Cylindrical Algebraic Decomposition and Real Polynomial Constraints

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Symbolic Computing with Real Polynomial Constraints

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- Tarski formulas are a natural language for real polynomial constraints. Restrictions (e.g. to just conjunctions or just equations) don't have nice closure properties.
- If we fix an order for the free variables in a Tarski formula, we have a natural association between Tarski formulas and geometric objects called semi-algebraic sets.

$$F(x_1, \ldots, x_n) \longleftrightarrow \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n \mid F(\alpha_1, \ldots, \alpha_n)\}$$

LOGIC

GEOMETRY

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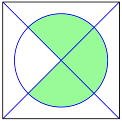
$x_1^2+x_2^2-1<0 \wedge \left[\begin{array}{c} x_2+x_1>0 \\ \vee \\ x_2-x_1<0 \end{array}\right]$		

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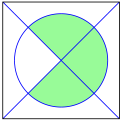
$x_1^2+x_2^2-1<0 \wedge \left[\begin{array}{c} x_2+x_1>0 \\ \vee \\ x_2-x_1<0 \end{array}\right]$		
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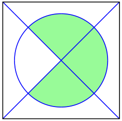
$x_1^2+x_2^2-1 < 0 \wedge \left[\begin{array}{c} x_2 + x_1 > 0 \\ \vee \\ x_2 - x_1 < 0 \end{array} \right]$		
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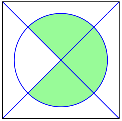
Tarski formula $$x_1^2+x_2^2-1<0 \wedge \left[\begin{array}{c} x_2+x_1>0 \\ \vee \\ x_2-x_1<0 \end{array}\right]$$	basic object	semi-algebraic set $S \subseteq \mathbb{R}^2$ 
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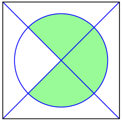
$x_1^2+x_2^2-1<0 \wedge \left[\begin{array}{c} x_2+x_1>0 \\ \vee \\ x_2-x_1<0 \end{array}\right]$		
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- Is one of these empty? Which?

$$F_1 := \left[\begin{array}{c} -91x + 10y^2 - 7 > 0 \wedge -54x - 43yx + 63 > 0 \wedge \\ -79x^2 + 4y - 61 > 0 \wedge -49x^2 + 42y + 97 > 0 \wedge \\ 27xy + 25y + 72 > 0 \wedge -47x - 57yx + 1 > 0 \end{array}\right]$$

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- What is the dimension of the set defined by this Tarski formula?

$$F_3 := \left[x^4 - 4x^3 + 6x^2 + y^2 - 4x + 4y + 5 \leq 0\right]$$

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The point: CAD provides just such an explicit representation

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- others?

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4. Applying NLSAT/SMT ideas to computer algebra

Part I: CAD Basics

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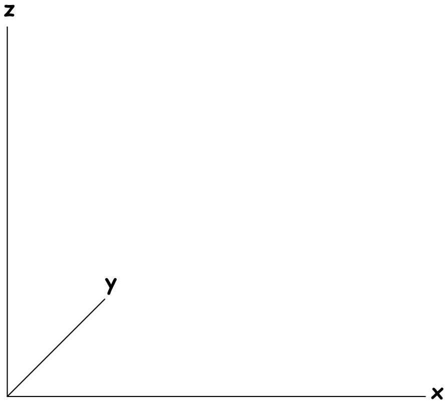

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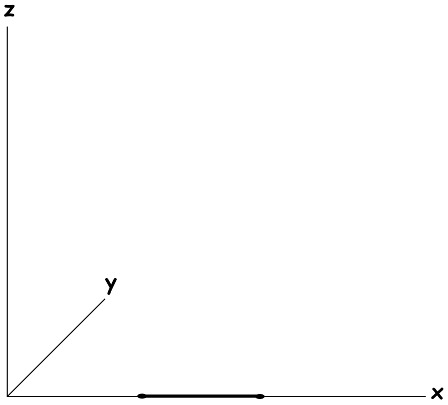

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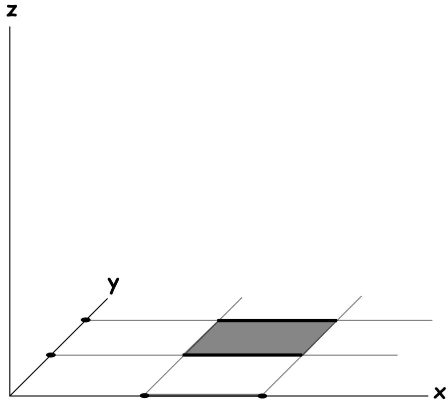

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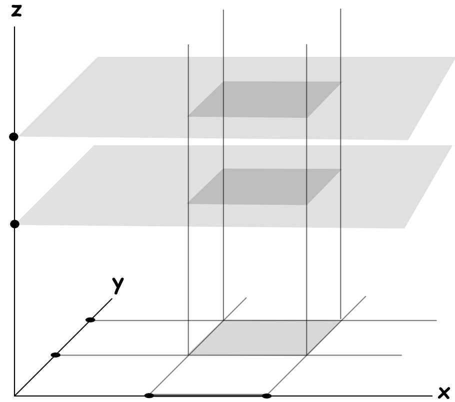

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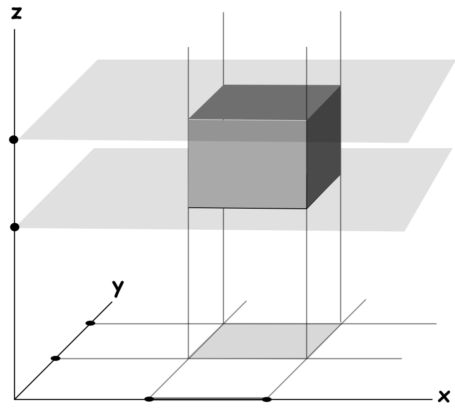

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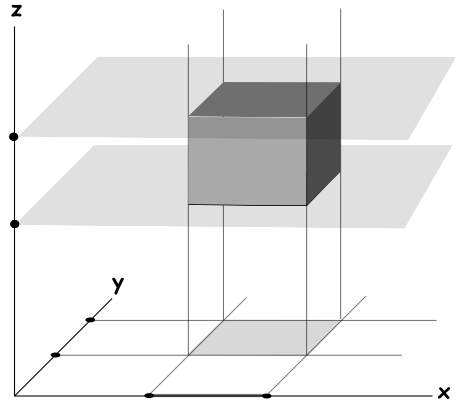

$$l_3 < z < u_3$$
$$l_2 < y < u_2$$
$$l_1 < x < u_1$$

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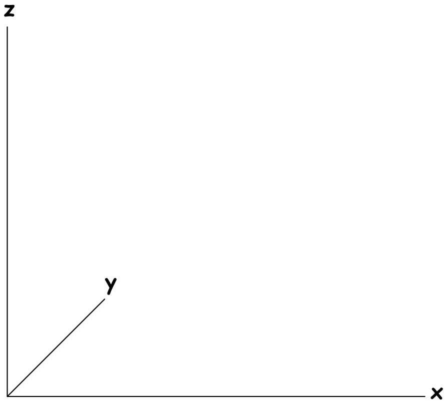

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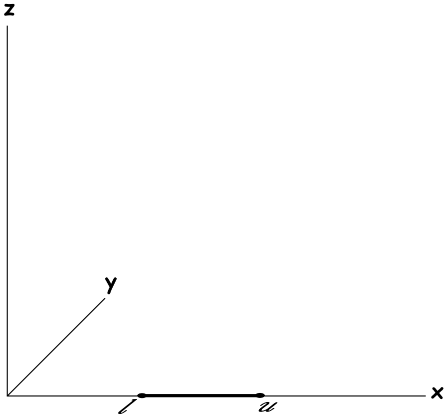

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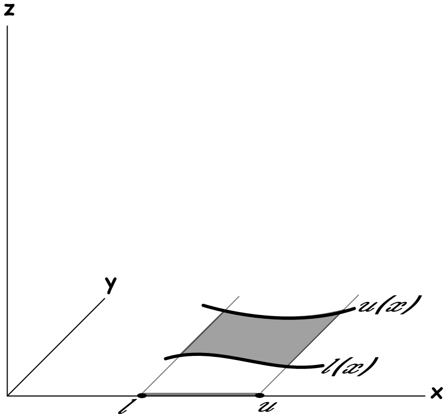

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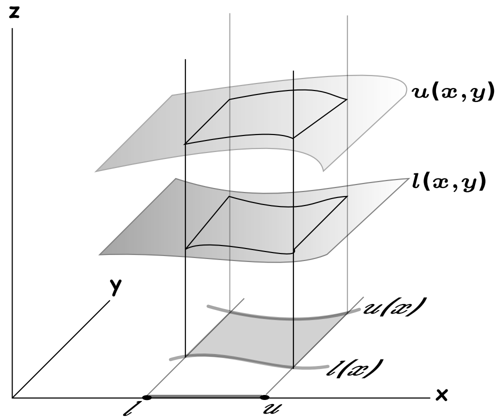

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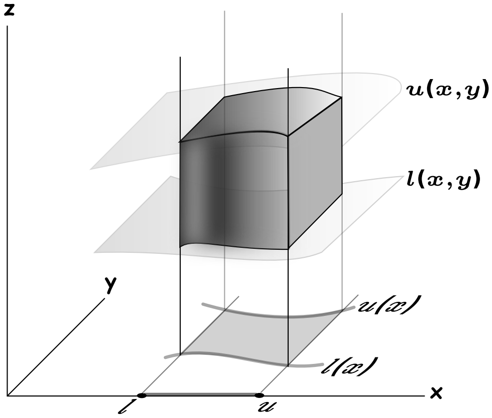

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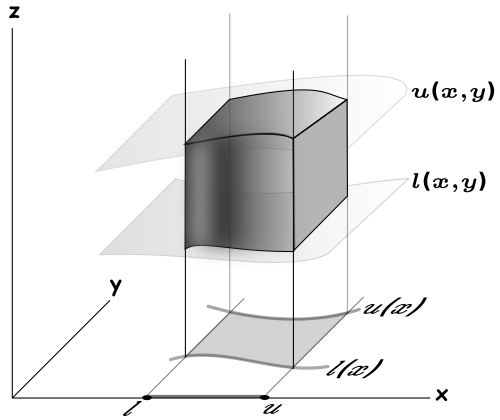

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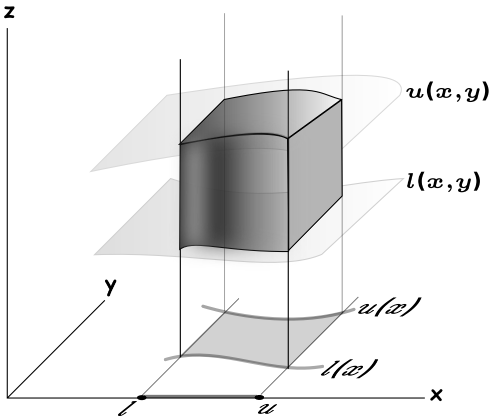

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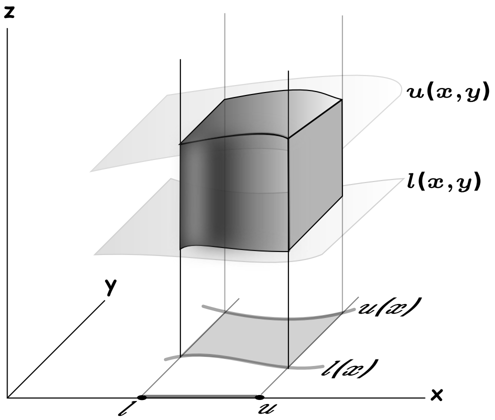

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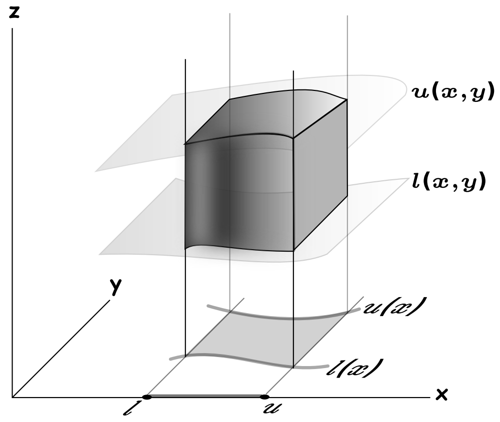

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Note:

We must fix a variable order (here $x < y < z$) to do this!
We define the level of a variable to be its place in this order.
E.g. the level of y is 2, the level of z is 3.

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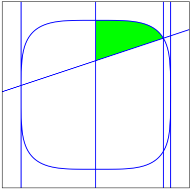

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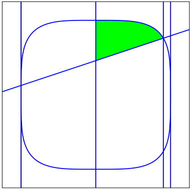

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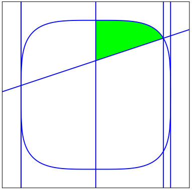

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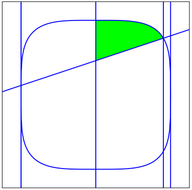

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Cylindrical Cell — indexed root expressions 2

Definition: The indexed root expression

$$x_k \ \sigma \ \mathrm{root}_i \ p(x_1, \ldots, x_k)$$

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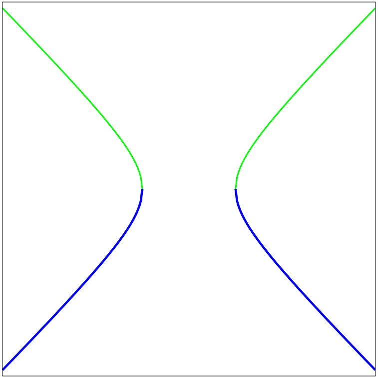

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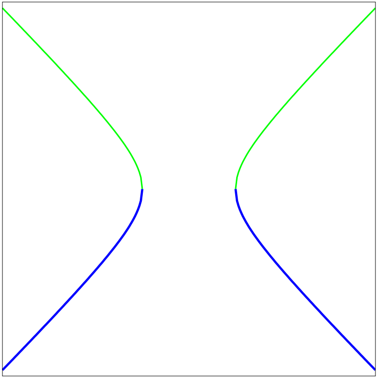

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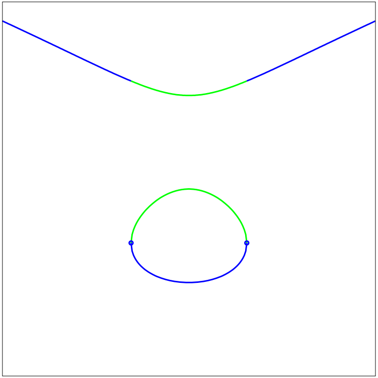

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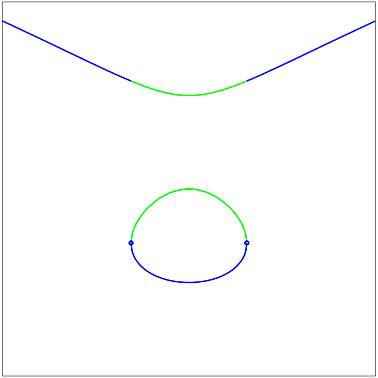

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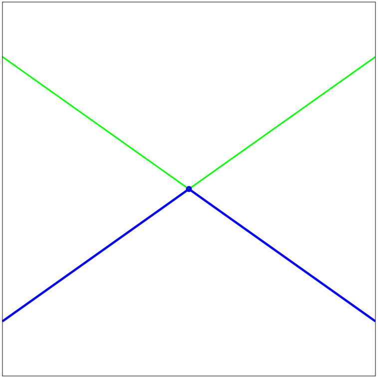

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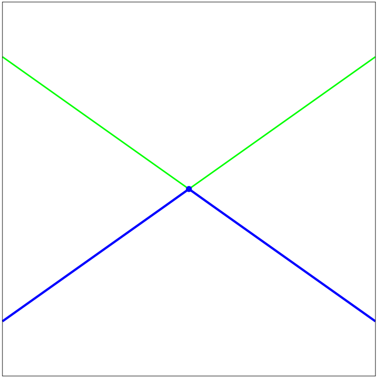

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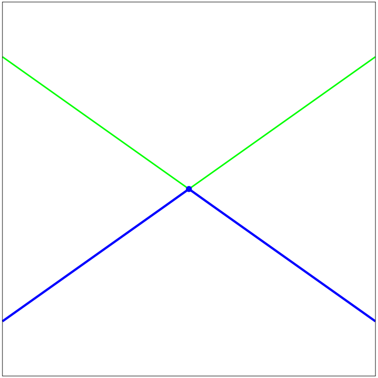

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- for all $\alpha \in S$ we have $f(\alpha, x_k) \neq 0$
- the the set of real roots of $f(\alpha, x_k)$, $\alpha \in S$ is either empty or it consists of finitely many continuous functions $\theta_1 < \cdots < \theta_t$ from S to $\mathbb{R}$, with $t \geq 1$; and, in the latter case
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The θ_i's are referred to as the sections of f over S.

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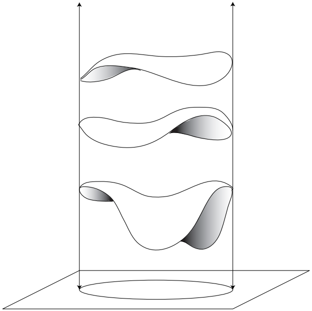

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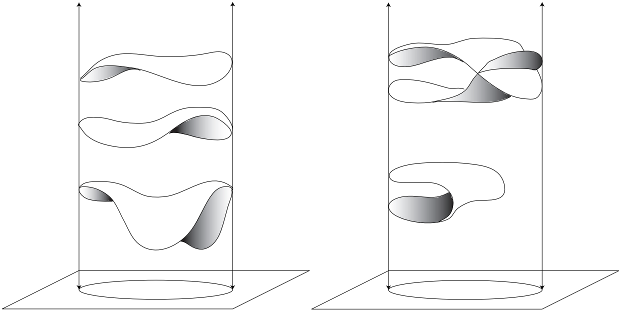

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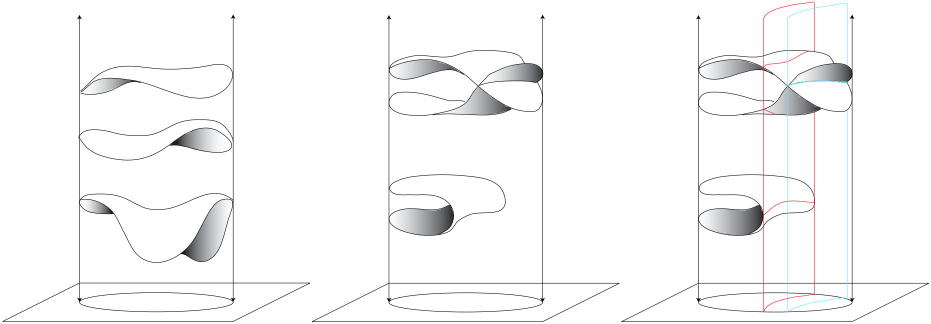
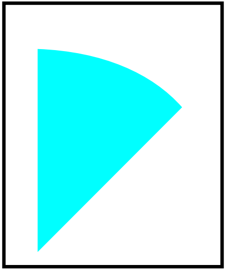


Cylindrical Cell — degenerate bounds



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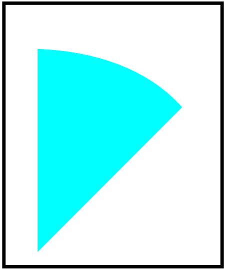

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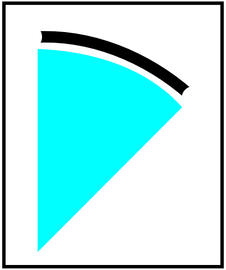

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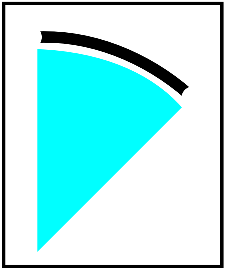

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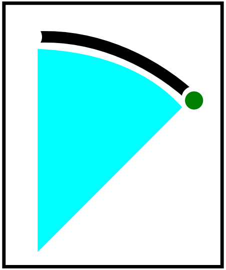

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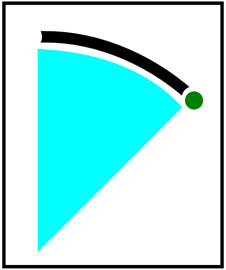

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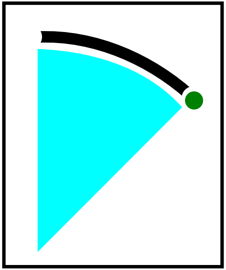

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If the k-level bound for a cell is given as an open interval, the cell is said to be a k-level sector. Otherwise it is a k-level section.

