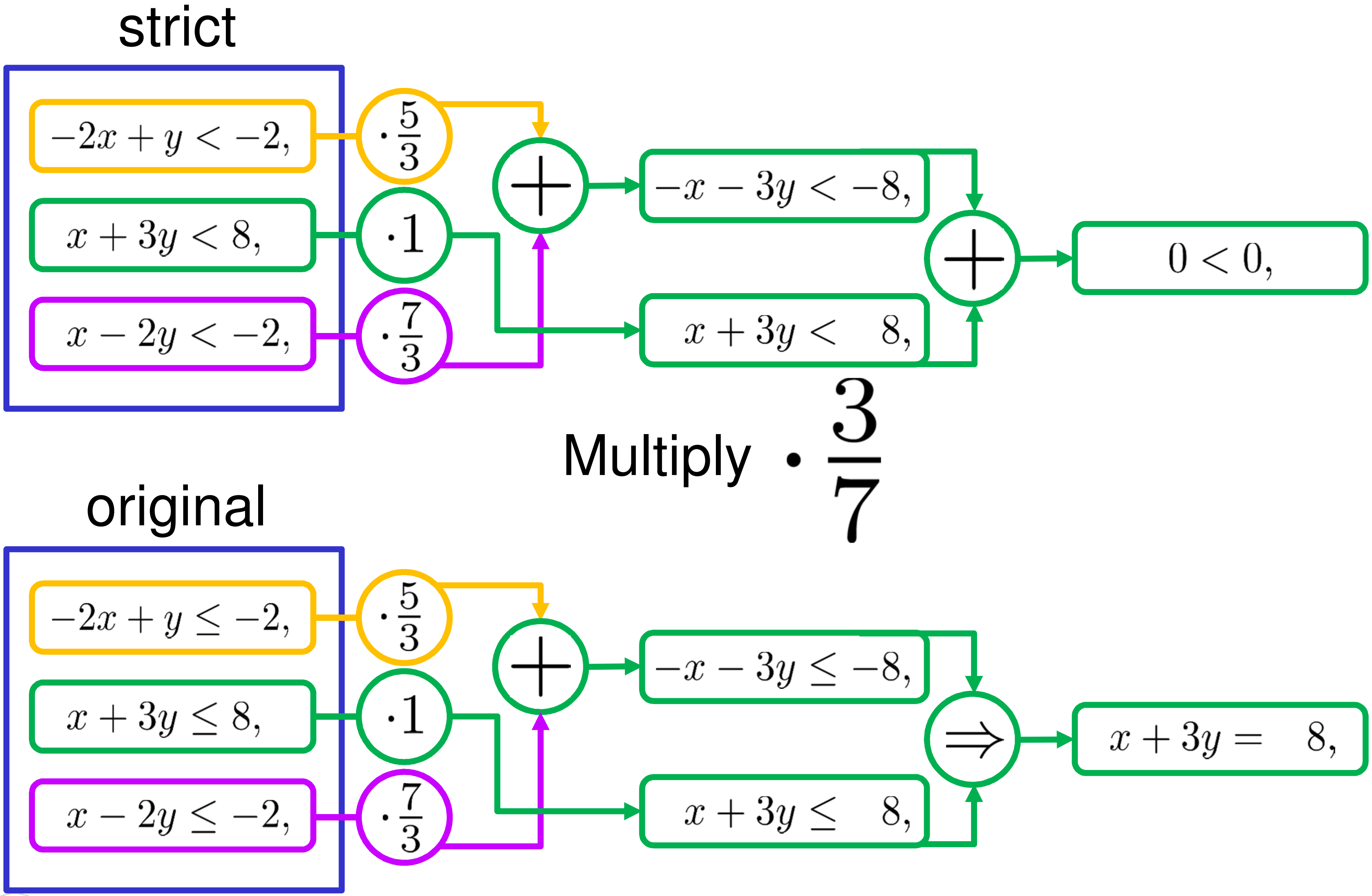
$\Rightarrow$

$-2x + y = -2,$

Equality Explanation

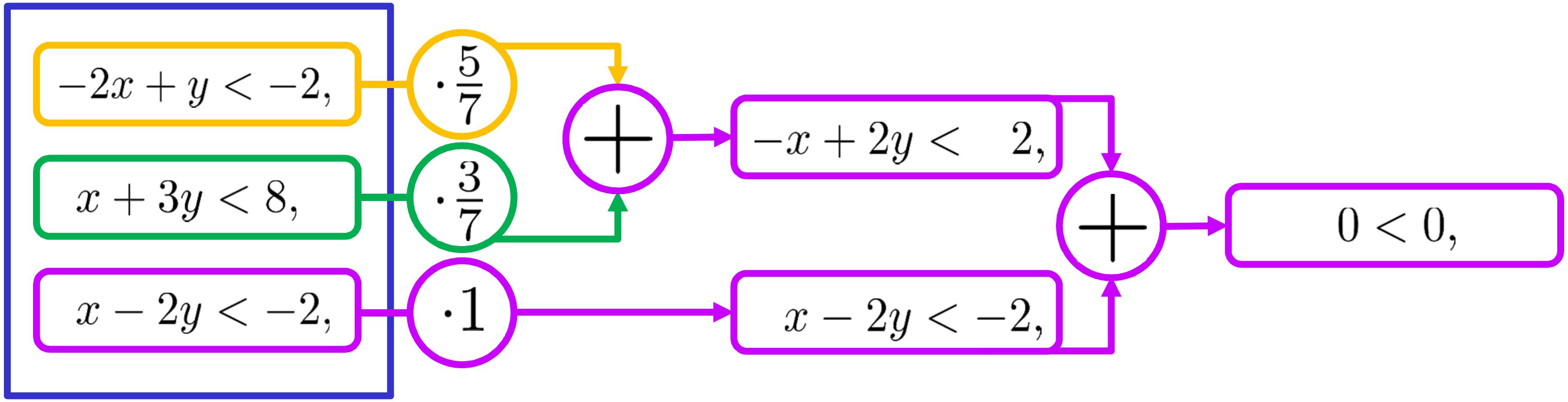


Equality Explanation

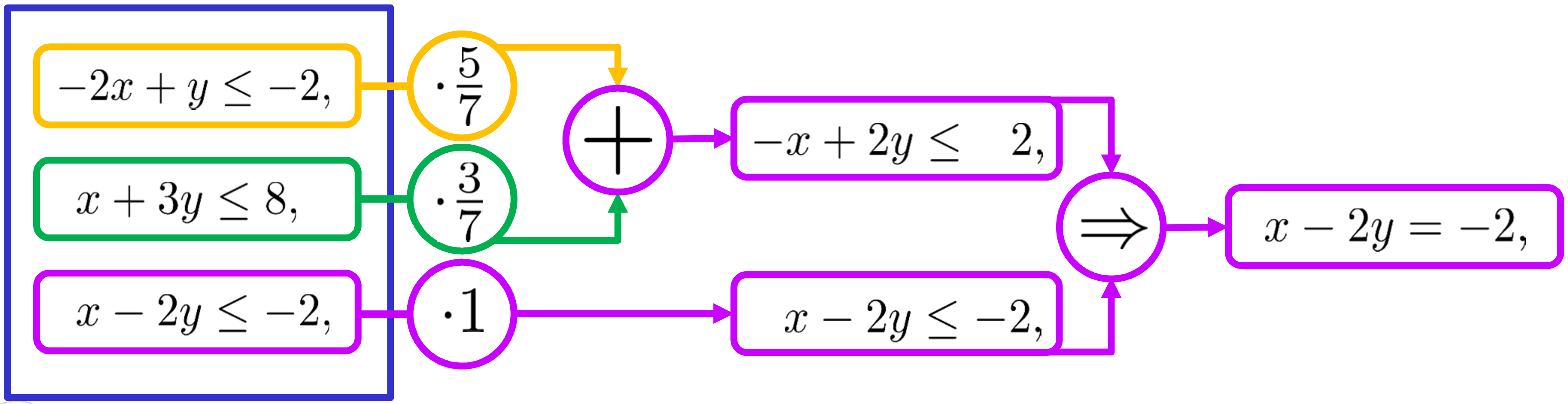


Equality Explanation

strict



original



Computing an Equality Basis

original

$$
\begin{aligned}
2u &\geq 2 + v - x,\\
3u &\leq 8 - v - 4x,\\
u &\leq -2 + 2v + x,\\
2y &\leq 9 + 3u - 2x,\\
2y &\geq 2v - x,\\
y &\leq 4 - v - u,\\
y &\geq 2x,
\end{aligned}
$$

sat

strict

$$
\boxed{
\begin{aligned}
2u &> 2 + v - x,\\
3u &< 8 - v - 4x,\\
u &< -2 + 2v + x,
\end{aligned}}
$$

$$
\begin{aligned}
2y &< 9 + 3u - 2x,\\
2y &> 2v - x,\\
y &< 4 - v - u,\\
y &> 2x,
\end{aligned}
$$

unsat

equalities

$$2u = 2 + v - x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$0 > 0,$$
