Comparing Different Projection Operators in the Cylindrical Algebraic Decomposition for SMT Solving

Tarik Viehmann, **Gereon Kremer**, Erika Ábrahám

SC^2 Workshop July 29th

Outline

1. Preliminaries
2. CAD
3. Experiments
   - Projections
   - SMT solving
   - Incompleteness of McCallum / Brown
   - Effects of squarefree basis
4. Conclusion

Nonlinear arithmetic QF_NRA

Definition (Nonlinear arithmetic)

Boolean combinations of **polynomial** constraints over reals

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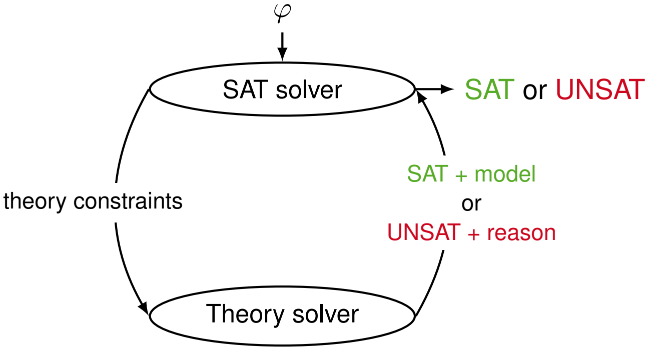
Example

$$\exists x, y. \qquad x^2 + y^2 - 4 \leq 0 \wedge (x^2 - y + 0.5 < 0 \vee x^2 + 5 \cdot y + 5 < 0)$$

SMT Solving


φ
SAT solver
SAT or UNSAT
theory constraints
SAT + model
or
UNSAT + reason
Theory solver

Cylindrical Algebraic Decomposition

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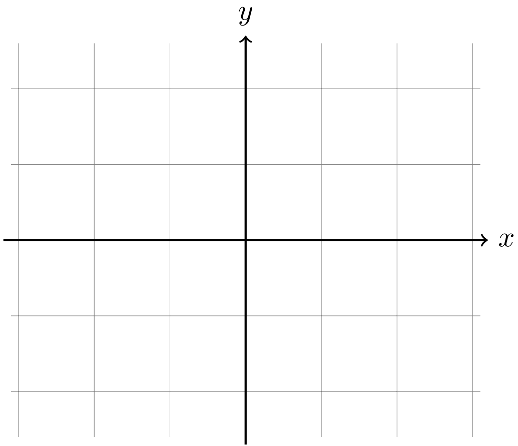

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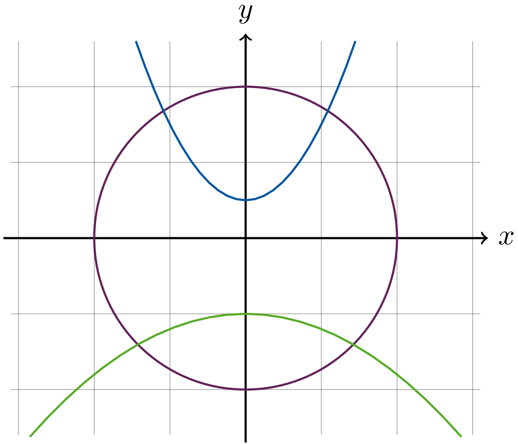

- Where are solutions?

$\rightarrow$ Sign-invariant regions

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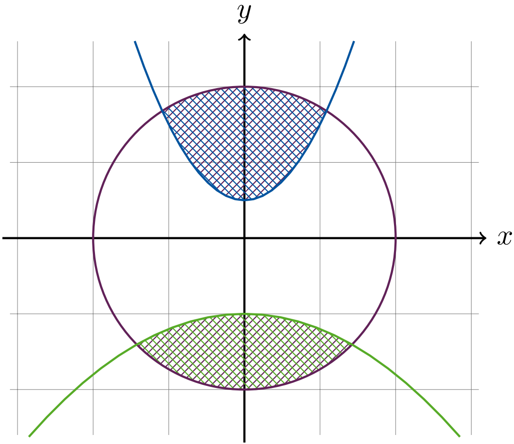