Cylindrical Cell - a formal definition

Definition: A cylindrical algebraic cell in $\mathbb{R}^k$.

1. $\mathbb{R}^0$ is a cylindrical algebraic cell in $\mathbb{R}^0$.
2. if C is a cylindrical algebraic cell in $\mathbb{R}^k$ then
   a) $\{(\alpha_1, \ldots, \alpha_k, \beta) \in C \times \mathbb{R} \mid \beta = \mathrm{root}_i f(\alpha, z)\}$ where f is delineable over C with at least i sections, is a cylindrical algebraic cell in $\mathbb{R}^{k+1}$,
   b) $\{(\alpha_1, \ldots, \alpha_k, \beta) \in C \times \mathbb{R} \mid \beta > \mathrm{root}_i f(\alpha, z) \text{ and } \beta < \mathrm{root}_j g(\alpha, z)\}$ where $\{f, g\}$ is delineable over C, f having at least i sections, g having at least j sections, and section $\mathrm{root}_i f$ everywhere less than section $\mathrm{root}_j g$, is a cylindrical algebraic cell in $\mathbb{R}^{k+1}$,
3. nothing else is a cylindrical cell.

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Substitute $x = -1/2$ into the bounds for y getting

$\text{root}_1\ y^2 + \left(-\frac{1}{2}\right)^2 - 1 < y < \text{root}_1\ 64\left(-\frac{1}{2}\right)^2 y^4 + 64\left(-\frac{1}{2}\right)^4 y^2 - 64\left(-\frac{1}{2}\right)^2 y^2 + 1$

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$-0.8660\ldots < y < -0.8090\ldots \longrightarrow$ choose $y = -53/64 = -0.828125$

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$$\begin{array}{rcccl} \text{root}_1\ z^2 + y^2 + x^2 - 1 & < & z & < & \text{root}_2\ z^2 + y^2 + x^2 - 1 \\ \text{root}_1\ y^2 + x^2 - 1 & < & y & < & \text{root}_1\ 64x^2y^4 + 64x^4y^2 - 64x^2y^2 + 1 \\ \text{root}_1\ 4x^3 - 4x - 1 & < & x & < & \text{root}_2\ 4x^3 - 4x - 1 \end{array}$$

Find a sample/model point $\alpha = (\,-1/2,\ -53/64, \qquad) \in C$

Substitute $x = -1/2$ into the bounds for y getting
$\text{root}_1\ y^2 + \left(-\frac{1}{2}\right)^2 - 1 < y < \text{root}_1\ 64\left(-\frac{1}{2}\right)^2 y^4 + 64\left(-\frac{1}{2}\right)^4 y^2 - 64\left(-\frac{1}{2}\right)^2 y^2 + 1$
Via unvivarate real root isolation we get
$-0.8660\ldots < y < -0.8090\ldots \longrightarrow$ choose $y = -53/64 = -0.828125$

Cylindrical Cell - a closing example

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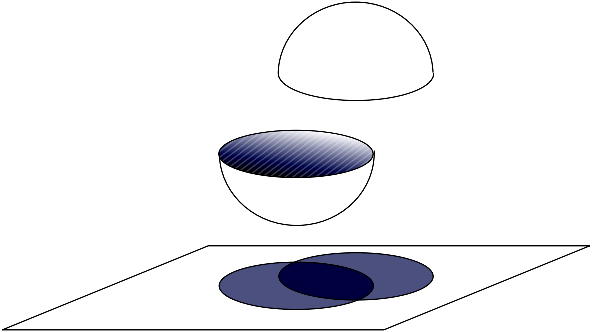

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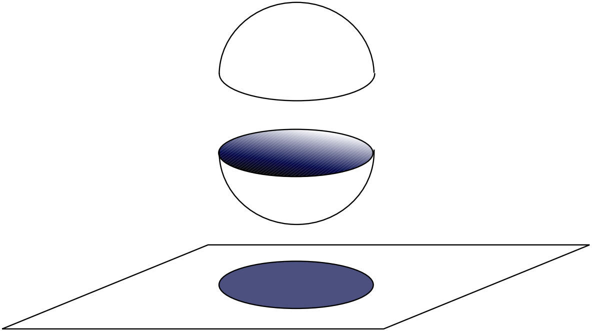

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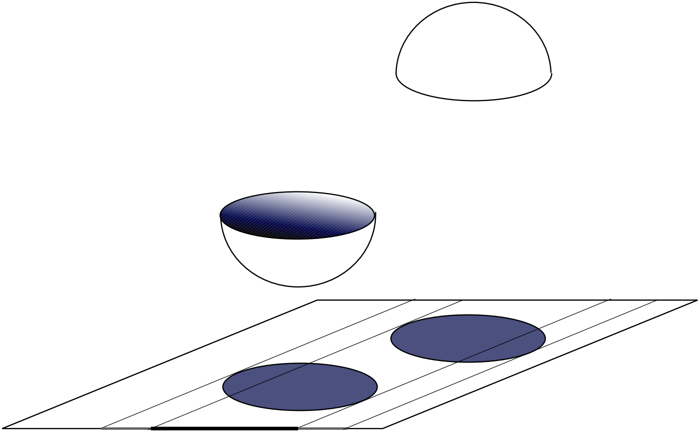

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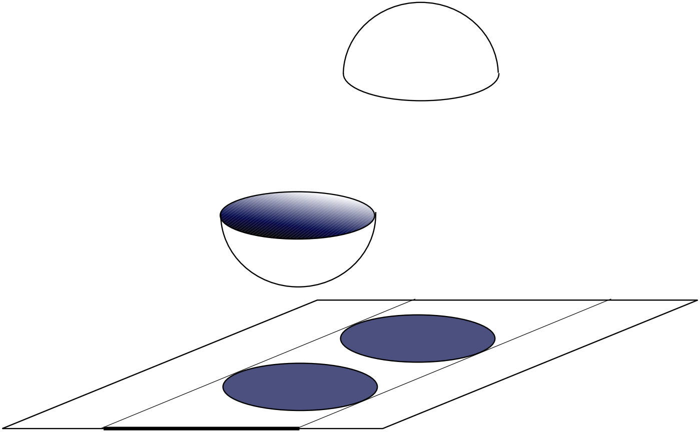

Cylindrically Arranged Cells

Theorem: Suppose semi-algebraic set S is definined as a union of elements in a set D_n of disjoint cylindrical cells in $\mathbb{R}^n$ that are cylindrically arranged. For $0 \leq k \leq n$, define $D_k = \{\pi_k(c) \mid c \in D_n\}$.

- The elements of D_k are disjoint cylindrical cells in $\mathbb{R}^k$ that are cylindrically arranged, and
- $\pi_k(S) = \bigcup_{c \in D_k} c$

Time out to compute

$$x^4 + y^2 < 1 \wedge (x+y)(x-y) < 0 \wedge y > -x - 1 \wedge (x + 1/2) > 0$$

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```
> solve([x^4 + y^2 < 1,(x + y)*(x - y) < 0,y > -x - 1,(x + 1/2) > 0],{x,y});
memory used=416.3MB, alloc=110.3MB, time=3.86
{-1/2 < x, x < 0, y < x, -x - 1 < y},

                                 4     1/2
    {-1/2 < x, x < 0, y < (-x  + 1)   , -x < y}, {x = 0, -1 < y, y < 0},

                                                    4     2
    {x = 0, 0 < y, y < 1}, {0 < x, x < RootOf(_Z  + _Z  - 1, 0.75 .. 0.8125),