$$5u < 10 - 5x,$$
$$-3u < -6 + 3x,$$
$$2y < 9 + 3u - 2x,$$
$$2y > -4 + 4u + x,$$
$$y < 6 - 3u - x,$$
$$y > 2x,$$

unsat

equalities

$$2u = 2 + v - x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$

Computing an Equality Basis

original

$$\begin{aligned} 2u &\geq 2 + v - x, \\ 3u &\leq 8 - v - 4x, \\ u &\leq -2 + 2v + x, \\ 2y &\leq 9 + 3u - 2x, \\ 2y &\geq 2v - x, \\ y &\leq 4 - v - u, \\ y &\geq 2x, \end{aligned}$$

sat

strict

$$\boxed{\begin{aligned} 5u &< 10 - 5x, \\ -3u &< -6 + 3x, \end{aligned}}$$

$$\begin{aligned} 2y &< 9 + 3u - 2x, \\ 2y &> -4 + 4u + x, \\ y &< 6 - 3u - x, \\ y &> 2x, \end{aligned}$$

unsat

equalities

$$\begin{aligned} 2u &= 2 + v - x, \\ 5u &= 10 - 5x, \end{aligned}$$

substitutions

$$\begin{aligned} v &\mapsto -2 + 2u + x, \\ u &\mapsto 2 - x, \end{aligned}$$

Computing an Equality Basis

original

$$
\begin{aligned}
2u &\geq 2 + v - x, \\
3u &\leq 8 - v - 4x, \\
u &\leq -2 + 2v + x, \\
2y &\leq 9 + 3u - 2x, \\
2y &\geq 2v - x, \\
y &\leq 4 - v - u, \\
y &\geq 2x,
\end{aligned}
$$

sat

⇨

strict

$$
\boxed{\begin{aligned}
0 &< 0, \\
0 &< 0,
\end{aligned}}
$$

$$
\begin{aligned}
2y &< 15 - 5x, \\
2y &> 4 - 3x, \\
y &< 2x, \\
y &> 2x,
\end{aligned}
$$

unsat

⇨

equalities

$$
\begin{aligned}
2u &= 2 + v - x, \\
5u &= 10 - 5x,
\end{aligned}
$$

⇩ **substitutions** ⇩

$$
\begin{aligned}
v &\mapsto -2 + 2u + x, \\
u &\mapsto 2 - x,
\end{aligned}
$$

⇦

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$2y < 15 - 5x,$$
$$2y > 4 - 3x,$$
$$\boxed{\begin{aligned} y &< 2x, \\ y &> 2x, \end{aligned}}$$