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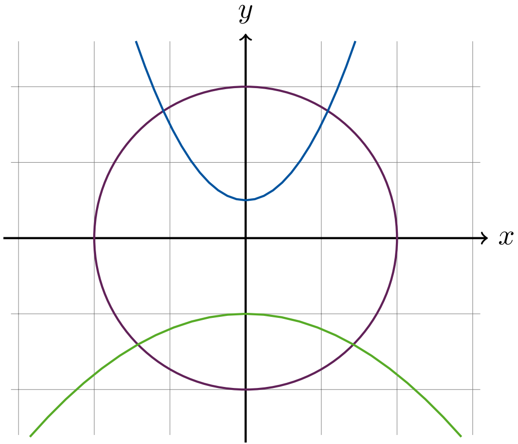

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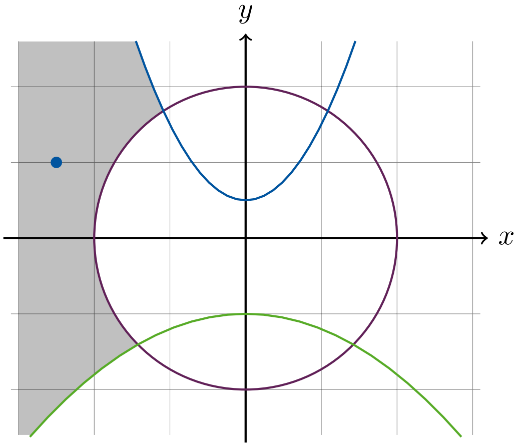

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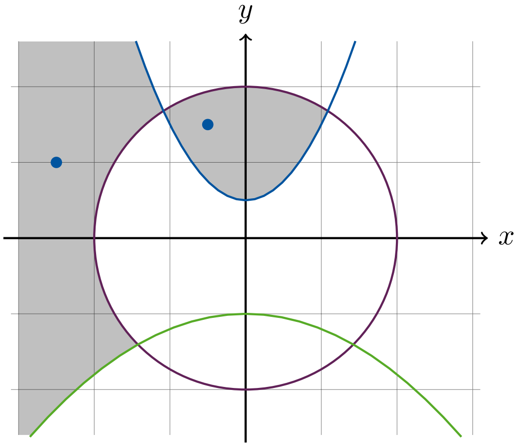

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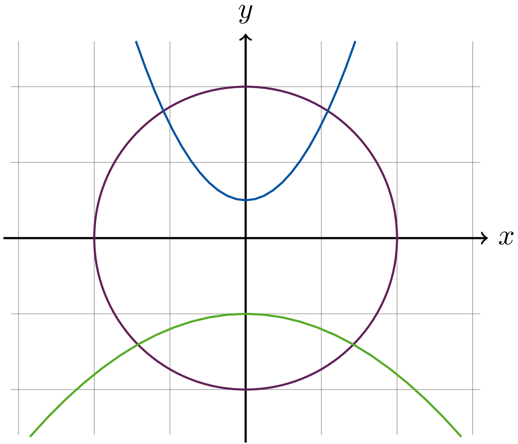

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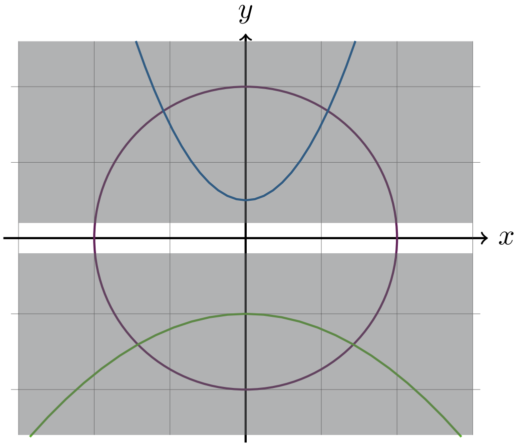