                  4     1/2
    y < -x, -(-x  + 1)    < y}, {0 < x, x < y,

                    4     2                                4     1/2
    x < RootOf(_Z  + _Z  - 1, 0.75 .. 0.8125), y < (-x  + 1)   }
```

Time out to compute

$$x^4 + y^2 < 1 \wedge (x+y)(x-y) < 0 \wedge y > -x - 1 \wedge (x + 1/2) > 0$$

In[84]:= `Reduce[x ^ 4 + y ^ 2 < 1 && (x + y) (x - y) < 0 && y > -x - 1 && (x + 1 / 2) > 0]`

Out[84]= $\left(-\frac{1}{2} < x \le 0 \ \&\& \ \left(-1 - x < y < x \ || \ -x < y < \sqrt{1 - x^4}\right)\right) \ ||$

$\left(0 < x < \text{Root}\left[-1 + \#1^2 + \#1^4 \ \&, \ 2\right] \ \&\& \ \left(-\sqrt{1 - x^4} < y < -x \ || \ x < y < \sqrt{1 - x^4}\right)\right)$

Cylindrically Arranged Cells

Cylindrical formula: start with the disjunction of defining formulas for the cells in D_k, and level by level factor out common subexpressions.

$$
\begin{array}{l}
0 < x < 1 \wedge y < -1 \\
\vee \\
0 < x < 1 \wedge 0 < y < \mathrm{root}_2\ y^2 + x^2 - 1 \\
\vee \\
x = 0 \wedge y < -1 \\
\vee \\
1 < x < \mathrm{root}_2 x^2 - 2 \wedge y < -1 \\
\vee \\
1 < x < \mathrm{root}_2 x^2 - 2 \wedge \mathrm{root}_1 (x-1)y - 1 < y
\end{array}
$$

$$\Downarrow$$

$$
\begin{array}{l}
0 < x < 1 \wedge \left(\begin{array}{l} y < -1 \vee \\ 0 < y < \mathrm{root}_2\ y^2 + x^2 - 1 \end{array} \right) \\
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\end{array}
$$

Cylindrically Arranged Cells — let's sum up

Describing a semi-algebraic set as a union of cylindrically arranged cells, whether factored into a cylindrical formula or not, is a natural idea.

1. It is an explicit description (SAT/dimension/model-points/projection are easy).
2. It gives us a natural "split by cases" description.
3. **However:**
   How could we compute complement in this representation?
   How we could construct such a thing from a Tarski formula?

So we have to look one step further to find an explicit representation that meets all of our goals ...

CAD — a definition ... finally

Defintion: A <u>Cylindrical Algebraic Decomposition</u> (CAD) is a decomposition of $\mathbb{R}^n$ into cylindrical cells that are cylindrically arranged.

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- On its own, a CAD does not represent a semi-algebraic set. But if we attach truth labels to each cell, then this "CAD+truth-labels" can be viewed as a representation of the semi-algebraic set defined by the union of the cells labeled true.

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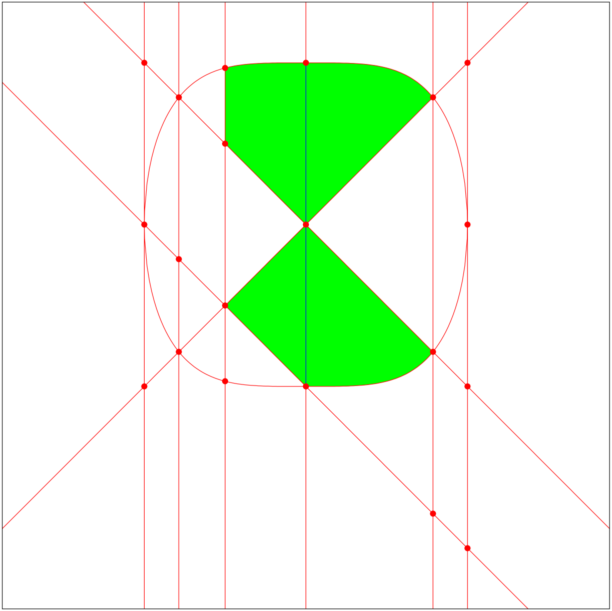
- In this representation, complement is easy: just toggle the cells' truth values! So all that's left is this: how, given a Tarski formula, do we construct a CAD+truth-values representation of the same semi-algebraic set?

CAD — A picture



CAD — the CAD construction problem

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- **Note 2:** When we say "produce a CAD", we intend that one has each cell represented so that a sample point or cylindrical formula defining the cell is either stored or can be easily produced.

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- If CAD D is <u>sign invariant for A</u> then it is <u>truth invariant for F</u>, where "sign invariant for A" means if $p_1, \ldots, p_k$ are the elements of A, D is invariant for the function $g(x) = (\mathrm{sgn}(p_1(x)), \ldots, \mathrm{sgn}(p_k(x))$.

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Focus on sign-invariant CADs!

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p_1 :	$+$	$+$	$+$	0	$-$	$-$	$-$	0	$+$	$+$	$+$

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F :	T	F	F	F	F	F	F	T	T	F	F

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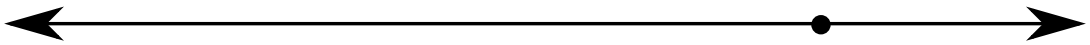

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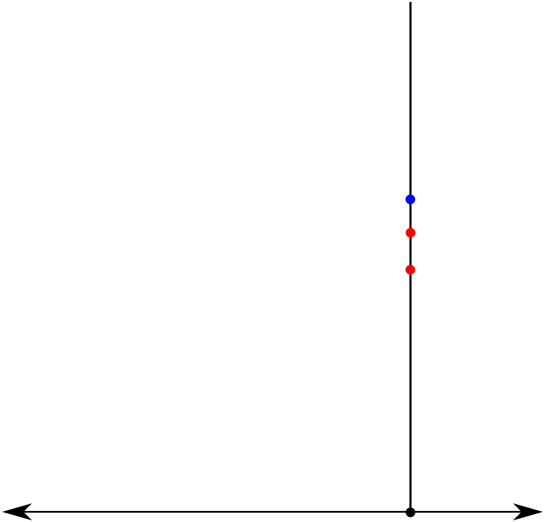

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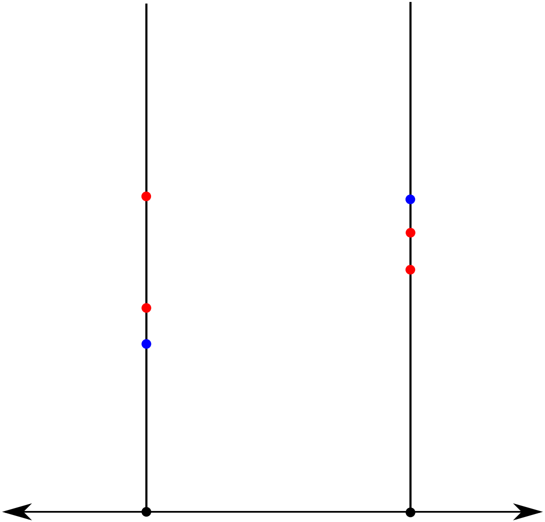