unsat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$9x < 15,$$
$$7x > 4,$$
$$0 < 0,$$
$$0 > 0,$$

unsat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$9x < 15,$$
$$7x > 4,$$

sat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

simplified

$$9x \leq 15,$$
$$7x \geq 4,$$

sat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Quantifier Elimination

$$\exists y \in \mathbb{R}.\ \boxed{\begin{array}{l} \boxed{2y \leq 2 + 3x} \wedge \boxed{5y \leq 25 - 3x} \wedge \boxed{y \leq 2x} \wedge \\ 2y \geq -2 + 5x \wedge 2y \geq 4 - 2x \wedge y \geq 2x \end{array}}$$

Virtual Substitution $\Downarrow$ Elimination Set: E

$$\exists y \in \mathbb{R}. F(y) \quad \leftrightarrow \quad \bigvee_{t \in E} F(t)$$

Quantifier Elimination

$$\exists y \in \mathbb{R}.\ \boxed{2y \le 2+3x} \wedge \boxed{5y \le 25-3x} \wedge \boxed{y \le 2x} \wedge 2y \ge -2+5x \wedge 2y \ge 4-2x \wedge y \ge 2x$$

Virtual Substitution $\Downarrow$ Elimination Set: Upper Bounds

$$\bigvee \begin{cases} 0 \le 0 \wedge 21x \le 40 \wedge 2 \le x \wedge 0 \ge 2x \wedge 5x \ge 2 \wedge 2 \ge x \\ 40 \le 21x \wedge 0 \le 0 \wedge 25 \le 13x \wedge 60 \ge 31x \wedge 4x \ge -30 \wedge 25 \ge 13x \\ x \le 2 \wedge 13x \le 25 \wedge 0 \le 0 \wedge 2 \ge x \wedge 6x \ge 4 \wedge 0 \ge 0 \end{cases}$$

$2y \le 2+3x \;\Downarrow\; y \mapsto 1 + \frac{3}{2}x$

$5y \le 25-3x \;\Downarrow\; y \mapsto 5 - \frac{3}{5}x$

$y \le 2x \;\Downarrow\; y \mapsto 2x$

Quantifier Elimination

$$\exists y \in \mathbb{R}.\ \boxed{2y \le 2 + 3x} \wedge \boxed{5y \le 25 - 3x} \wedge \boxed{y = 2x} \wedge 2y \ge -2 + 5x \wedge 2y \ge 4 - 2x \wedge \boxed{y \ge 2x}$$

Virtual Substitution $\Downarrow$ Elimination Set: Upper Bounds

$$\bigvee \begin{cases} 0 \le 0 \wedge 21x \le 40 \wedge 2 \le x \wedge 0 \ge 2x \wedge 5x \ge 2 \wedge 2 \ge x & \boxed{2y \le 2 + 3x \Downarrow y \mapsto 1 + \frac{3}{2}x} \\ 40 \le 21x \wedge 0 \le 0 \wedge 25 \le 13x \wedge 60 \ge 31x \wedge 4x \ge -30 \wedge 25 \ge 13x & \boxed{5y \le 25 - 3x \Downarrow y \mapsto 5 - \frac{3}{5}x} \\ x \le 2 \wedge 13x \le 25 \wedge 0 \le 0 \wedge 2 \ge x \wedge 6x \ge 4 \wedge 0 \ge 0 & \boxed{y \le 2x \Downarrow y \mapsto 2x} \end{cases}$$

Quantifier Elimination

$$\exists y \in \mathbb{R}.\ \boxed{2y \le 2 + 3x \wedge 5y \le 25 - 3x \wedge y = 2x \wedge 2y \ge -2 + 5x \wedge 2y \ge 4 - 2x}$$