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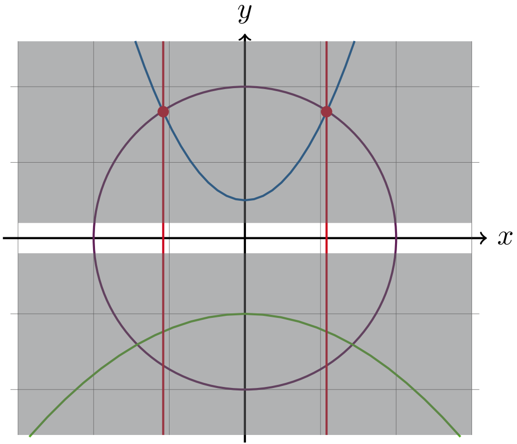

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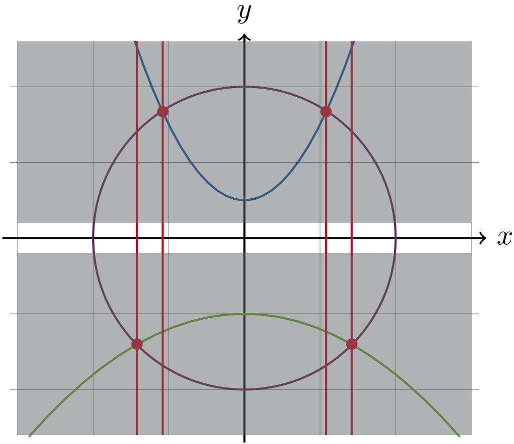

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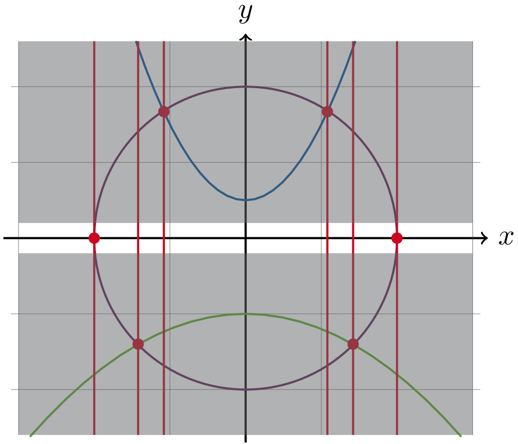

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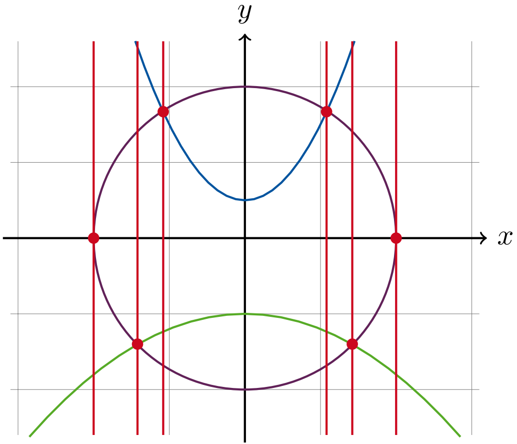

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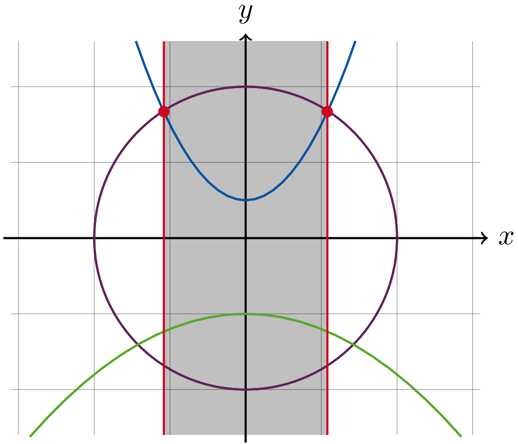

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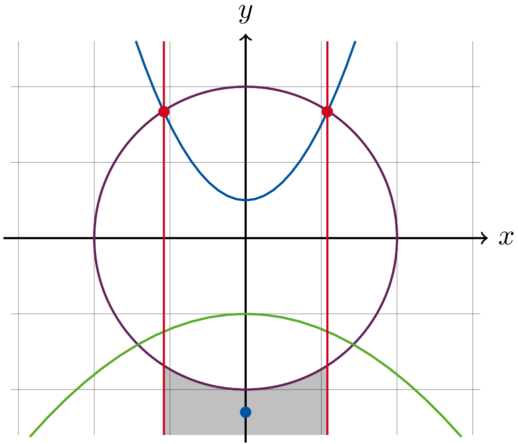