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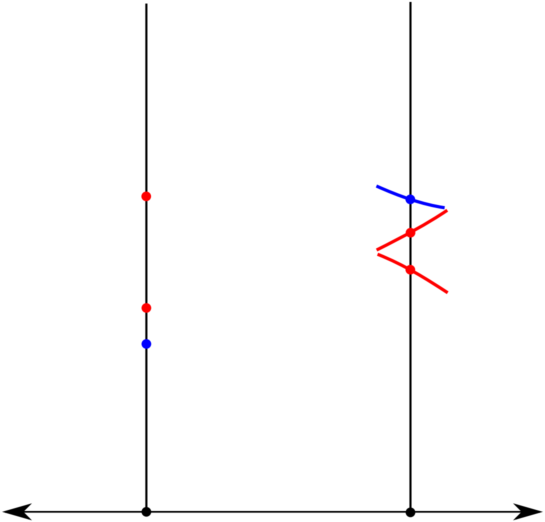

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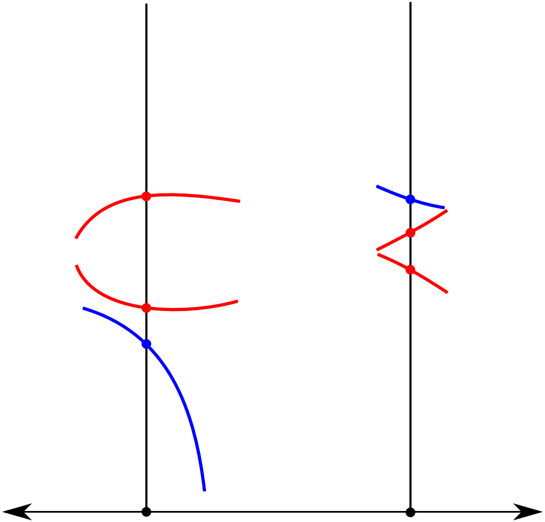

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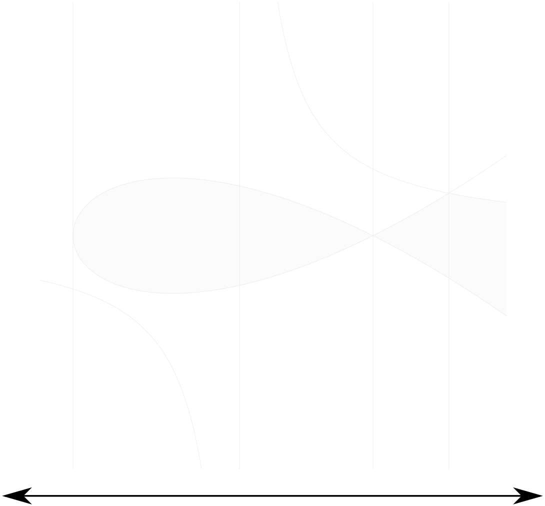

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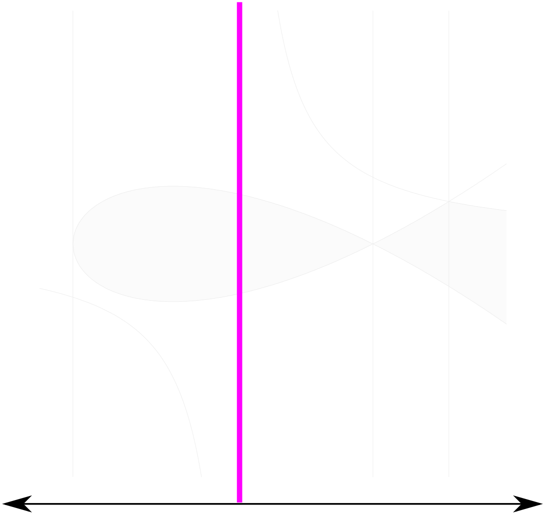

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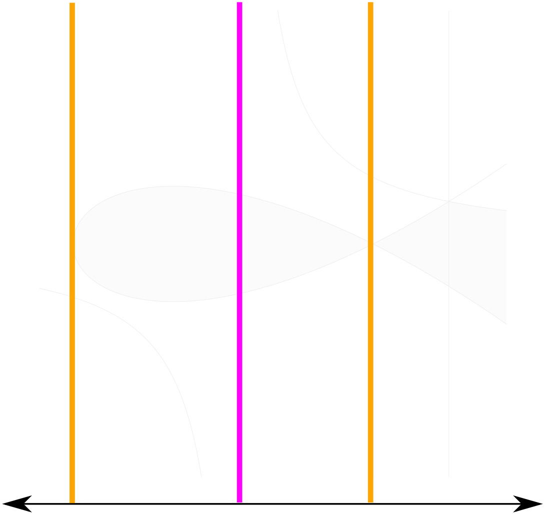

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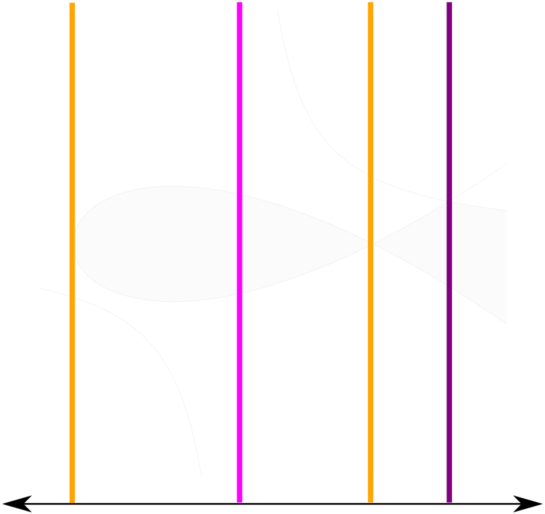

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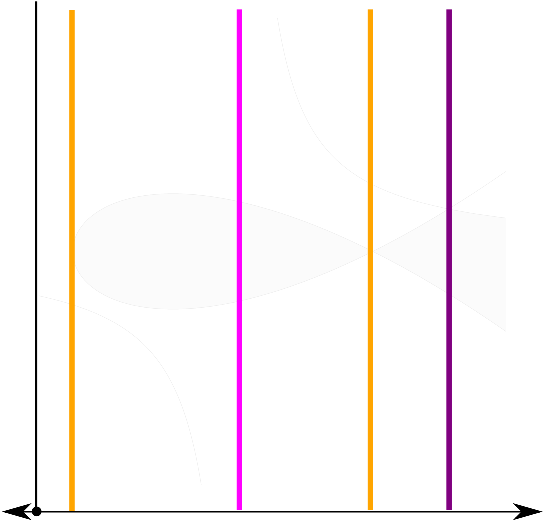

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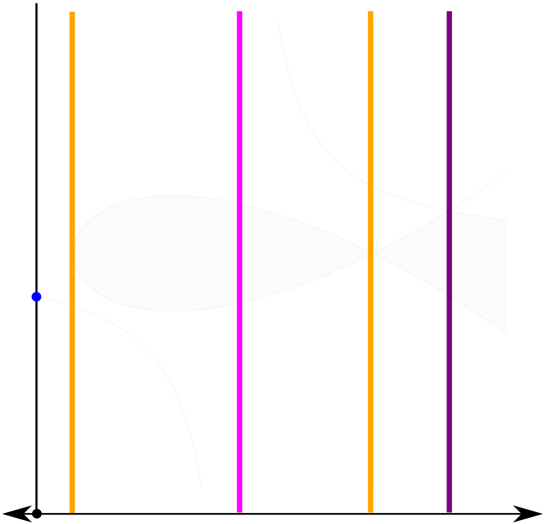

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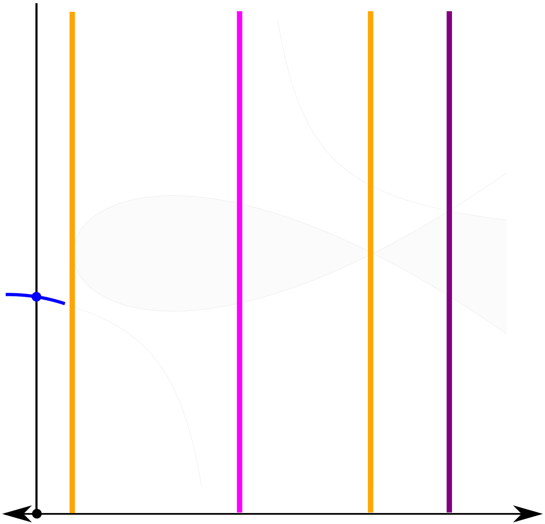

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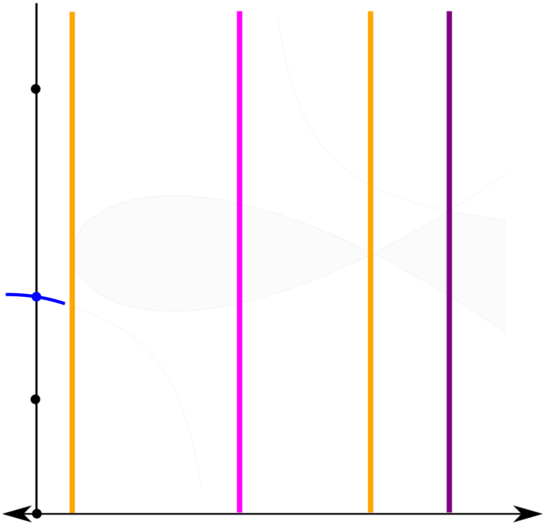

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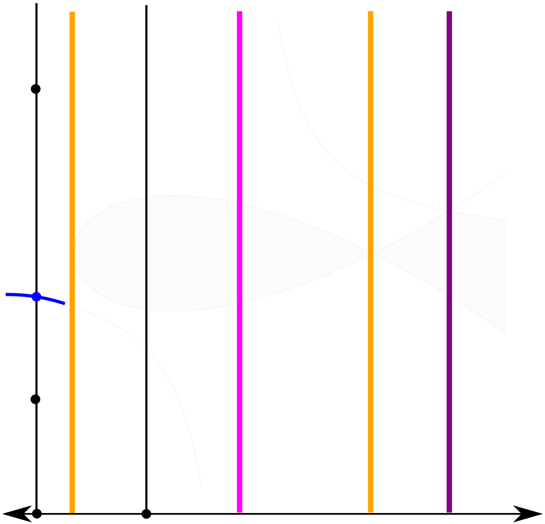

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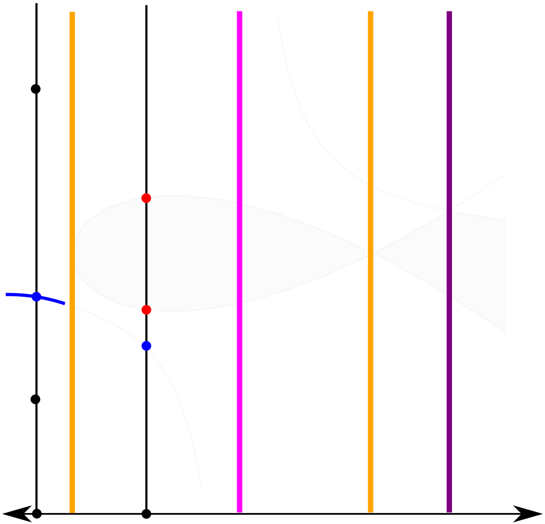

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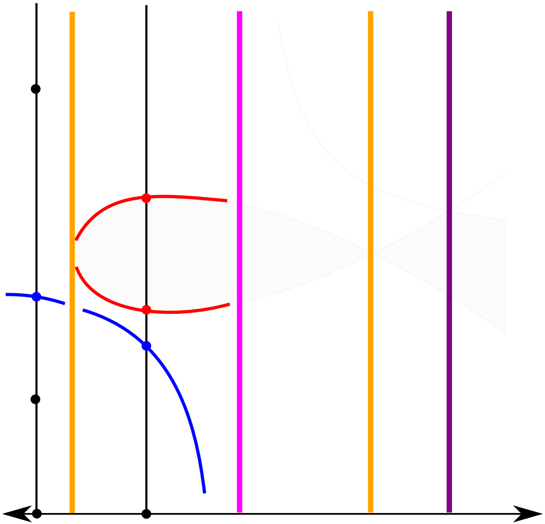

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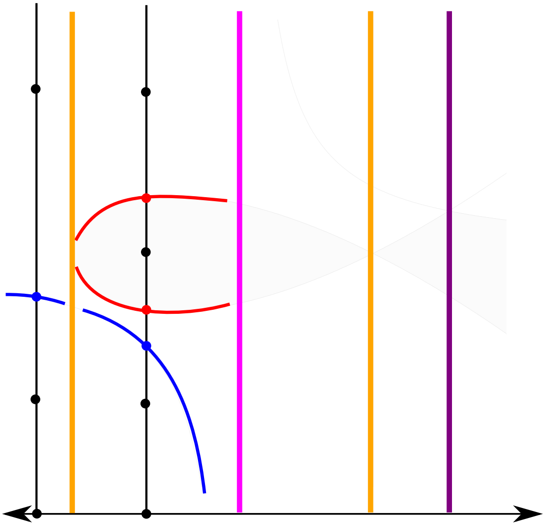

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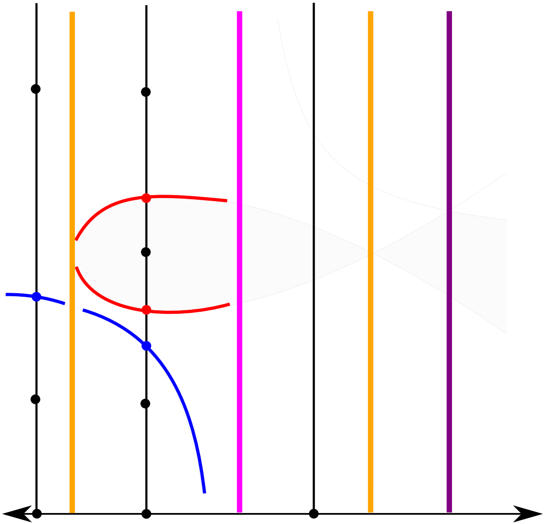

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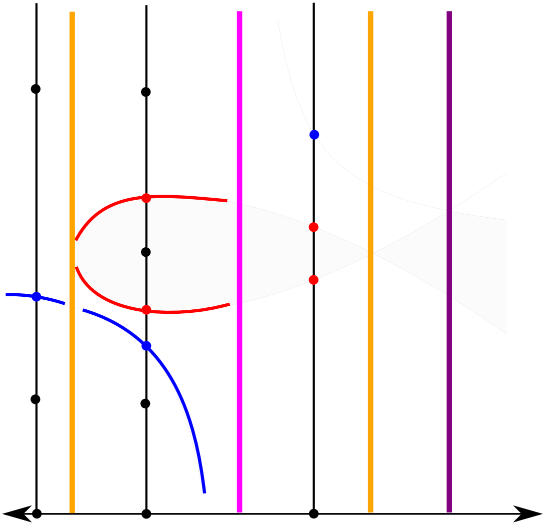

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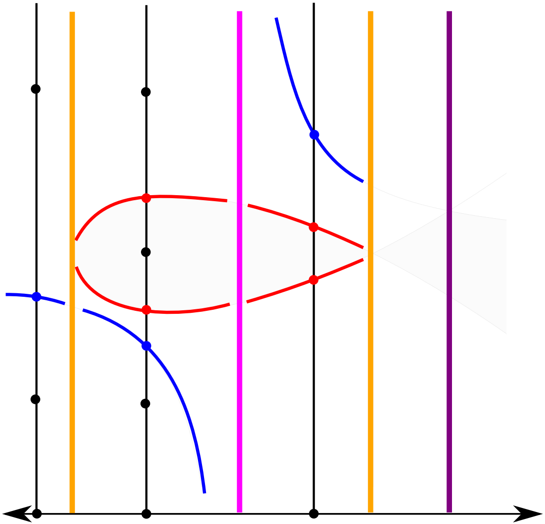

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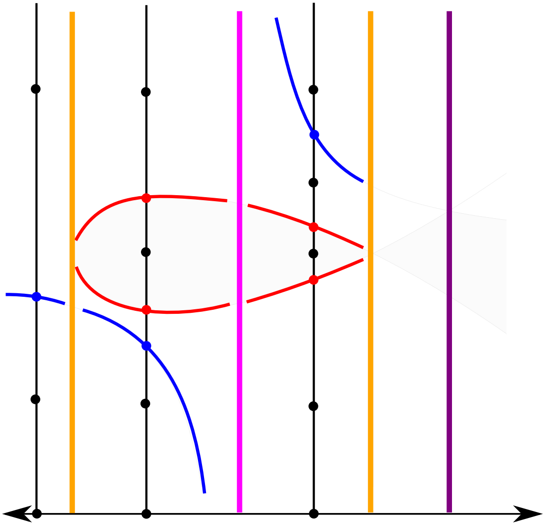

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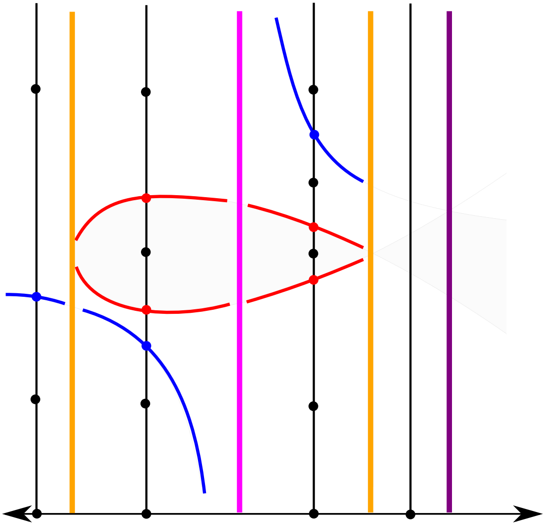

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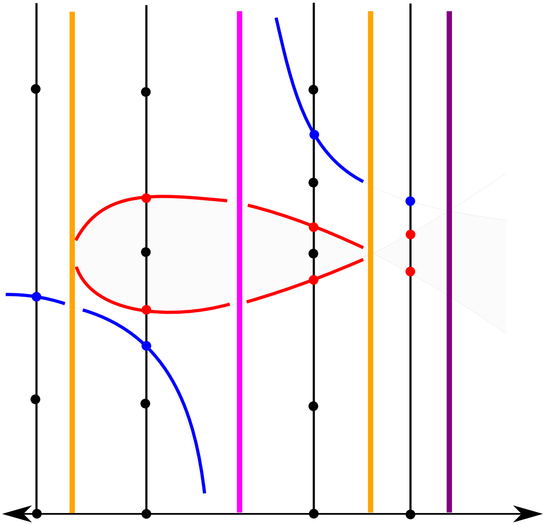

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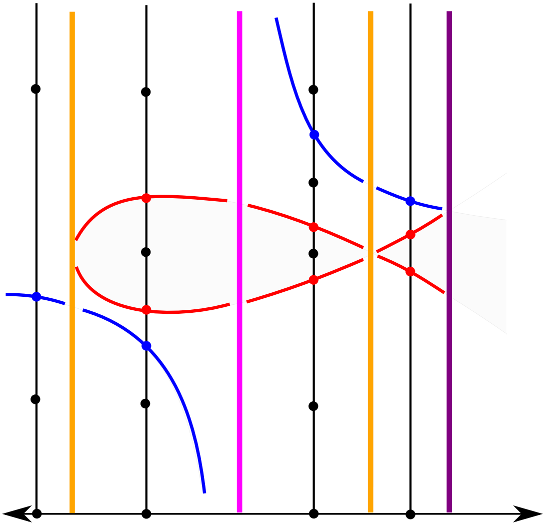

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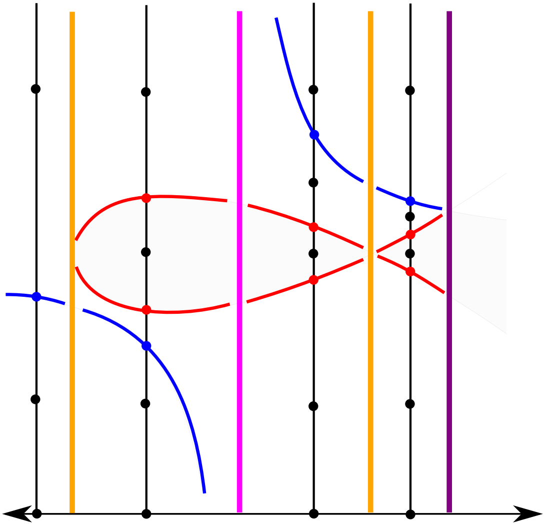

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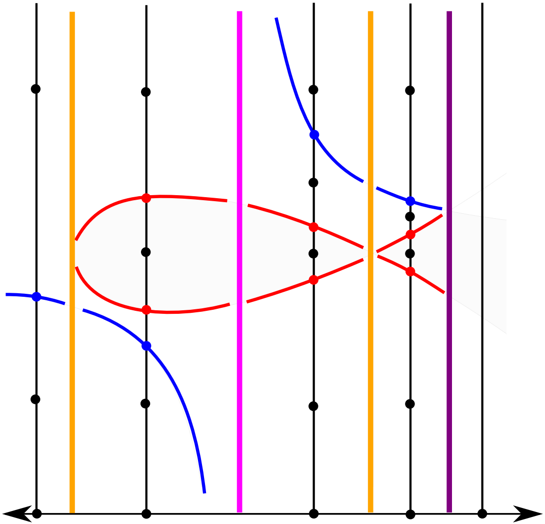

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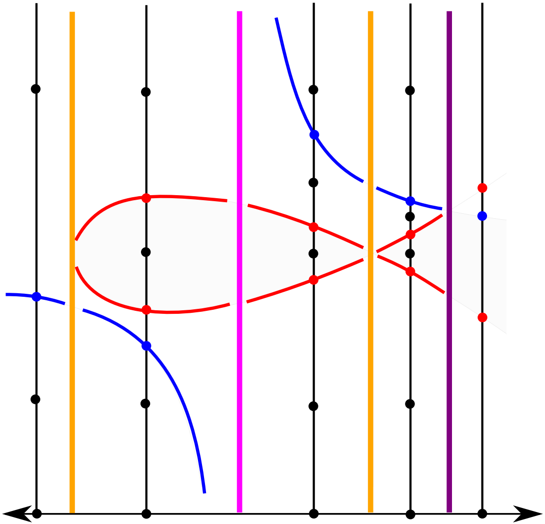

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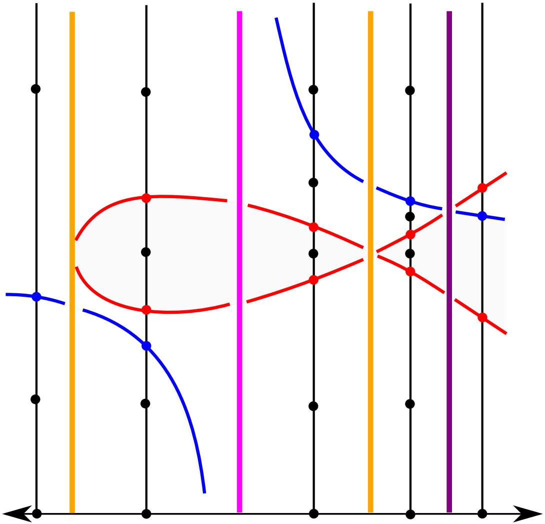

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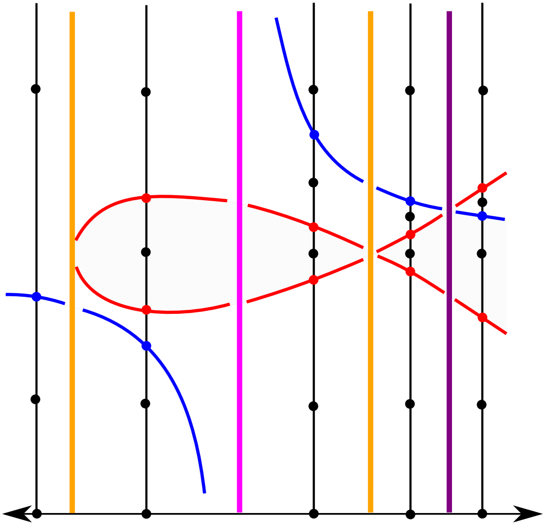

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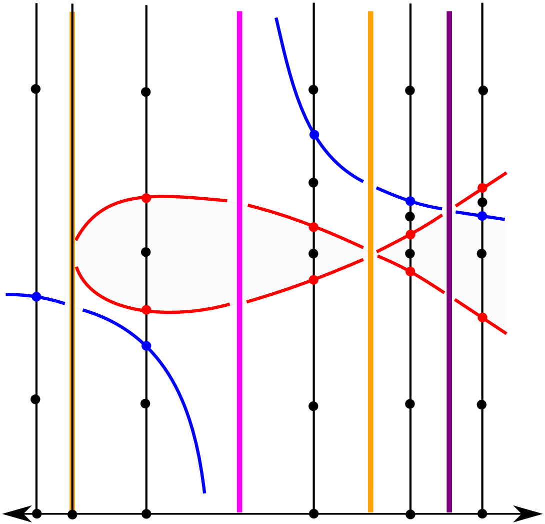

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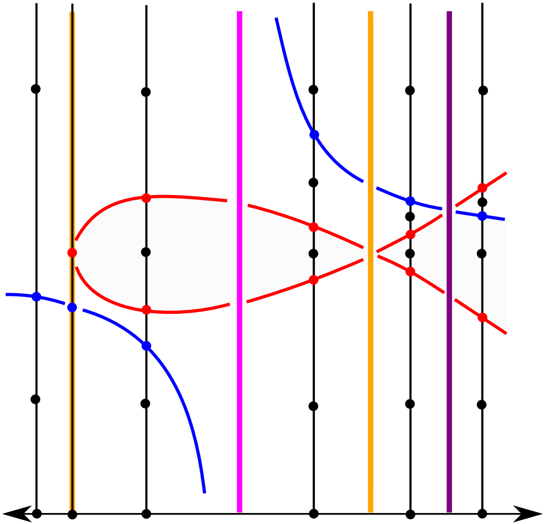

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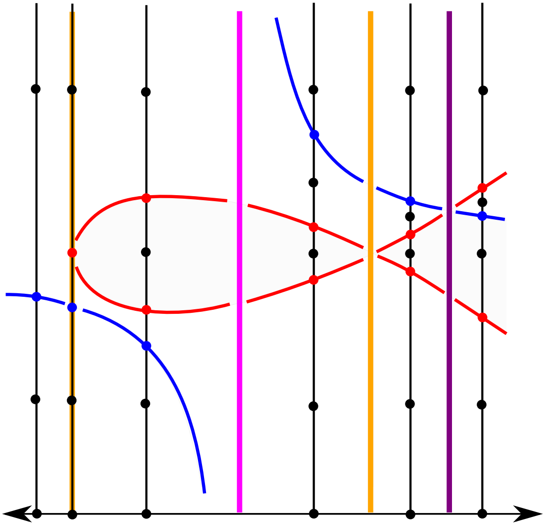

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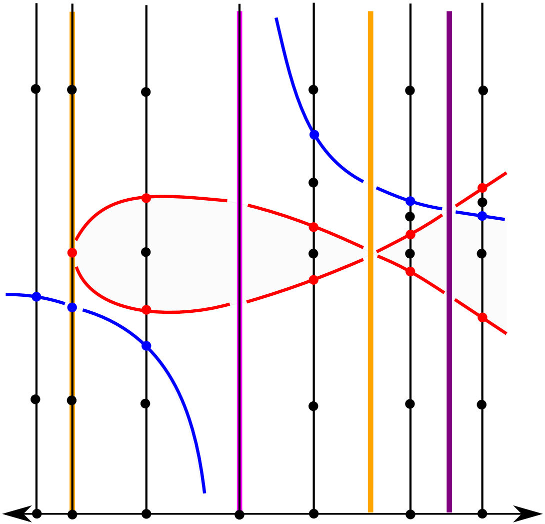

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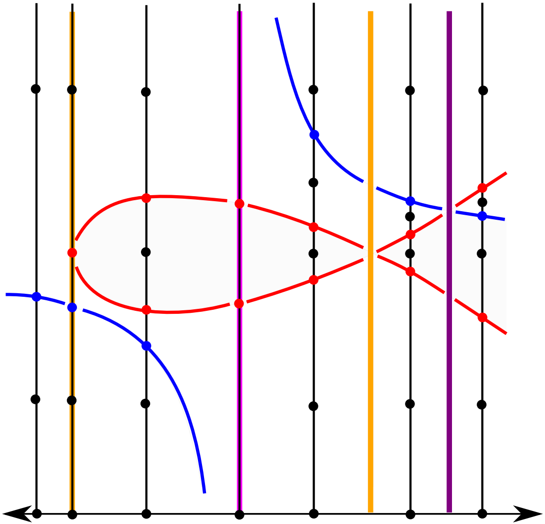

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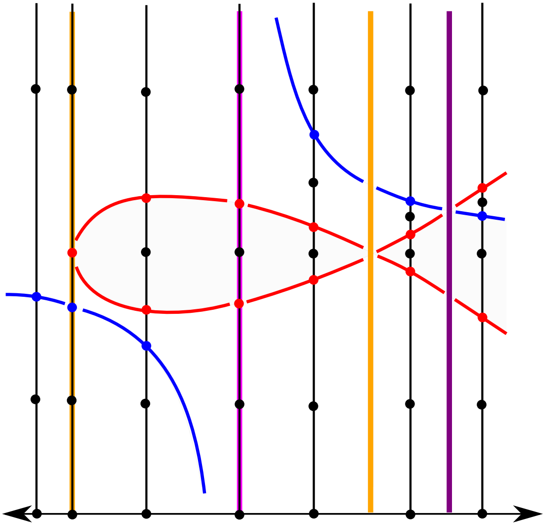

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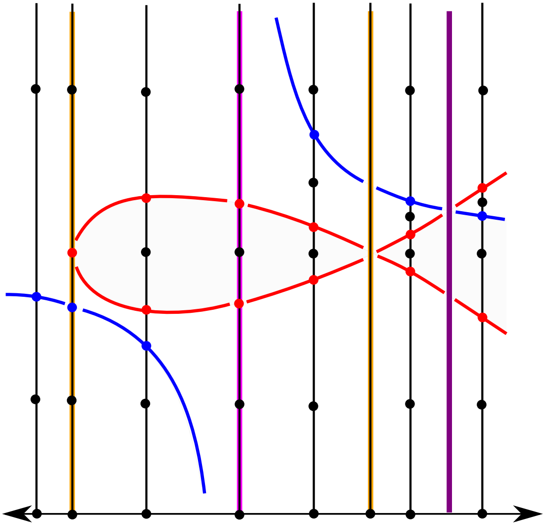

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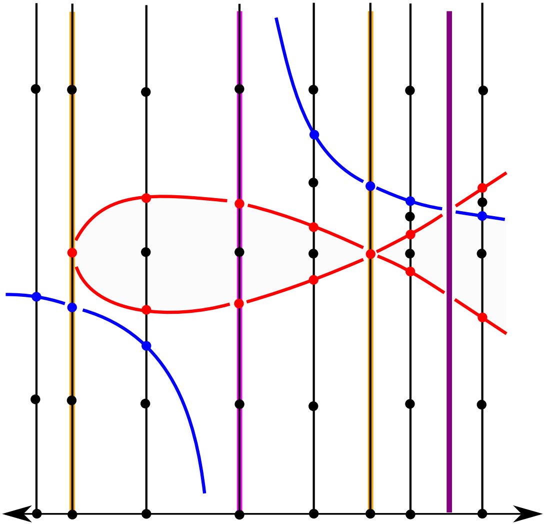

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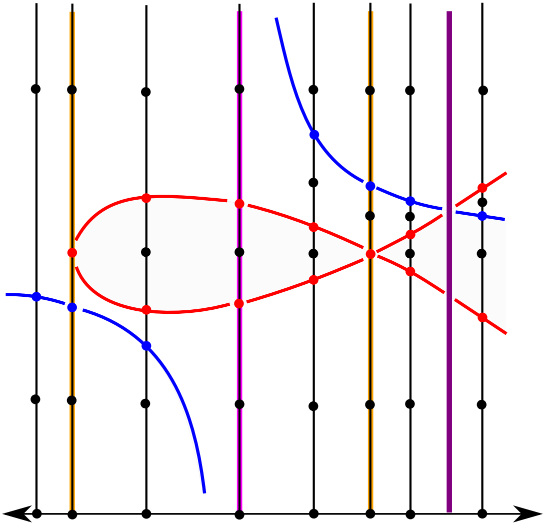

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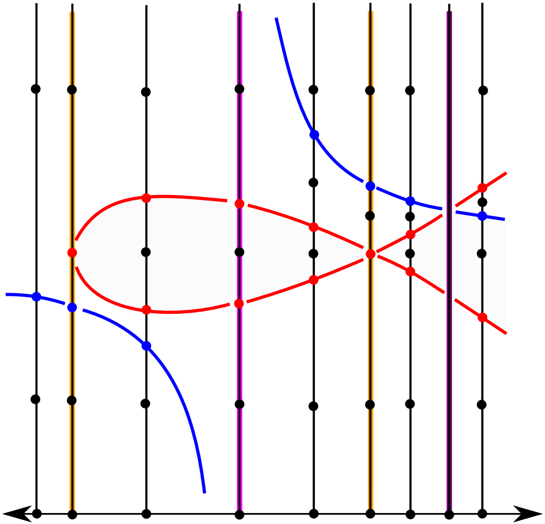

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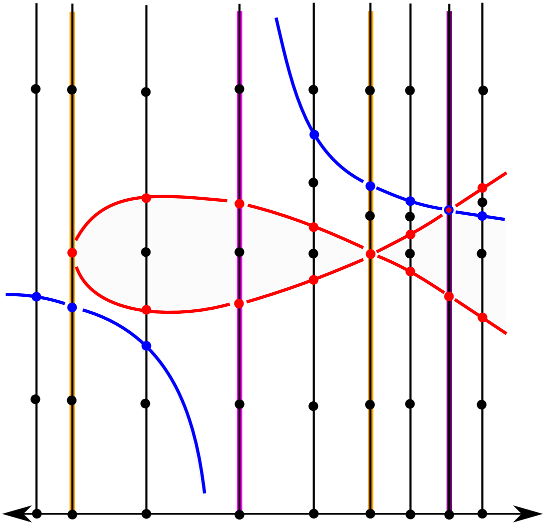

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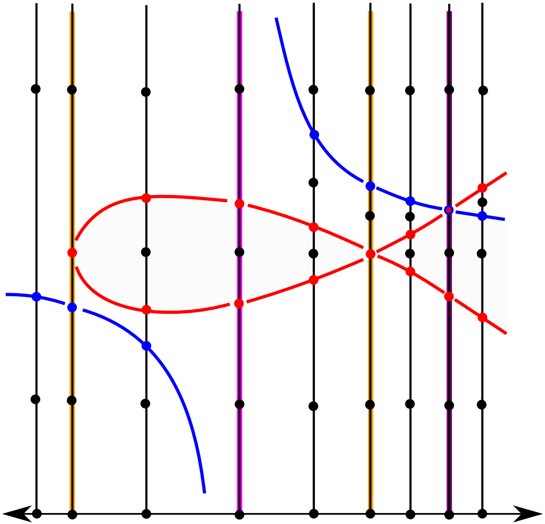

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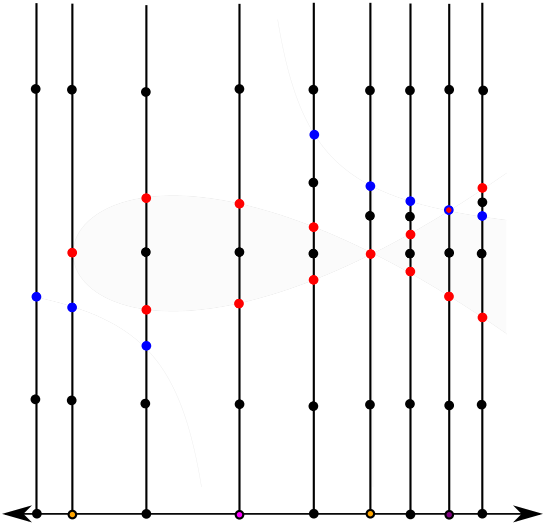

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Wanted: cells in which A_n are sign-invariant

Compute: $P_n(A_n) \in \mathbb{Z}[x_1, \ldots, x_{n-1}]$ s.t. if cell $S \in \mathbb{R}^{n-1}$ is sign-invariant for $P_n(A_n)$, then A_n are delineable over S