Virtual Substitution $\Downarrow$ Elimination Set: Upper Bounds

$$\bigvee \left\{ \begin{array}{ll}
\boxed{\begin{array}{llll} 0 \le 0 & \wedge\ 21x \le 40 & \wedge\ 2 \le x & \wedge \\ 0 \ge 2x & \wedge\ 5x \ge 2 & \wedge\ 2 \ge x & \end{array}} & \boxed{\begin{array}{c} 2y \le 2 + 3x \\ \Downarrow \\ y \mapsto 1 + \frac{3}{2}x \end{array}} \\
\boxed{\begin{array}{llll} 40 \le 21x & \wedge\ 0 \le 0 & \wedge\ 25 \le 13x & \wedge \\ 60 \ge 31x & \wedge\ 4x \ge -30 & \wedge\ 25 \ge 13x & \end{array}} & \boxed{\begin{array}{c} 5y \le 25 - 3x \\ \Downarrow \\ y \mapsto 5 - \frac{3}{5}x \end{array}} \\
\boxed{\begin{array}{llll} x \le 2 & \wedge\ 13x \le 25 & \wedge\ 0 \le 0 & \wedge \\ 2 \ge x & \wedge\ 6x \ge 4 & \wedge\ 0 \ge 0 & \end{array}} & \boxed{\begin{array}{c} y \le 2x \\ \Downarrow \\ y \mapsto 2x \end{array}}
\end{array} \right.$$

Quantifier Elimination

$$\exists y \in \mathbb{R}. \quad \boxed{2y \leq 2 + 3x} \wedge \boxed{5y \leq 25 - 3x} \wedge \boxed{y = 2x} \wedge 2y \geq -2 + 5x \wedge 2y \geq 4 - 2x$$

Virtual Substitution $\Downarrow$ Elimination Set: 1 Equation

$\cancel{\vee}$

$$x \leq 2 \wedge 13x \leq 25 \wedge 0 \leq 0 \wedge 2 \geq x \wedge 6x \geq 4 \wedge 0 \geq 0$$

$$\boxed{\begin{array}{c} y = 2x \\ \Downarrow \\ y \mapsto 2x \end{array}}$$

Quantifier Elimination

$$\exists y \in \mathbb{R}. \quad \boxed{2y \leq 2 + 3x} \wedge \boxed{5y \leq 25 - 3x} \wedge \boxed{y = 2x} \wedge 2y \geq -2 + 5x \wedge 2y \geq 4 - 2x$$

Virtual Substitution $\Downarrow$ Elimination Set: 1 Equation

$$x \leq 2 \wedge 13x \leq 25 \wedge 0 \leq 0 \wedge 2 \geq x \wedge 6x \geq 4 \wedge 0 \geq 0$$

$$\boxed{\begin{array}{c} y = 2x \\ \Downarrow \\ y \mapsto 2x \end{array}}$$

$\cancel{\bigvee}$

$$\exists x_1 \in \mathbb{R}. \left[F(x_1) \quad \wedge \quad \boxed{\bigwedge_{i=1}^{m} a_i^T \mathbf{x} \circ_i b_i}\right]$$

Puplications

Original paper:

Computing a Complete Basis for Equalities Implied by a System of LRA Constraints
Bromberger & Weidenbach
SMT Workshop 2016

Extended journal version:

New Techniques for Linear Arithmetic: Cubes and Equalities
Bromberger & Weidenbach
FMSD 2017

Implementation via Simplex

- **Extension** of Dutertre & de Moura's version of the Dual Simplex Algorithm
- **Highly incremental**
- **Substitution and Normalization** is automatically done **via Pivoting**
- **Additional optimizations** (Eliminating unnecessary inequalities with the help of test points)

Conclusions

Finding Equalities:

$Ax \leq b \Rightarrow Ax < b \Rightarrow$ Conflict $\Rightarrow$ Equality

A Basis for Equalities:

$Ax \leq b \Rightarrow Ax < b \Rightarrow$ Equality $\Rightarrow$ Basis

Substitution

Applications:

- Simplify for LIA
- Nelson-Oppen for LRA

Thank you for your attention!