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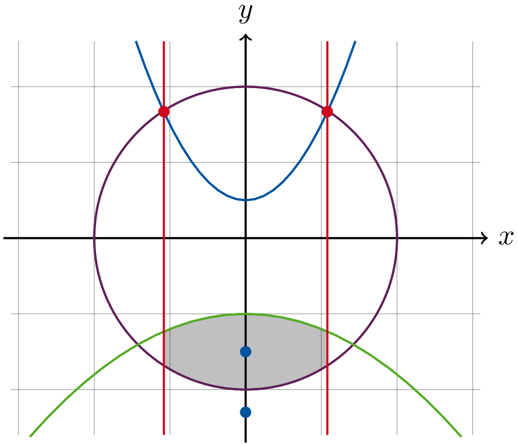

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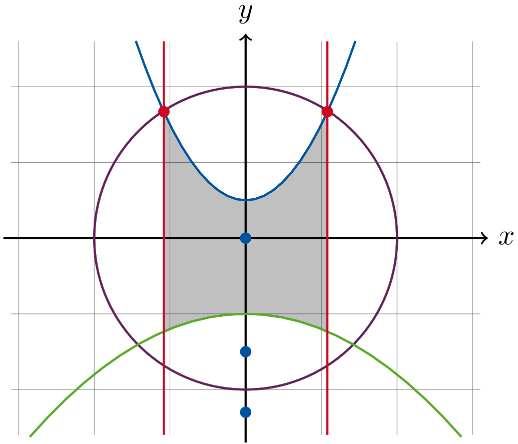

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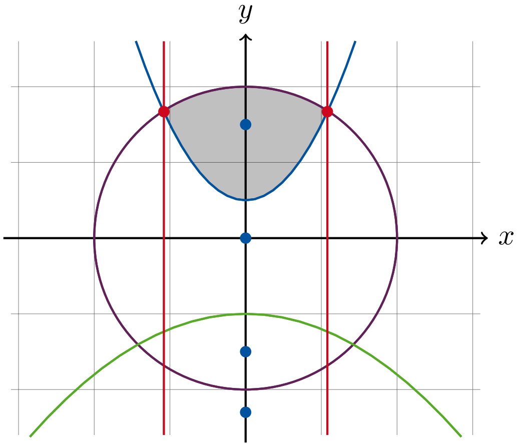

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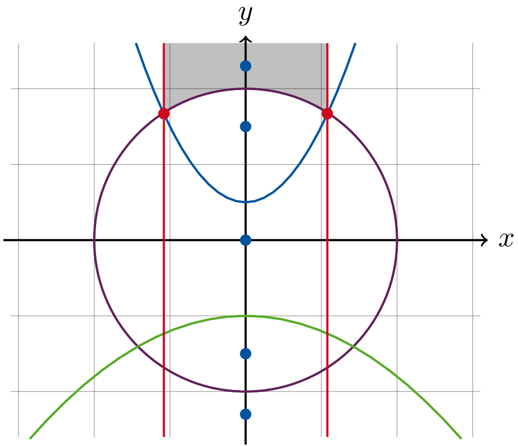

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Cylindrical Algebraic Decomposition

$$
\begin{array}{c}
P_n \subseteq \mathbb{Z}[x_1, \ldots, x_n] \\
\text{project} \big\downarrow \\
P_{n-1} \subseteq \mathbb{Z}[x_1, \ldots, x_{n-1}] \\
\text{project} \big\downarrow \\
\vdots \\
\text{project} \big\downarrow \\
P_1 \subseteq \mathbb{Z}[x_1]
\end{array}
$$