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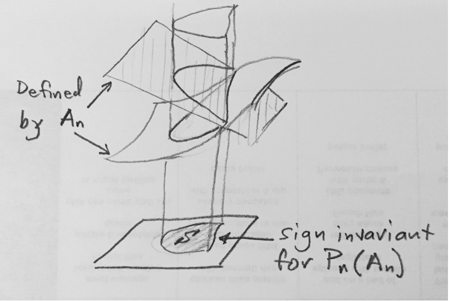

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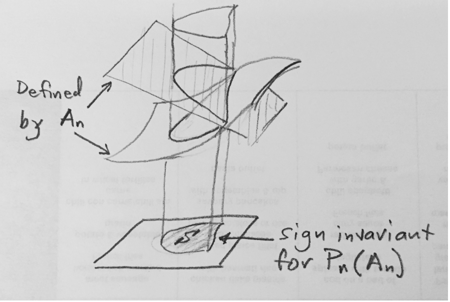

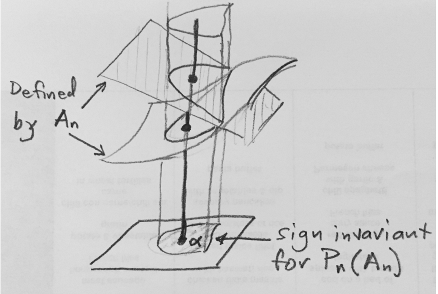

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Each cell d in D_c represents a cell in the CAD of $\mathbb{R}^n$!

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Collins' Original Projection Operator:

$$\begin{aligned}
\mathrm{ProjC}(A,n) &:= \mathrm{ProjC}_1(A,n) \cup \mathrm{ProjC}_2(A,n), \text{ where} \\
\mathrm{ProjC}_1(A,n) &:= \bigcup_{f\in A} \bigcup_{g \in \mathrm{RED}_{x_n}(f)} \left(\{\mathrm{ldcf}_{x_n}(g)\} \cup \mathrm{PSC}_{x_n}(g,g')\right) \\
\mathrm{ProjC}_2(A,n) &:= \bigcup_{f_1,f_2\in A} \bigcup_{\substack{g_1 \in \mathrm{RED}_{x_n}(f_1) \\ g_2 \in \mathrm{RED}_{x_n}(f_2)}} \mathrm{PSC}_{x_n}(g_1,g_2)
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$A \cup B \cup P(B_2)$ is called the "projection factor set"

Sign-invariant CAD: construction & projection

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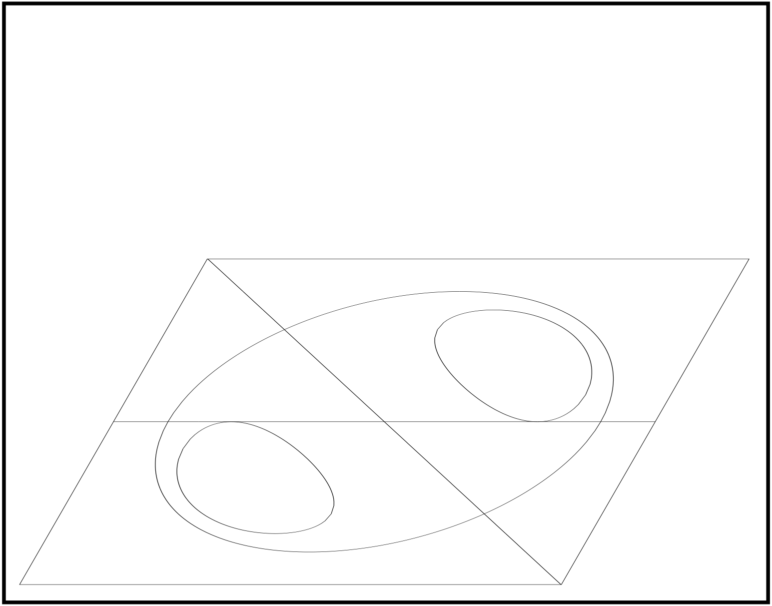

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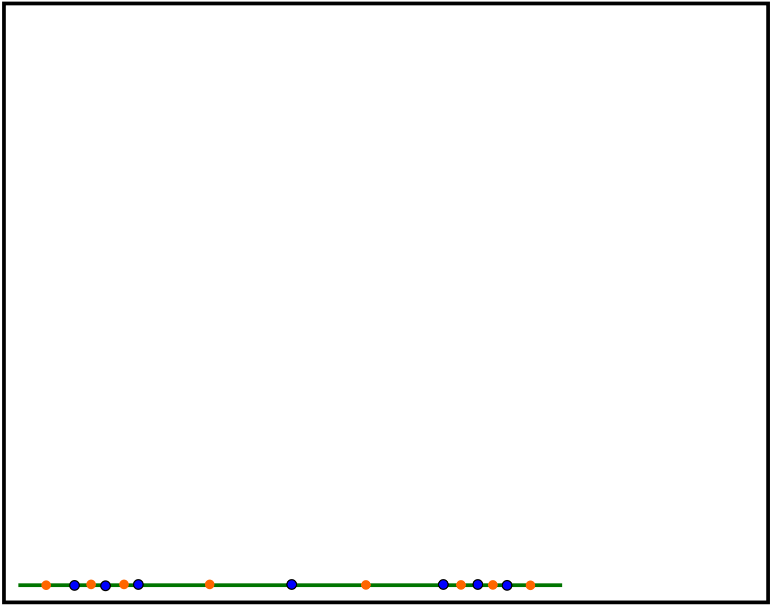

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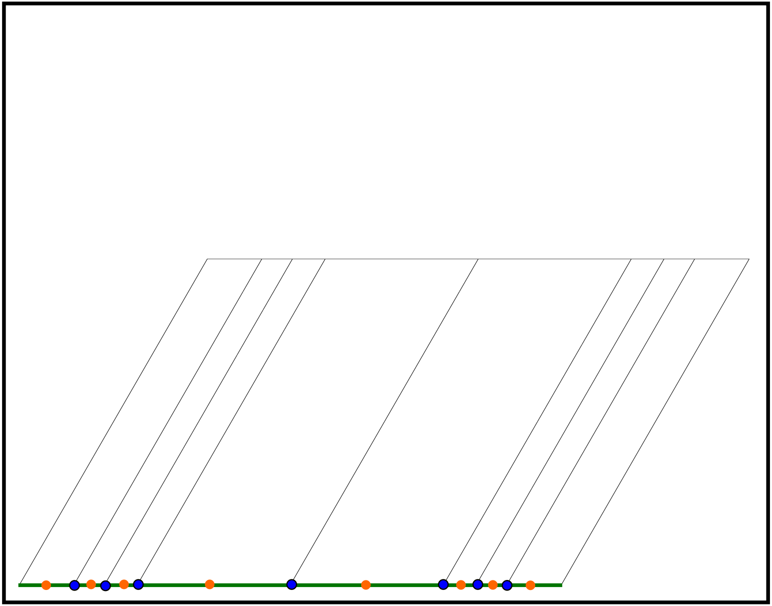

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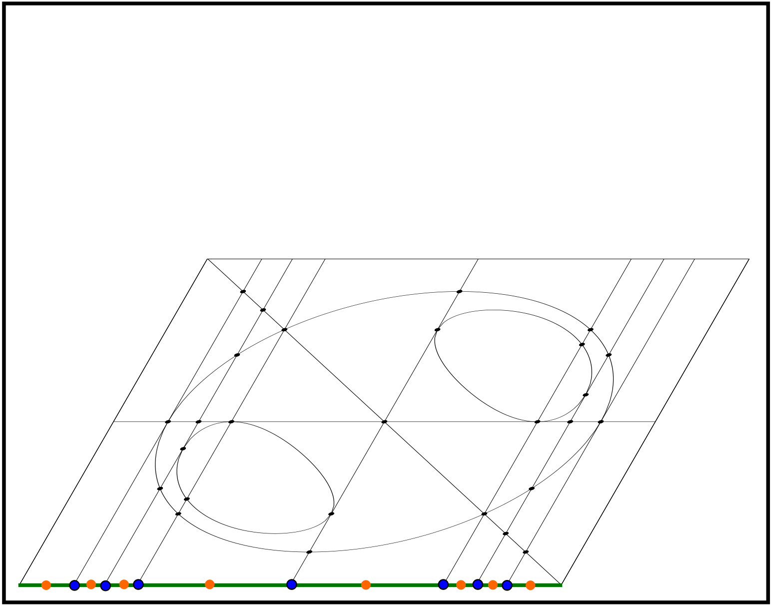

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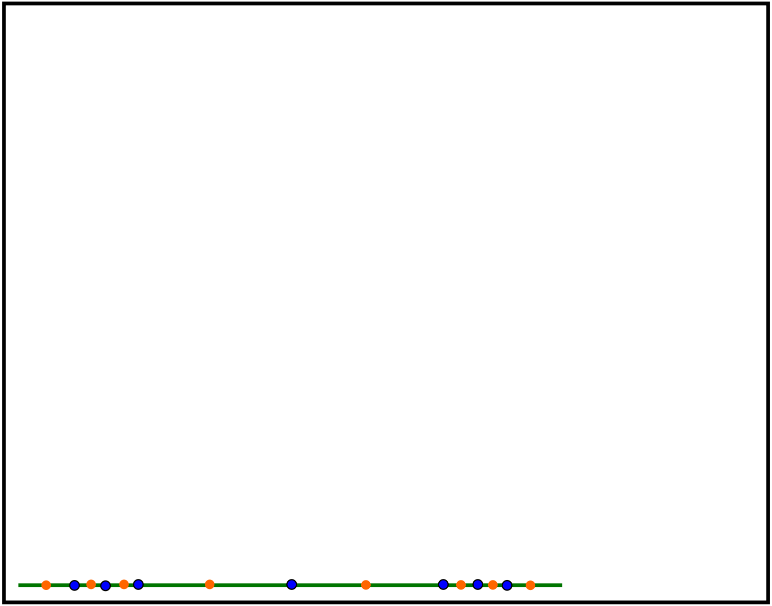

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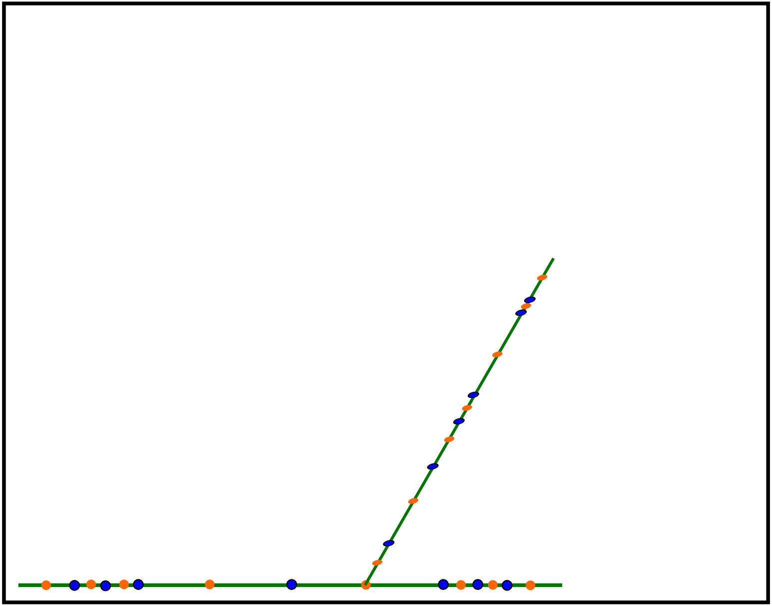

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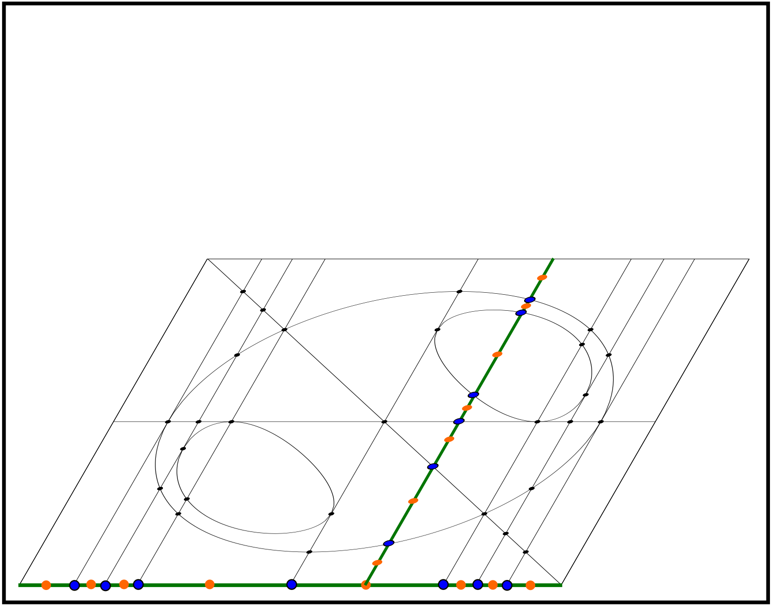

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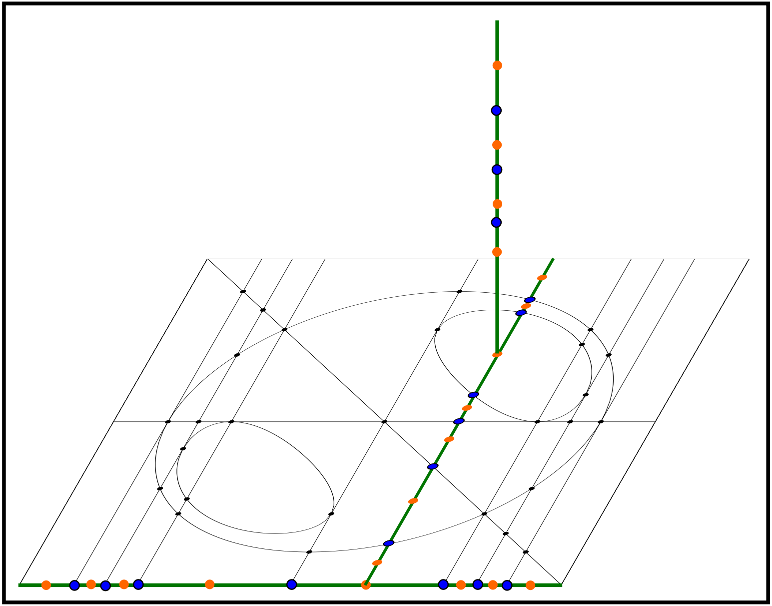

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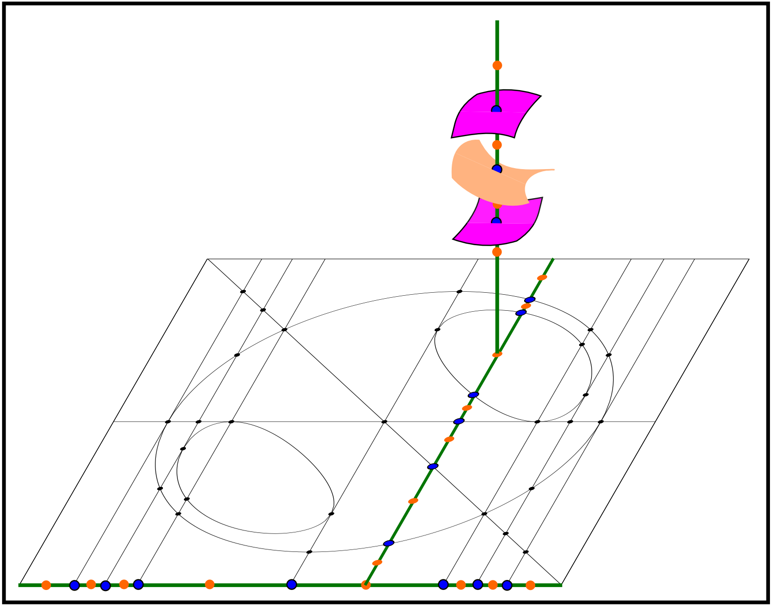

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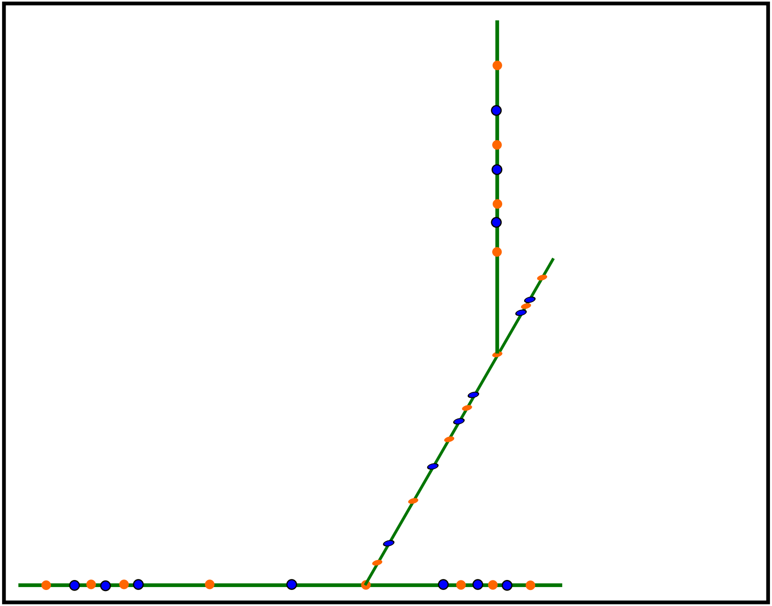

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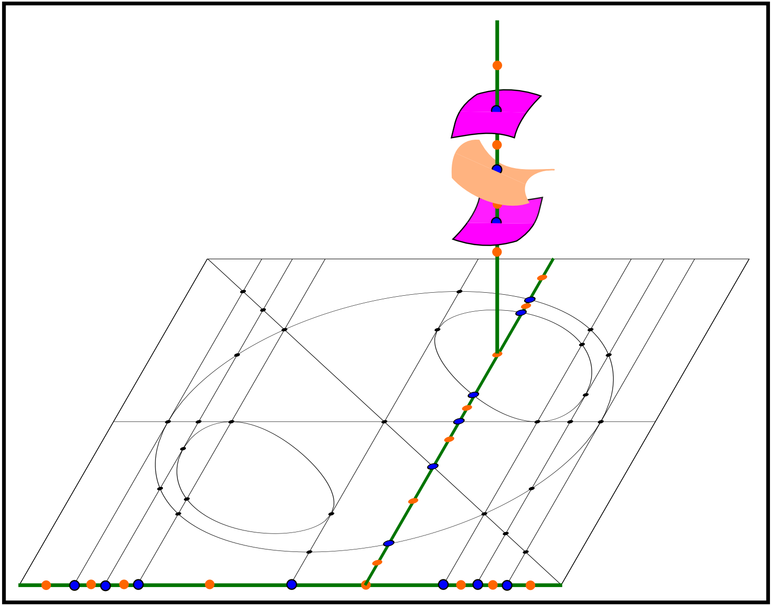

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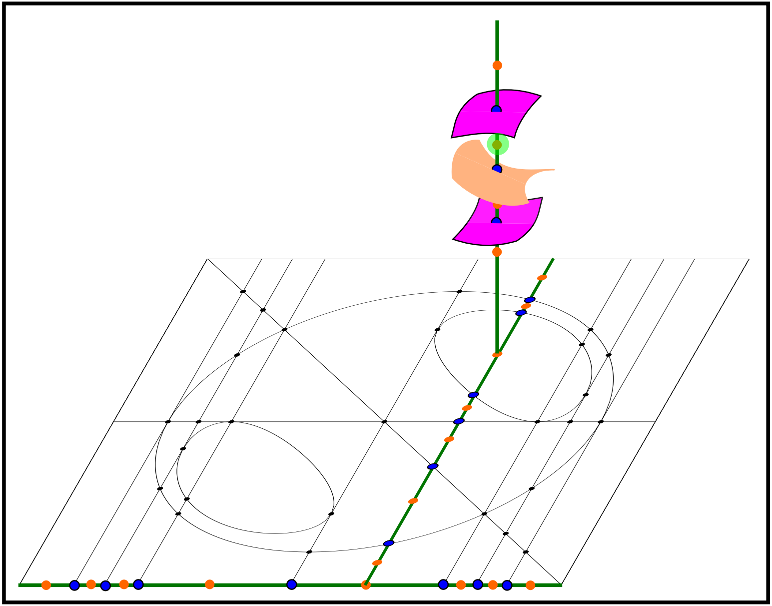

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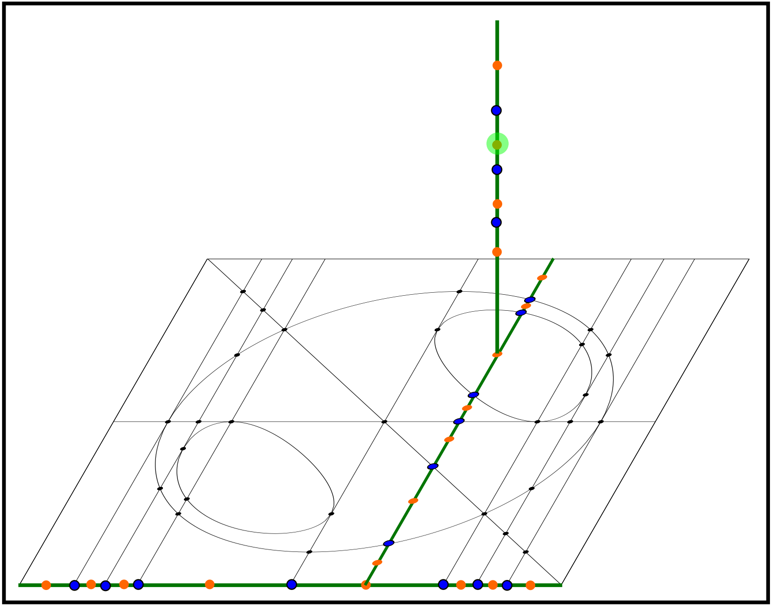

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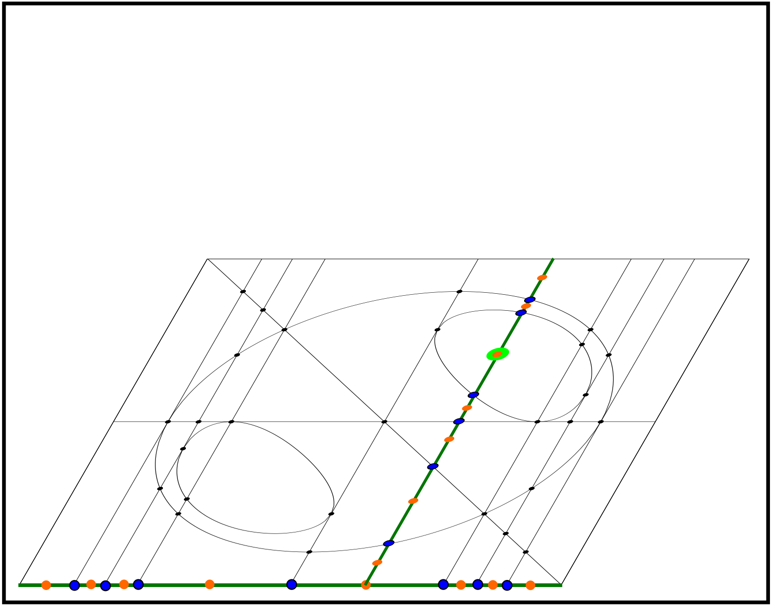

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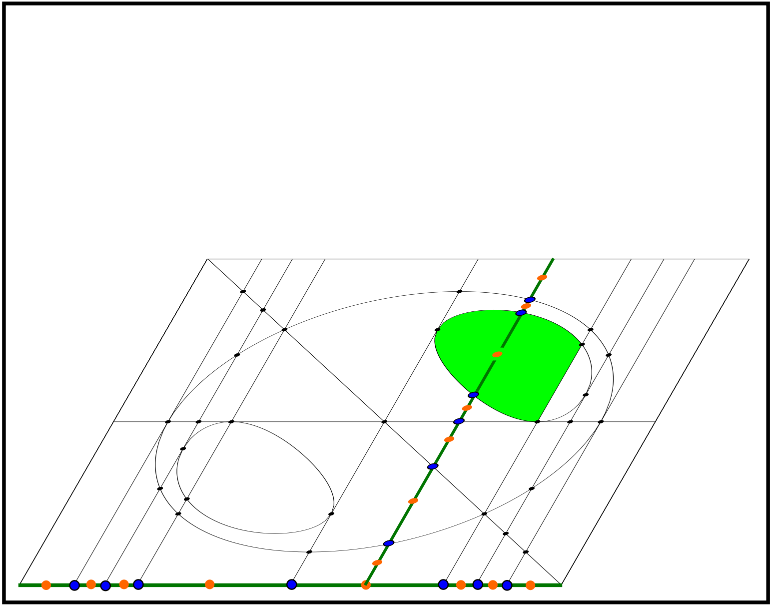

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- How to compute discriminants, resultants, PSC's and other polynomial operations used by projection operators

What's next?

Part II: Problem formulation

Example b1: Let's see CAD work

Suppose we had the following quantifier-elimination problem:

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Only works for orders! (a, b, x, y), (b, a, x, y), (a, b, y, x), (b, a, y, x)

Prenex and Variable Ordering Constraints

- Prenex form — basic Collins-style CAD requires

$$\left(Q_1 x_{i_1}, \ldots x_{i_2-1}\right) \left(Q_2 x_{i_2}, \ldots x_{i_3-1}\right) \cdots (Q_r x_{i_r}, \ldots x_n) \left[F(x_1, \ldots, x_n)\right]$$

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- Variables can only be reordered within their block
- Given prenex formula with r quantifier blocks, we have a strict partial order that constrains CAD variable order:

$$x < y \text{ if } x \text{ free and } y \text{ bound, or } \mathit{block}(x) \text{ before } \mathit{block}(y)$$

Example b3: Order matters

Consider a CAD for the formula below using the Collins projection:

$$zw^2+wx+w-2y+z+4x-1 < 0 \wedge 2yw^2-3wz-4w-2z+4x-3y+y < 0 \wedge x > 0 \wedge y > 0 \wedge z > 0$$

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If we use the order w, z, y, x, we get 24 projection factors!

Choosing a variable ordering

The choice of variable ordering is hugely important!