Equality Test

Original System of Constraints $\mathcal{P}$

actual inequalities	implying inequalities	strict inequalities
$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$
	Equality Explanation	

$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$
	Minimal Conflict	

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$

actual inequalities	implying inequalities	strict inequalities
$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$
	Equality Explanation	

$a_i^T \mathbf{x} < b_i$	$a_i^T \mathbf{x} < b_i$	
	Minimal Conflict	

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$

non-strict inequalities		strict inequalities
$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$

$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$		
non-strict inequalities $\boxed{y \geq 2x,}$		strict inequalities
$a_i^T \mathbf{x} \leq b_i$ $\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$ $\boxed{2y \geq 2v - x,}$	$a_i^T \mathbf{x} \leq b_i$ $\mathcal{P} \rightarrow a_i^T \mathbf{x} = b_i$	$a_i^T \mathbf{x} < b_i$ $\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$

Satisfiable Assignment for $\mathcal{P}$

$$x \mapsto 1 \qquad y \mapsto 2 \qquad v \mapsto 1$$

$$\boxed{2y \geq 2v - x,} \Rightarrow \boxed{2 \cdot 2 \geq 2 \cdot 1 - 1,} \Rightarrow \boxed{4 \geq 3,}$$

$$\boxed{y \geq 2x,} \Rightarrow \boxed{2 \geq 2 \cdot 1,} \Rightarrow \boxed{2 \geq 2,}$$

Equality Test

Original System of Constraints $\mathcal{P}$

non-strict inequalities $\boxed{y \geq 2x,}$

$a_i^T\mathbf{x} \leq b_i$

$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$

$\boxed{2y \geq 2v - x,}$

$a_i^T\mathbf{x} \leq b_i$

$\mathcal{P} \rightarrow a_i^T\mathbf{x} = b_i$

strict inequalities

$a_i^T\mathbf{x} < b_i$

$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$

$\Downarrow$

$a_i^T\mathbf{x} < b_i$

$a_i^T\mathbf{x} < b_i$

$\boxed{y > 2x,}$

System for Equality Test

Inequality Representation

$$a_i^T \mathbf{x} \leq b_i \qquad \text{for } i = 1, \ldots, m$$

Tableau & Bounds Representation

$$l_k \leq x_k \leq u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Equalitiy Test

$$a_i^T \mathbf{x} < b_i \qquad \text{for } i = 1, \ldots, m$$

$$l_k < x_k < u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

substitution

Equalitiy Basis

$$a_i^T \mathbf{x} = b_i \quad \text{for } i \in I \subseteq \{1, \ldots, m\}$$

$$x_k = l_k \text{ or } x_k = u_k$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Inequality Representation	Tableau & Bounds Representation
$a_i^T \mathbf{x} \leq b_i \quad$ for $i = 1, \ldots, m$	$l_k \leq x_k \leq u_k \quad$ for $k \in \mathcal{B} \cup \mathcal{N}$ $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad$ for $i \in \mathcal{B}$

Equalitiy Test

$a_i^T \mathbf{x} < b_i \quad$ for $i = 1, \ldots, m$	$l_k < x_k < u_k \quad$ for $k \in \mathcal{B} \cup \mathcal{N}$ $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad$ for $i \in \mathcal{B}$

substitution **Equalitiy Basis** pivoting

$a_i^T \mathbf{x} = b_i \quad$ for $i \in I \subseteq \{1, \ldots, m\}$	$l_k = u_k \quad$ for $k \in I \subseteq \mathcal{N}$ $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad$ for $i \in \mathcal{B}$

Bounds-and-Tableau Representation

basic variables $\mathcal{B}$ ← non-basic variables $\mathcal{N}$

$\in \mathbb{Q} \cup \{-\infty\}$ $\in \mathbb{Q} \cup \{+\infty\}$

Bounds: $l_k \leq x_k \leq u_k$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$ for $i \in \mathcal{B}$