Cylindrical Algebraic Decomposition

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\begin{tikzcd}
P_n \subseteq \mathbb{Z}[x_1, \ldots, x_n] \arrow[d, "\text{project}"'] & Z_n \subseteq Z_{n-1} \times \mathbb{R} \\
P_{n-1} \subseteq \mathbb{Z}[x_1, \ldots, x_{n-1}] \arrow[d, "\text{project}"'] & \vdots \arrow[u, "\mathit{roots}(P_n \text{ at } Z_{n-1})"'] \\
\vdots \arrow[d, "\text{project}"'] & Z_2 \subseteq Z_1 \times \mathbb{R} \arrow[u, "\mathit{roots}(P_3 \text{ at } Z_2)"'] \\
P_1 \subseteq \mathbb{Z}[x_1] \arrow[r, bend right, "\mathit{roots}(P_1)"'] & Z_1 \subseteq \mathbb{R} \arrow[u, "\mathit{roots}(P_2 \text{ at } Z_1)"']
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Cylindrical Algebraic Decomposition

Projection

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project $\downarrow$

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$$P_1 \subseteq \mathbb{Z}[x_1]$$

$$P_1 \xrightarrow{roots(P_1)} Z_1$$

$$Z_1 \subseteq \mathbb{R}$$

$\uparrow$ $roots(P_2 \text{ at } Z_1)$

$$Z_2 \subseteq Z_1 \times \mathbb{R}$$

$\uparrow$ $roots(P_3 \text{ at } Z_2)$

$$\vdots$$

$\uparrow$ $roots(P_n \text{ at } Z_{n-1})$

$$Z_n \subseteq Z_{n-1} \times \mathbb{R}$$

Projection Operators

We consider:

- Collins
- Hong
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Not considered:

- Lazard
- Seidl & Sturm
- Strzeboński
- Brown & Košta
- ...

Collins & Hong

Definition (Collins' operator [Collins75])

$$proj_C^1 := \bigcup_{p \in P} \left(coeffs(p) \cup \bigcup_{r \in RED(p)} PSC(r, r') \right)$$

$$proj_C^2 := \bigcup_{p,q \in P} \bigcup_{\substack{r_p \in RED(p) \\ r_q \in RED(q)}} PSC(r_p, r_q)$$

$$proj_C := proj_C^1 \cup proj_C^2$$

Collins & Hong

Definition (Hong's operator [Hong90])

$$proj_C^1 := \bigcup_{p \in P} \left(coeffs(p) \cup \bigcup_{r \in RED(p)} PSC(r, r') \right)$$

$$proj_H^2 := \bigcup_{p,q \in P} \bigcup_{r_p \in RED(p)} PSC(r_p, q)$$

$$proj_H := proj_C^1 \cup proj_H^2$$

McCallum & Brown

Definition (McCallum's operator [McCallum84])

Let P be a squarefree basis.

$$proj_M^1 := \bigcup_{p \in P} \{disc(p)\} \cup coeffs(p)$$

$$proj_M^2 := \bigcup_{p,q \in P} \{res(p, q)\}$$

$$proj_M := proj_M^1 \cup proj_M^2$$

Incomplete!

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McCallum & Brown

Definition (Brown's operator [Brown01])

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- Analyzed:
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- Not analyzed yet:
  - Different variable orderings
  - Different lifting orders

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- Project **all polynomials**, ignore boolean structure
- 5698 benchmarks such that all projections terminated

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- Hong may be viable if **incompleteness** of McCallum is an issue

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Operator	Solved	Timeout
Collins	5041	657
Hong	5125	573
McCallum	5284	414
Brown	5299	399

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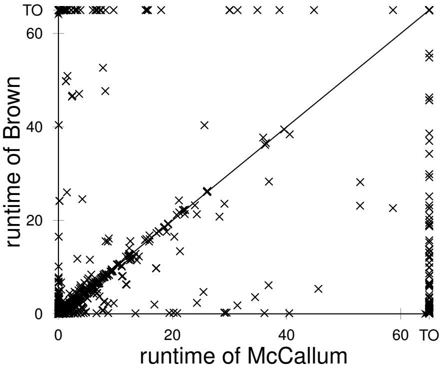

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- $\Rightarrow$ not a pressing issue **on our SMT benchmarks**

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- Using CoCoALib
- Overall solving is about 10% **slower**
- But **less timeouts!** McCallum: $889 \to 739$, Brown: $842 \to 739$
- McCallum closes in on Brown!