- In general, one can only reorder variables within the same block
- Several ways to choose ordering (see [Dolzman, Seidl, Sturm 2004], [Huang, Davenport, England ... 2014], ...)
- Simple hueristic
  1. Descending order by degree of variable, breaking ties with
  2. Descending order by highest total-degree term in which the variable appears, breaking ties with
  3. Descending order by number of terms containing the variable

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$$x = 0, y = x, 3x - 2y = 1, ax + b = 0$$

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Punchline: Do linear substituions <u>before</u> CAD construction!

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A Word about defining formulas

Finishing off the previous example ...

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Example b6: $(\exists y)[x^2 + y^2 < 1 \wedge x + y > 0] \to x < 1 \wedge x > \mathrm{root}_1 2x^2 - 1$

End of Part II

What have we learned?

- The importance of variable ordering
- Carrying out simplification (e.g. linear substitutions) prior to CAD construction
- The value of splitting up inputs and the distinction betweeen Tarski formulas for results vs. formulas with indexed root expressions.

Complexity

Constructing a CAD via Collins' original algorithm takes time

where $r = \#$ of variables,
$n =$ max degree of input in any variable,
$m = \#$ of input polynomials,
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This is really unfortunate complexity.

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How can we improve upon the original Collins CAD?

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   - Divide & Conquer / incremental CAD (Strzebonski, Kremer, ...)

Part III: SMT meets computer algebra in NLSAT

Big picture to keep in mind

SAT/SMT Strategy

1. incrementally build model
2. learn by generalizing conflicts

Collins Strategy

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The NLSAT algorithm of Jovanović and de Moura (2012) is a beautiful synthesis of the two!

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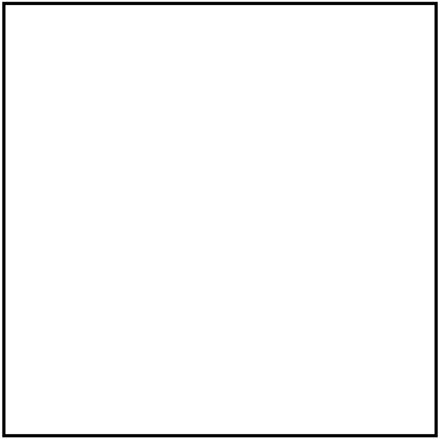
$$\overbrace{(x > 1 \vee x < -1)}^{C11} \wedge \overbrace{x^2 < 3}^{C12} \wedge \overbrace{(y^2 - 2xy + x > 0 \vee y < -2)}^{C21} \wedge \overbrace{x^3 + y^2 < 2xy}^{C22}$$

$S||$



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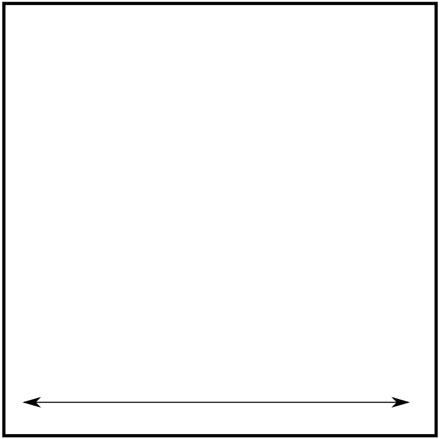

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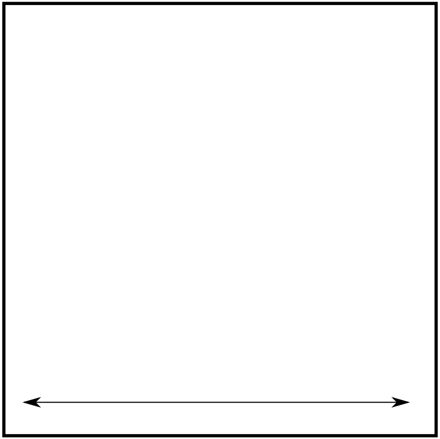

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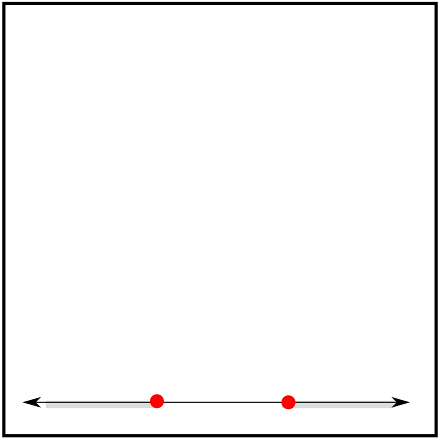

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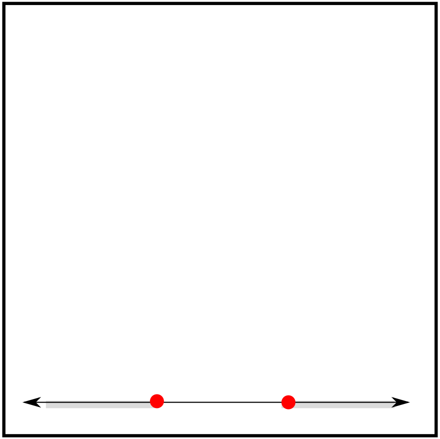

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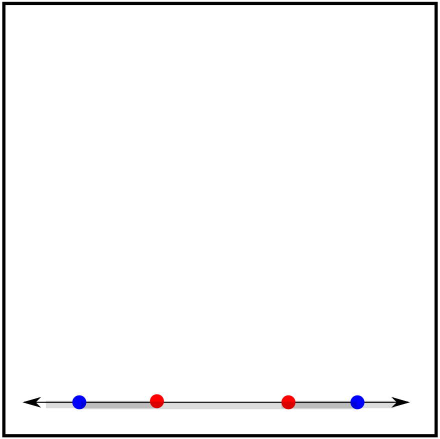

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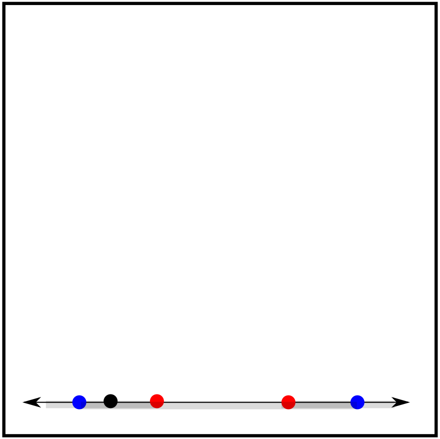

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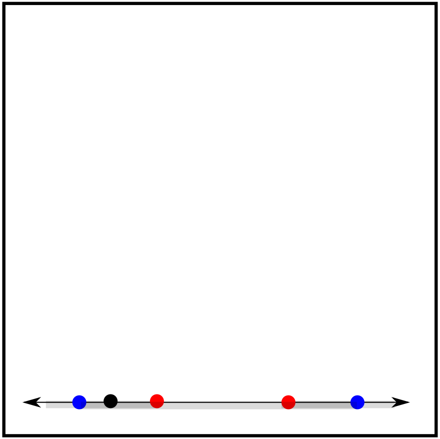

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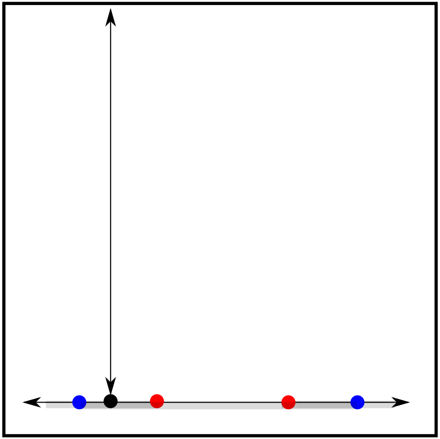

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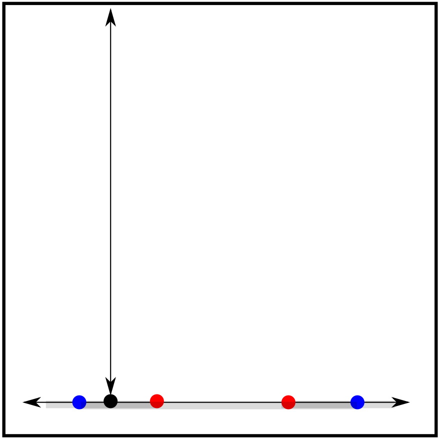

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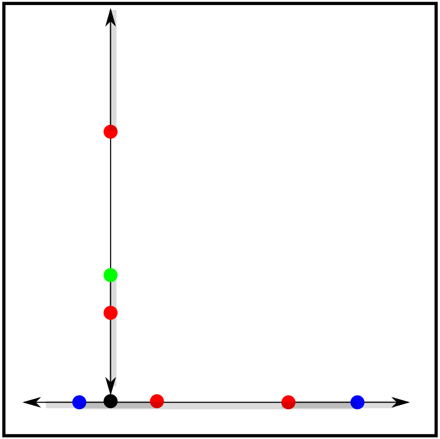

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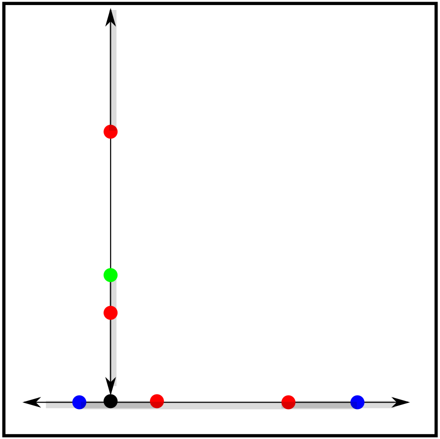

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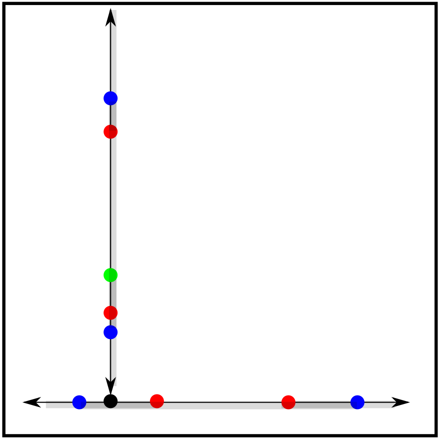

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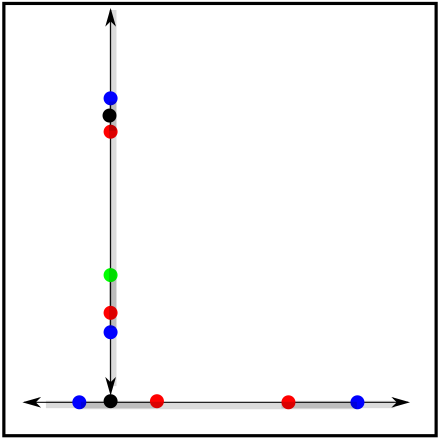

NLSAT Example 1: incrementally build a model

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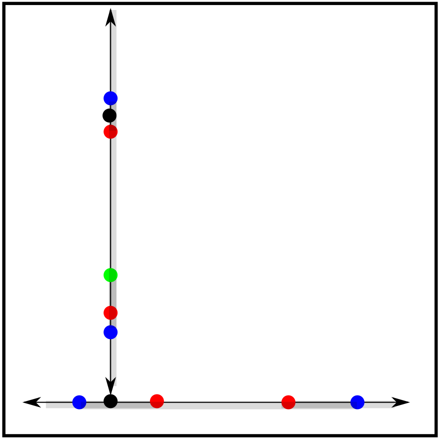
$$S \| \; C11 \; C12 \; \left(x \leftarrow -\tfrac{3}{2}\right) \; C21 \; C22 \; \left(y \leftarrow \tfrac{5}{8}\right)$$



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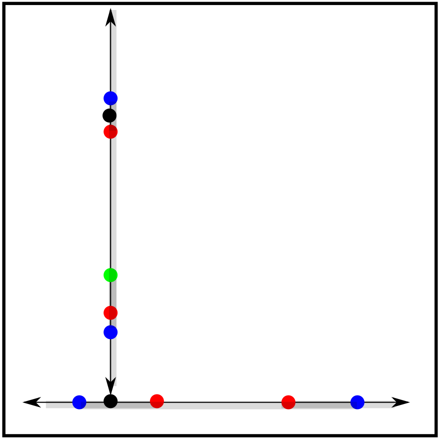