<u>Inequality Representation:</u>

$a_i^T \mathbf{x} \leq b_i$ for $i = 1, \ldots, m$

Bounds-and-Tableau Representation

basic variables $\mathcal{B}$ ← non-basic variables $\mathcal{N}$

$\in \mathbb{Q} \cup \{-\infty\}$ $\in \mathbb{Q} \cup \{+\infty\}$

Bounds: $l_k \leq x_k \leq u_k$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$ for $i \in \mathcal{B}$

<u>Inequality Representation:</u>

$a_i^T \mathbf{x} \leq b_i$ for $i = 1, \ldots, m$

$\boxed{3u + v + 4x} \leq 8$

slacking

$s = 3u + v + 4x$

$-\infty \leq s \leq 8$

Dual Simplex

Bounds: $l_k \le x_k \le u_k$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$ for $i \in \mathcal{B}$

Assignment: $\beta(x_k) \in \mathbb{Q}$ for $k \in \mathcal{B} \cup \mathcal{N}$

Invariant: $l_k \le \beta(x_k) \le u_k$ for $k \in \mathcal{N}$

$$\beta(x_i) := \sum_{j \in \mathcal{N}} a_{ij} \beta(x_j) \quad \text{for } i \in \mathcal{B}$$

Dual Simplex

Bounds: $l_k \leq x_k \leq u_k$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$ for $i \in \mathcal{B}$

Pivoting: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$

Invariant: $l_k \leq \beta(x_k) \leq u_k$ for $k \in \mathcal{N}$

$$\beta(x_i) := \sum_{j \in \mathcal{N}} a_{ij} \beta(x_j) \quad \text{for } i \in \mathcal{B}$$

Dual Simplex

Bounds: $l_k \leq x_k \leq u_k \quad$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad$ for $i \in \mathcal{B}$

Pivoting: $x_k = \frac{1}{a_{ik}} \cdot x_i - \sum_{j \in \mathcal{N} \setminus \{k\}} \frac{a_{ij}}{a_{ik}} \cdot x_j$

Invariant: $l_k \leq \beta(x_k) \leq u_k \quad$ for $k \in \mathcal{N}$

$$\beta(x_i) := \sum_{j \in \mathcal{N}} a_{ij} \beta(x_j) \quad \text{for } i \in \mathcal{B}$$

Dual Simplex

Bounds: $l_k \leq x_k \leq u_k$ for $k \in \mathcal{B} \cup \mathcal{N}$

Tableau: $x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j$ for $i \in \mathcal{B}$

Pivoting: $x_k \mapsto \frac{1}{a_{ik}} \cdot x_i - \sum_{j \in \mathcal{N} \setminus \{k\}} \frac{a_{ij}}{a_{ik}} \cdot x_j$

Invariant: $l_k \leq \beta(x_k) \leq u_k$ for $k \in \mathcal{N}$

$$\beta(x_i) := \sum_{j \in \mathcal{N}} a_{ij} \beta(x_j) \quad \text{for } i \in \mathcal{B}$$

Dual Simplex

Basic:

$$l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k$$

Tableau:

$$s = 3u + v + 4x \qquad -2x + y \leq -2,$$
$$s = 3u + v + 4x \qquad x + 3y \leq 8,$$
$$s = 3u + v + 4x \qquad x - 2y \leq -2,$$
$$s = 3u + v + 4x \qquad x - 2y \leq -2,$$
$$s = 3u + v + 4x$$
$$s = 3u + v + 4x$$

Non-Basic:

$$l_k \leq \beta(x_k) = 3 \leq u_k \qquad l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k \qquad l_k \leq \beta(x_k) = 3 \leq u_k$$
$$l_k \leq \beta(x_k) = 3 \leq u_k \qquad l_k \leq \beta(x_k) = 3 \leq u_k$$

Dual Simplex

Basic:

$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$

Tableau:

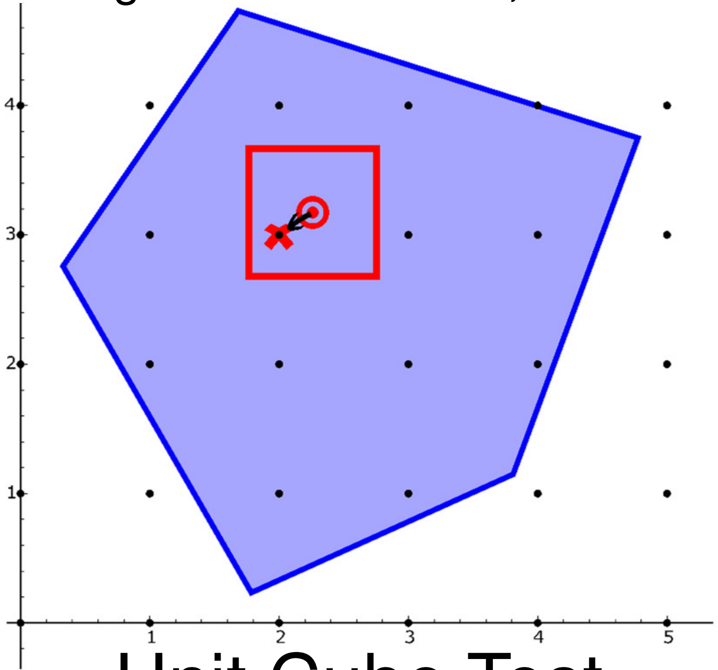
$$s = 3u + v + 4x \qquad -2x + y \leq -2,$$
$$s = 3u + v + 4x \qquad x + 3y \leq 8,$$
$$s = 3u + v + 4x \qquad x - 2y \leq -2,$$
$$s = 3u + v + 4x \qquad x - 2y \leq -2,$$
$$s = 3u + v + 4x$$
$$s = 3u + v + 4x$$

$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k \qquad \beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k \qquad \beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$
$$\beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k \qquad \beta(x_k) = 3 \qquad l_k \leq x_k \leq u_k$$

Non-Basic:

Fast Cube Tests
for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)

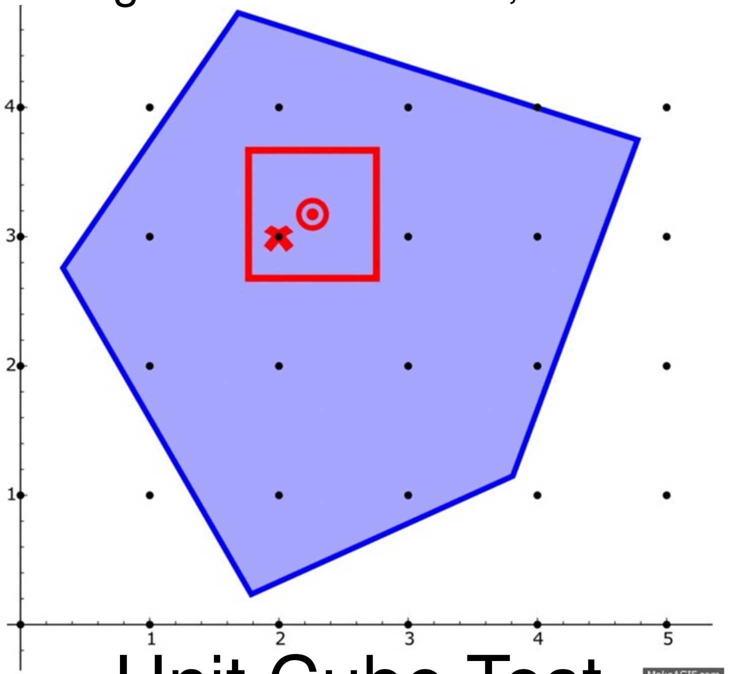


Unit Cube Test

- without optimization
- unit cubes: cubes with edge length 1

Fast Cube Tests for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)



Unit Cube Test

- without optimization
- unit cubes: cubes with edge length 1
- unit cube => guaranteed integer solution