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Effects of squarefree basis

- McCallum / Brown require P_k to be a **squarefree basis**
- **Difficult** to compute ignored until now

- Using CoCoALib
- Overall solving is about 10% **slower**
- But **less timeouts!** McCallum: $889 \to 739$, Brown: $842 \to 739$
- McCallum closes in on Brown!

- $\Rightarrow$ usually detrimental, sometimes essential
  required for correctness!

Conclusion

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- Individual examples **may vary wildly**
- $\Rightarrow$ Portfolio?

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- Adapt variable ordering?
- Effects of delineating polynomials and additional points?

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References

- Chistopher W. Brown, **Improved projection for cylindrical algebraic decomposition**, JSC **32** (2001).
- George E. Collins, **Quantifier elimination for real closed fields by cylindrical algebraic decompostion**, ATFL'75.
- Hoon Hong, **An improvement of the projection operator in cylindrical algebraic decomposition**, ISSAC '90.
- Scott McCallum, **An improved projection operation for cylindrical algebraic decomposition**, Ph.D. thesis, 1984.

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Notation

Definition (Polynomials)

$p = \sum_{i=0}^{m} a_i \cdot x_n^i$ in **main variable** x_n and $a_i \in \mathbb{R}[x_1, ..., x_{n-1}]$.

Definition (Simple properties)

$$coeffs(p) := \{a_0, ..., a_m\} \qquad lcf(p) := a_m$$

$$red_k(p) := \sum_{i=0}^{m-k} a_i \cdot x_n^i \qquad red(p) := \{red_k(p) \mid k = 0 \ldots m\}$$

Building blocks

$$Syl(p, q) := \begin{pmatrix} a_k & \cdots & a_0 & & & \\ & a_k & \cdots & a_0 & & \\ & & \ddots & & \ddots & \\ & & & a_k & \cdots & a_0 \\ b_l & \cdots & b_0 & & & \\ & b_l & \cdots & b_0 & & \\ & & \ddots & & \ddots & \\ & & & b_l & \cdots & b_0 \end{pmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} l \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} k \end{array}$$

Building blocks

$$M_j(p,q) := \begin{pmatrix} a_k & & \cdots & & a_0 & & & \\ & a_k & & \cdots & & a_0 & & \\ & & \ddots & & & & \ddots & \\ & & & a_k & & \cdots & & a_0 \\ b_l & & \cdots & & b_0 & & & \\ & b_l & & \cdots & & b_0 & & \\ & & \ddots & & & & \ddots & \\ & & & b_l & & \cdots & & b_0 \end{pmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} l-j \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} k-j \end{matrix}$$

Building blocks

$$M_j(p,q) := \begin{pmatrix} a_k & \cdots & a_0 & & & & \\ & a_k & \cdots & a_0 & & & \\ & & \ddots & & \ddots & & \\ & & & a_k & \cdots & a_0 & \\ b_l & \cdots & b_0 & & & & \\ & b_l & \cdots & b_0 & & & \\ & & \ddots & & \ddots & & \\ & & & b_l & \cdots & b_0 & \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ \ddots \\ a \end{matrix}}\right\} l-j \\ \left.\vphantom{\begin{matrix} b \\ b \\ \ddots \\ b \end{matrix}}\right\} k-j \end{matrix}$$

Definition (Principal subresultant coefficients)

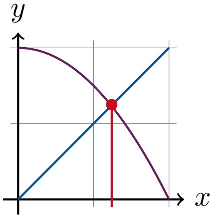
$$psc_i(p,q) := \det(M_i)$$
$$PSC(p,q) := \{psc_i \mid i = 0 \ldots \min(k,l)\}$$

Building blocks

Definition (Resultant)

$$res(p, q) := \det(Syl(p, q))$$

p, q have a **common root** $\Leftrightarrow$ $res(p, q)$ has a root



Definition (Discriminant)

$$disc(p) := res(p, p')$$

p has a **multiple root** $\Leftrightarrow$ $disc(p)$ has a root