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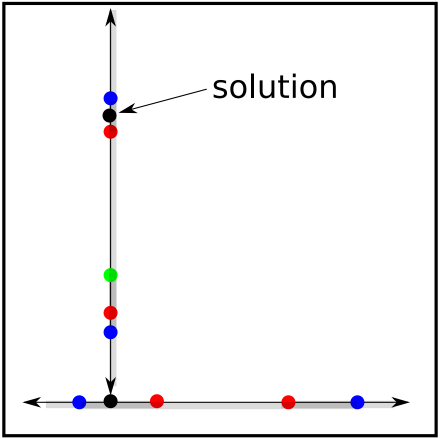

NLSAT Example 2: learn by generalizing conflicts

$$\overbrace{x^2 - 2 < 0}^{C11} \wedge \overbrace{y^2 - x^2 + x + 1 \leq 0}^{C21} \wedge (\overbrace{y > 1 \vee y < -2}^{C22}) \wedge \overbrace{z > \frac{3}{2}}^{C31} \wedge \overbrace{z - x^2 y - 1 < 0}^{C32} \wedge \overbrace{z^2 + x - y < 0}^{C33}$$

$S||$

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$S || C11$

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$S || C11 \ (x \leftarrow 0)$

NLSAT Example 2: learn by generalizing conflicts

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$S || C11\ (x \leftarrow 0)\ C21$

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$S || C11\ (x \leftarrow 0)\ C21$ $\times$

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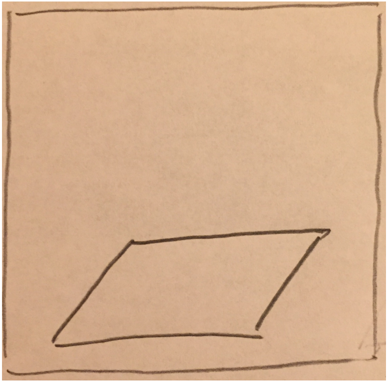
idea

CAD: construct cylindrical cell, then <u>specialize</u> to some model point α



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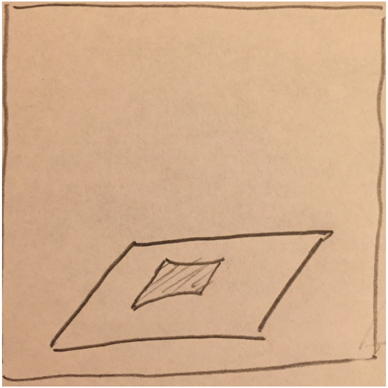

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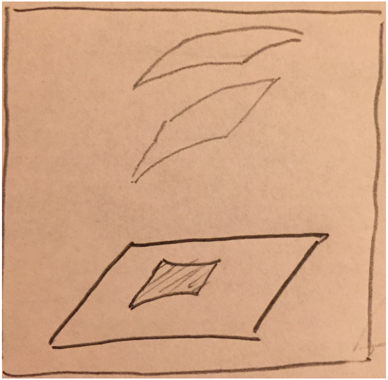

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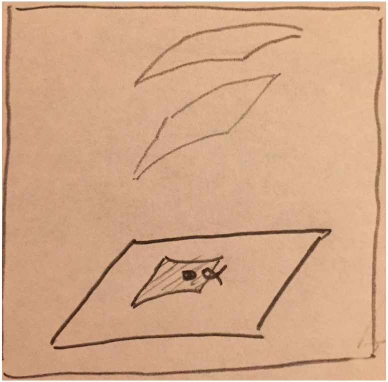

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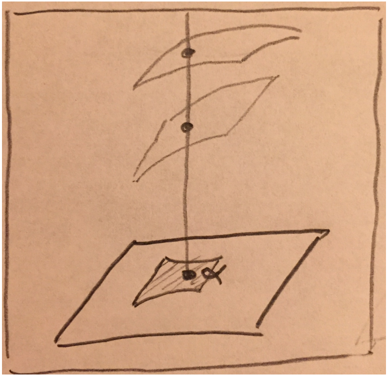

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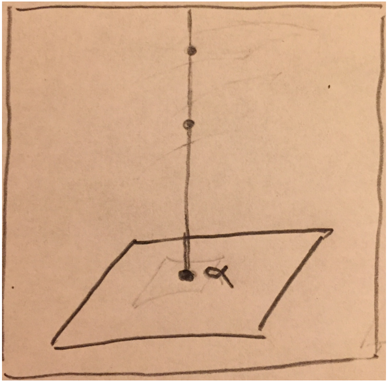

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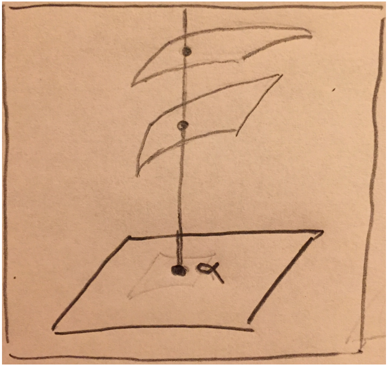
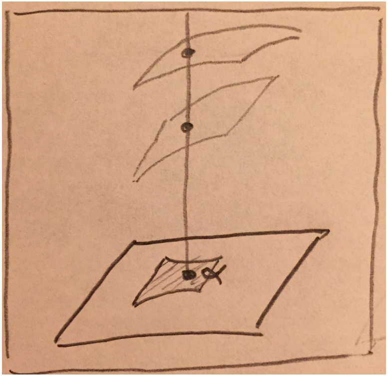


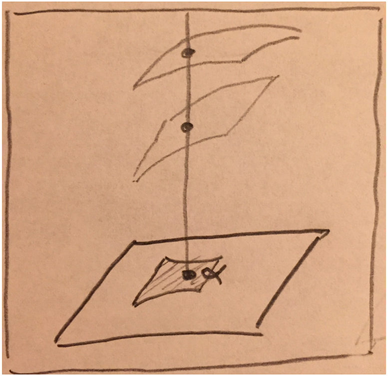
idea

CAD: construct cylindrical cell, then <u>specialize</u> to some model point α



idea

CAD: construct cylindrical cell, then specialize to some model point α
NLSAT: construct model point α, then generalize to a cylindrical cell



NLSAT Example 2: learn by generalizing conflicts

$$\overbrace{x^2 - 2 < 0}^{C11} \wedge \overbrace{y^2 - x^2 + x + 1 \leq 0}^{C21} \wedge (\overbrace{y > 1 \vee y < -2}^{C22}) \wedge \overbrace{z > \frac{3}{2}}^{C31} \wedge \overbrace{z - x^2 y - 1 < 0}^{C32} \wedge \overbrace{z^2 + x - y < 0}^{C33}$$

$S || C11\ (x \leftarrow 0)\ C21 \times \leftarrow$ $C21$ at $x = 0$ is $y^2 + 1 \leq 0$

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generalize conflict to cell $x > \text{root}_1 x^2 - x - 1 \wedge x < \text{root}_2 x^2 - x - 1$

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$$\overbrace{x^2-2<0}^{C11} \wedge \overbrace{y^2-x^2+x+1\le 0}^{C21} \wedge \overbrace{(y>1 \vee y<-2)}^{C22} \wedge \overbrace{z>\frac{3}{2}}^{C31} \wedge \overbrace{z-x^2y-1<0}^{C32} \wedge \overbrace{z^2+x-y<0}^{C33}$$
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NLSAT Example 2: learn by generalizing conflicts

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$S||C11$

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∘ above $\left(-\frac{9}{8},\frac{11}{10}\right)$ constraints C31 and C33 conflict independent of C32

“Learn by generalizing conflicts” in detail

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- we know point $\alpha = (-\frac{9}{8}, \frac{11}{10})$

"Learn by generalizing conflicts" in detail

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- we compute cylindrical cell C that contains α and is sign-invariant for $\{y - x, 4y - 4x - 9\}$,

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- we compute cylindrical cell C that contains α and is sign-invariant for $\{y - x, 4y - 4x - 9\}$, $C := -\infty < x \wedge x < \infty \wedge x < y \wedge y < x + \frac{9}{4}$

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  Note: C generalizes conflicting point α

"Learn by generalizing conflicts" in detail

- we know point $\alpha = (-\frac{9}{8}, \frac{11}{10})$
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- we compute $P_3(\{z - \frac{3}{2}, z^2 + x - y\}) = \{y - x, 4y - 4x - 9\}$
  ... so over any region s.t. $\{y - x, 4y - 4x - 9\}$ sign-invariant
  ... $\{z - \frac{3}{2}, z^2 + x - y\}$ are delineable
  ... so $z > \frac{3}{2} \wedge z^2 + x - y < 0$ is violated exactly as it is above α
- we compute cylindrical cell C that contains α and is sign-invariant for $\{y - x, 4y - 4x - 9\}$, $C := -\infty < x \wedge x < \infty \wedge x < y \wedge y < x + \frac{9}{4}$
  Note: C generalizes conflicting point α
- we learn clause $\neg C = (x \geq y \vee y \geq x + \frac{9}{4})$

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NLSAT: an SC^2 success story (before SC^2)

SAT/SMT Strategy

1. incrementally build model
2. learn by generalizing conflicts

Collins Strategy

1. univariate real root isolation
2. projection (eliminate variables)

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NLSAT is conflict-driven, incremental, optimistic, lazy ...

What have we learned?

Be conflict-driven, incremental, optimistic, and lazy!

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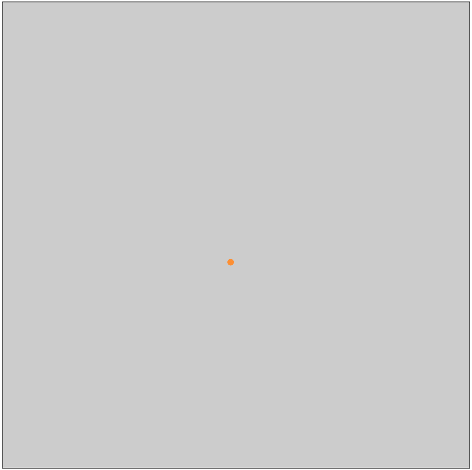
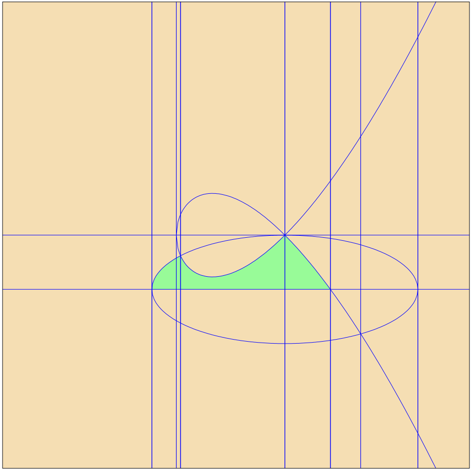
Can we do something conflict-driven, incremental, optimistic, lazy ... ?

Applying "generalize-from-a-model-point" to CAD





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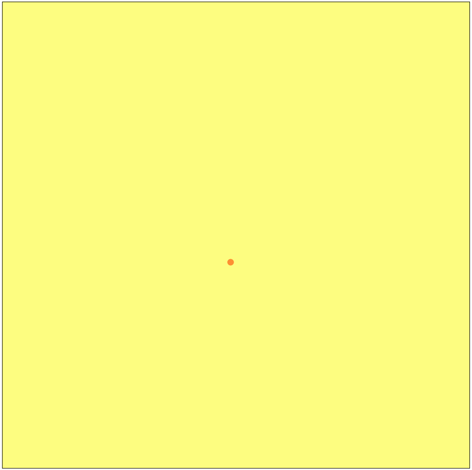

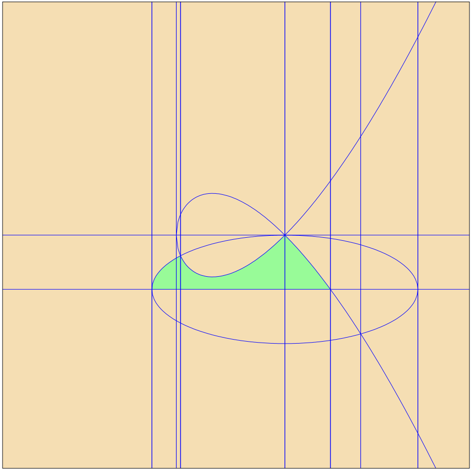

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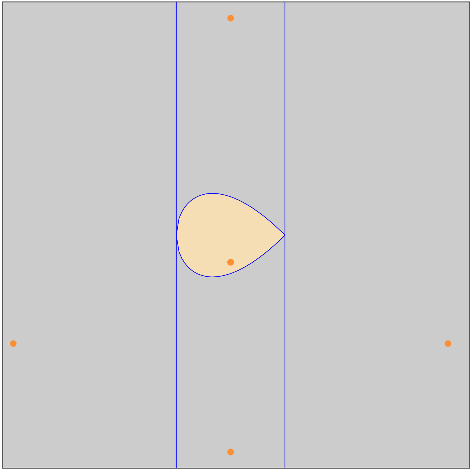

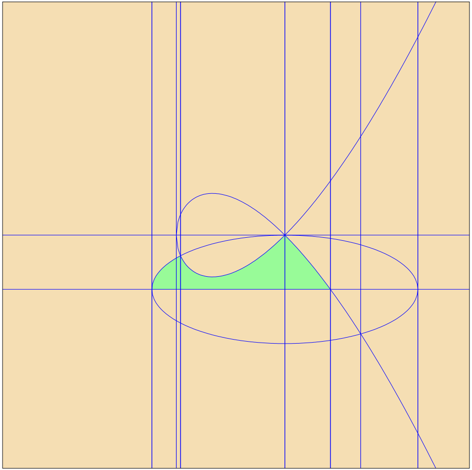

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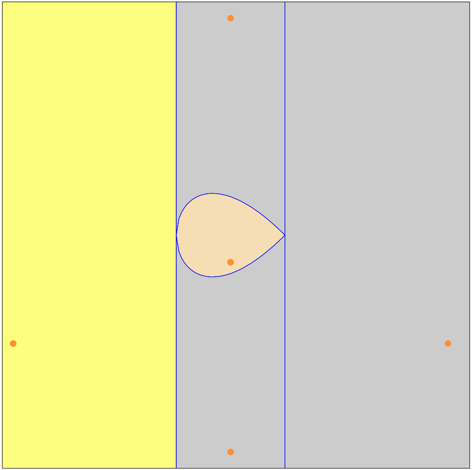

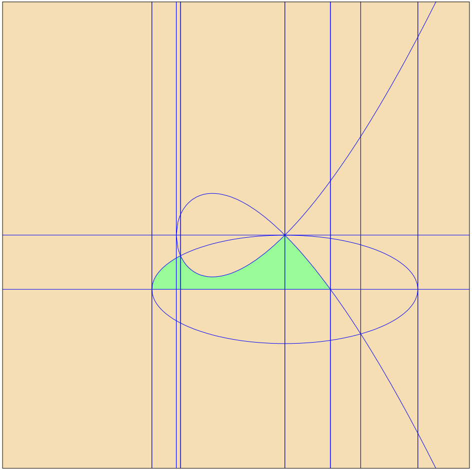

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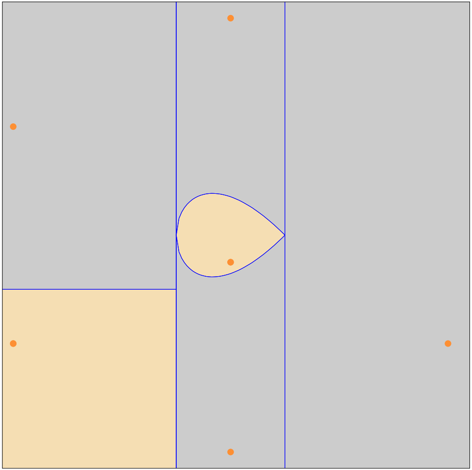

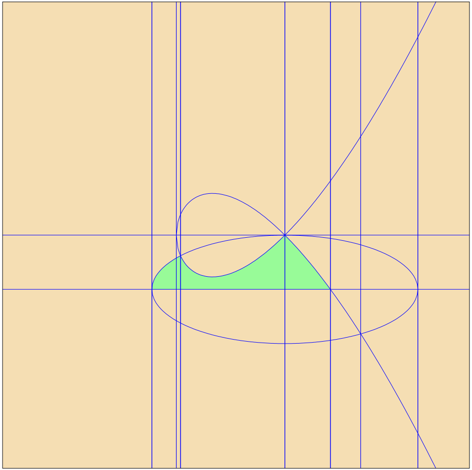

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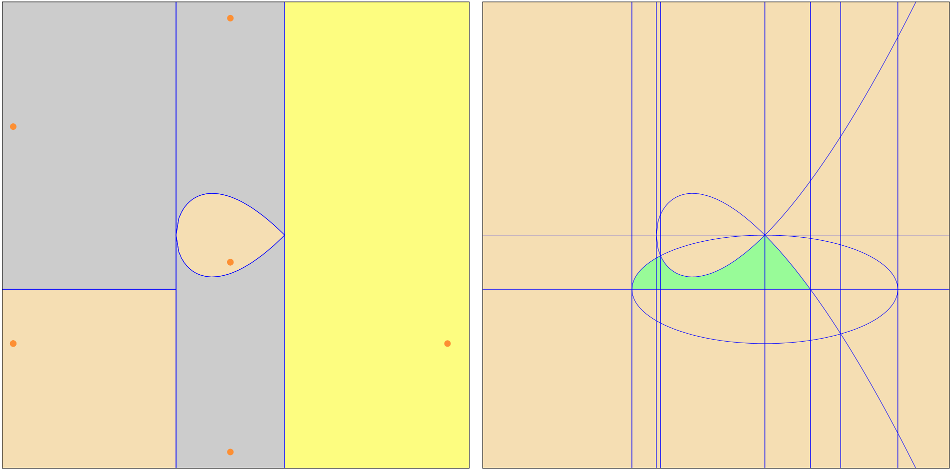

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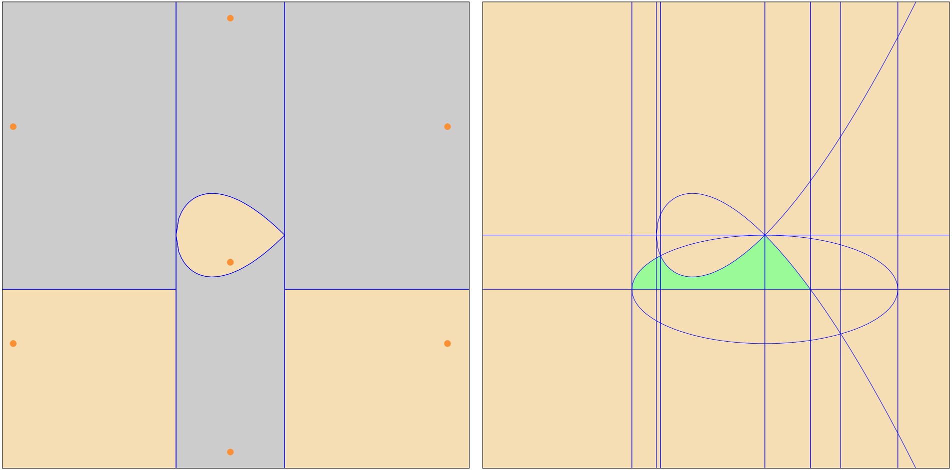

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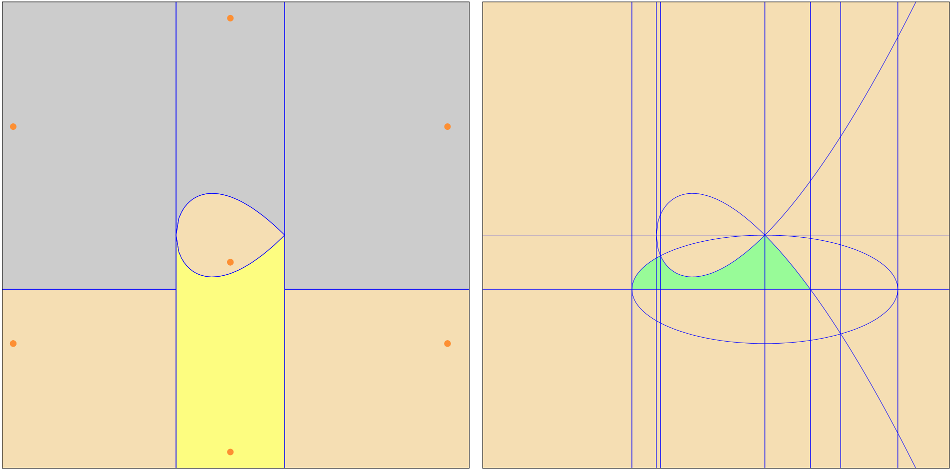

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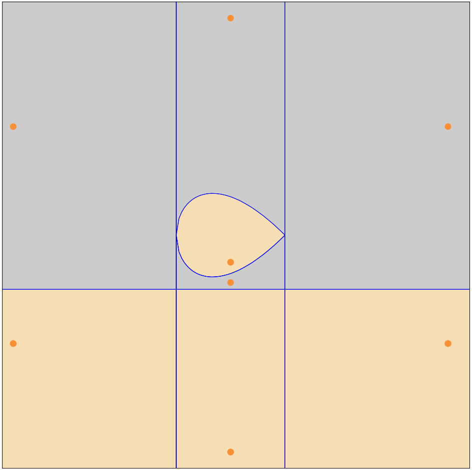

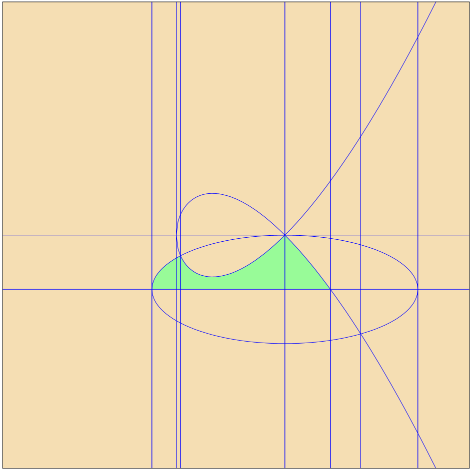

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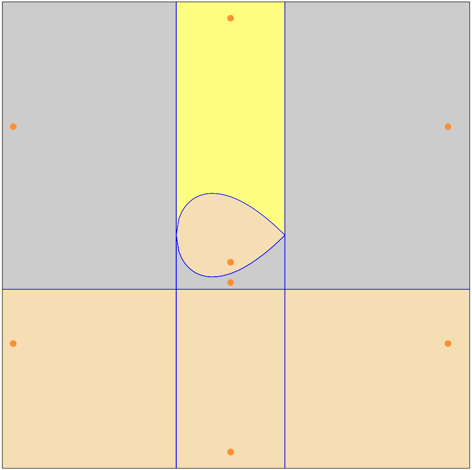

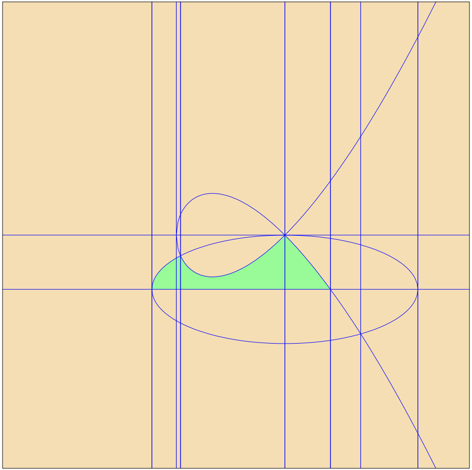

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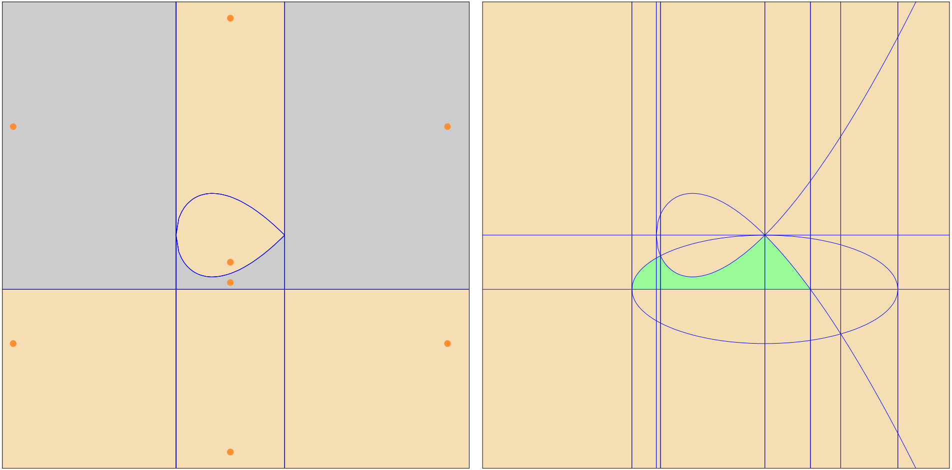

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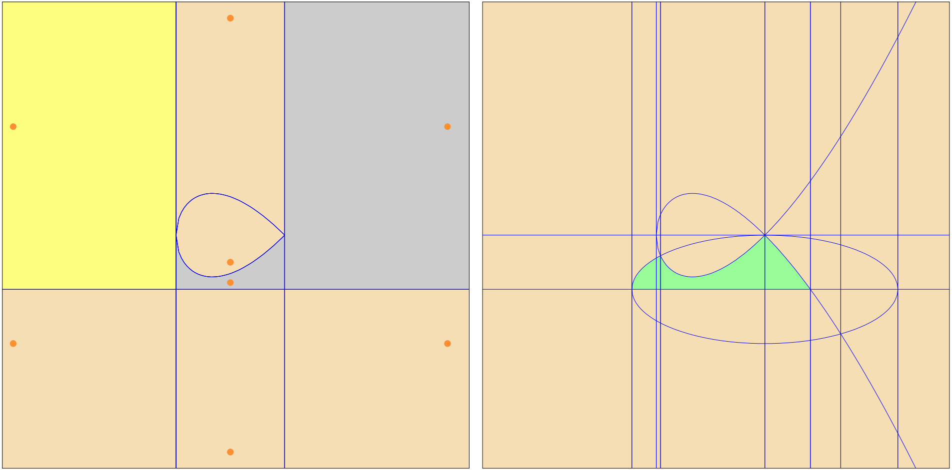

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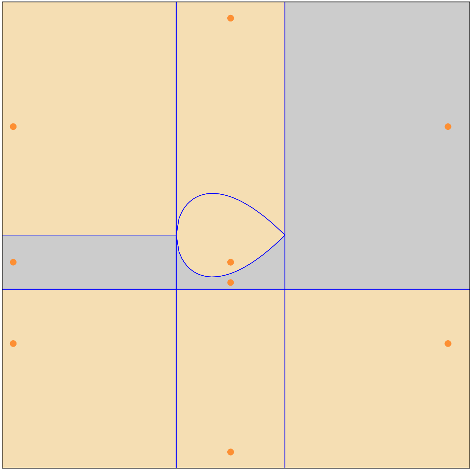

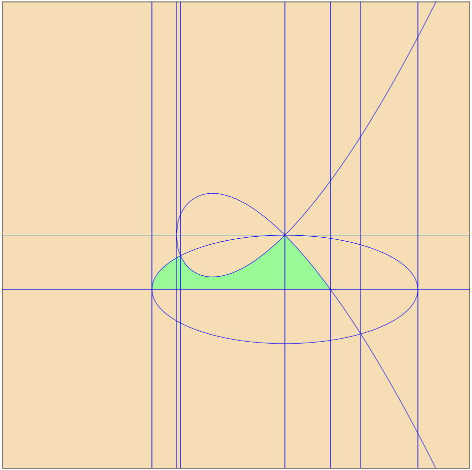

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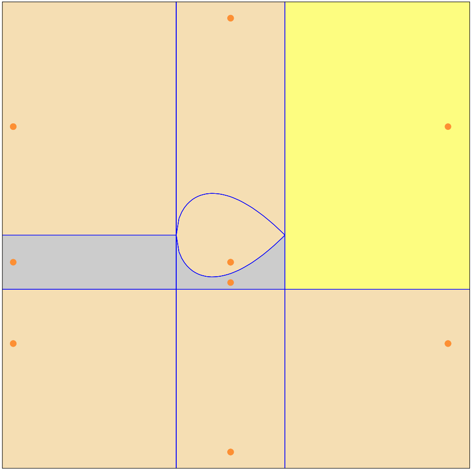

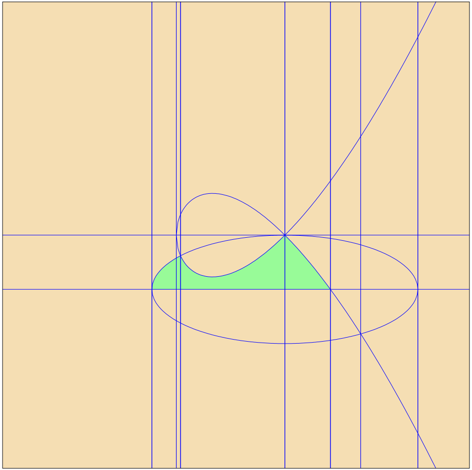

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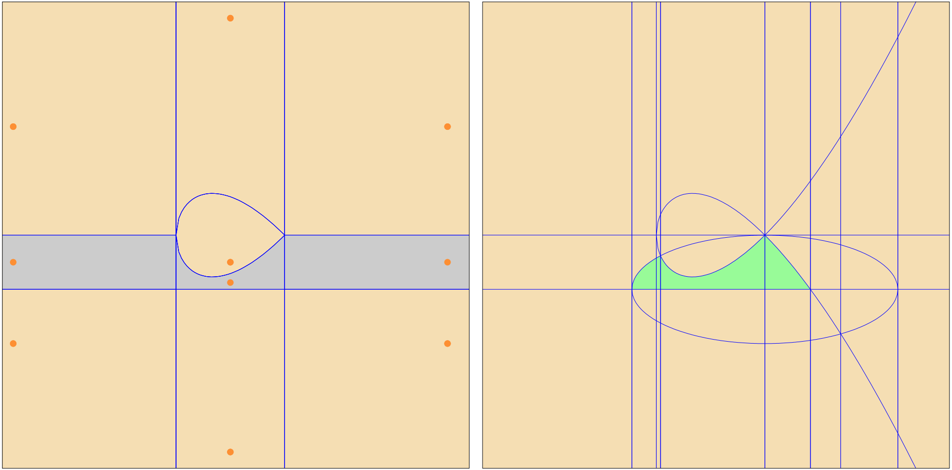

$$y < 0 \wedge y + \tfrac{1}{2} > 0 \wedge y^2 - (x + \tfrac{1}{2})(x - \tfrac{1}{2})^2 > 0 \wedge \tfrac{1}{6}(x - \tfrac{1}{2})^2 + (y + \tfrac{1}{2})^2 - \tfrac{1}{4} < 0$$

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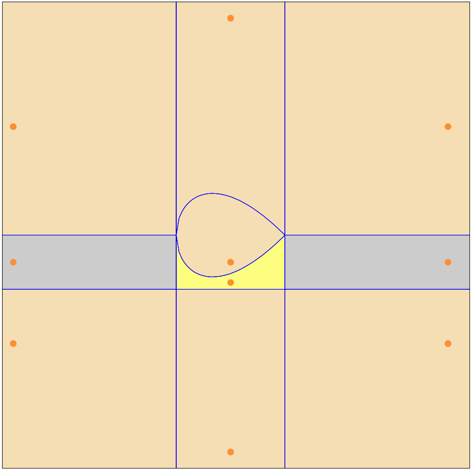

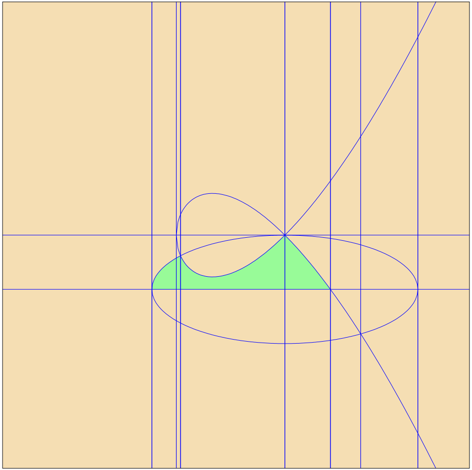

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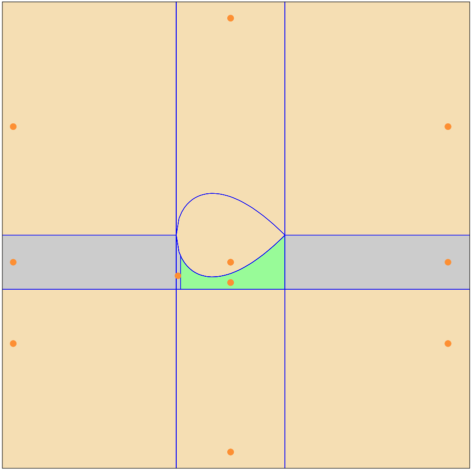

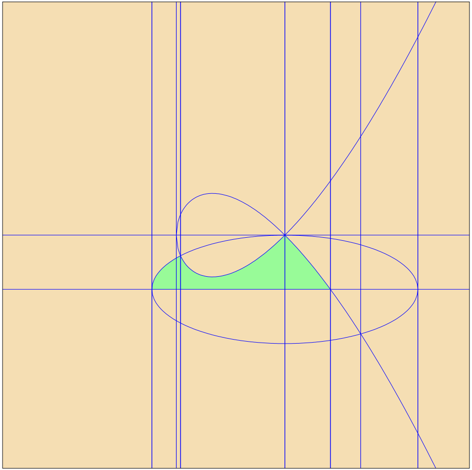

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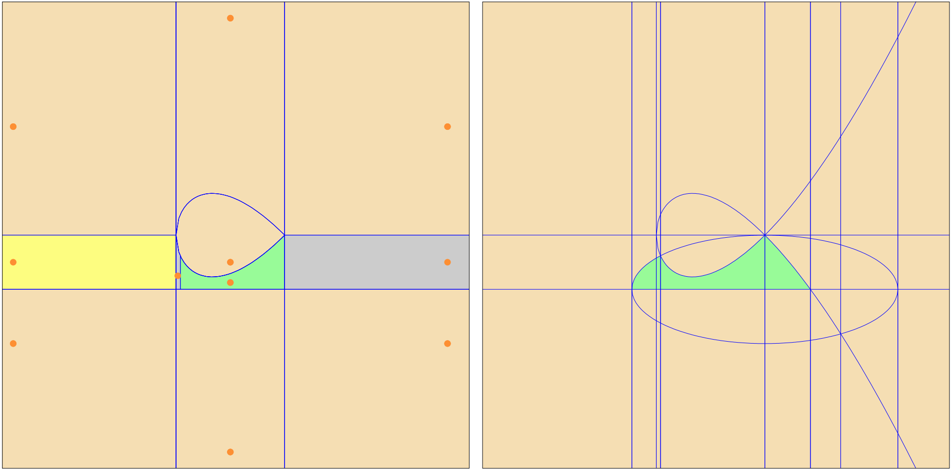

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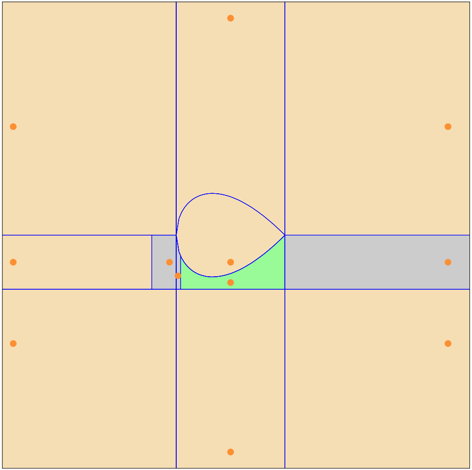

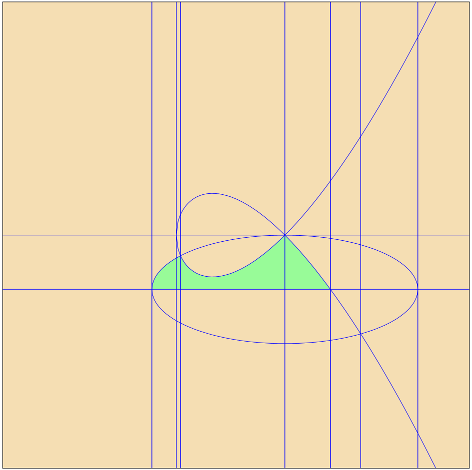

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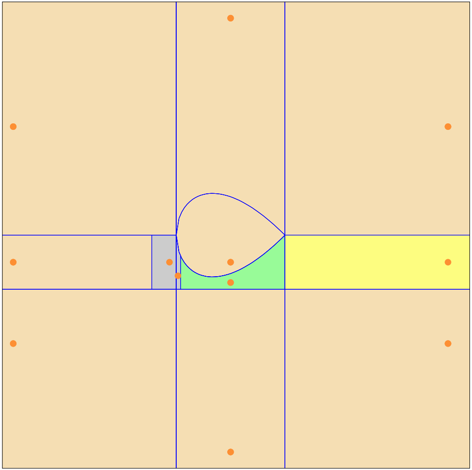

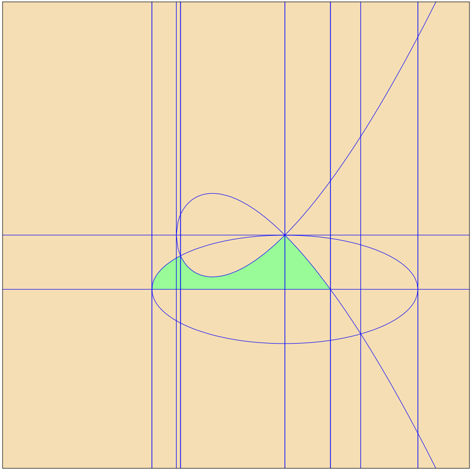

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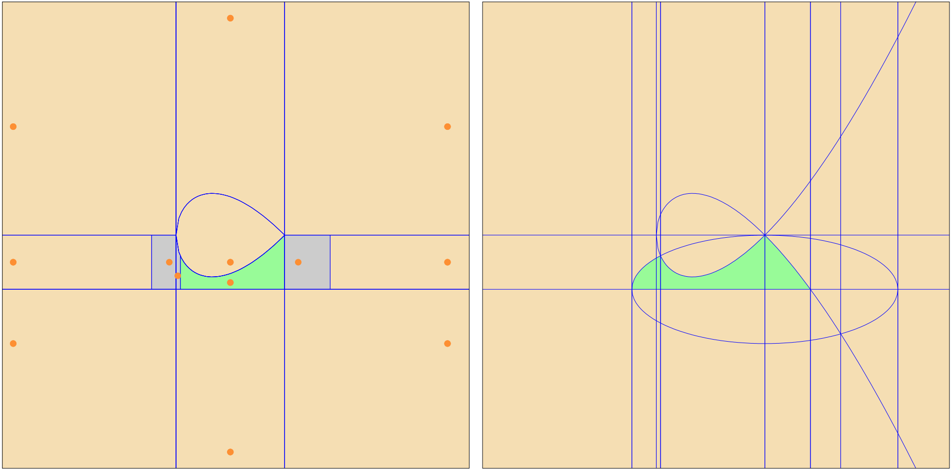

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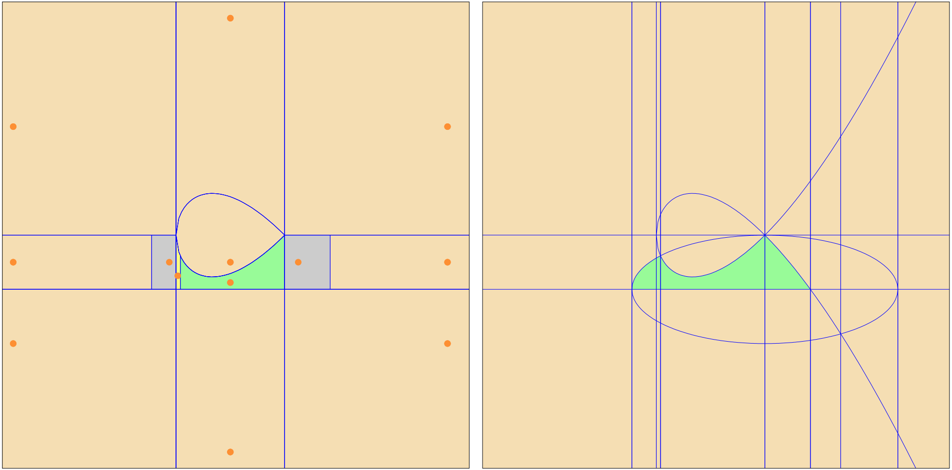

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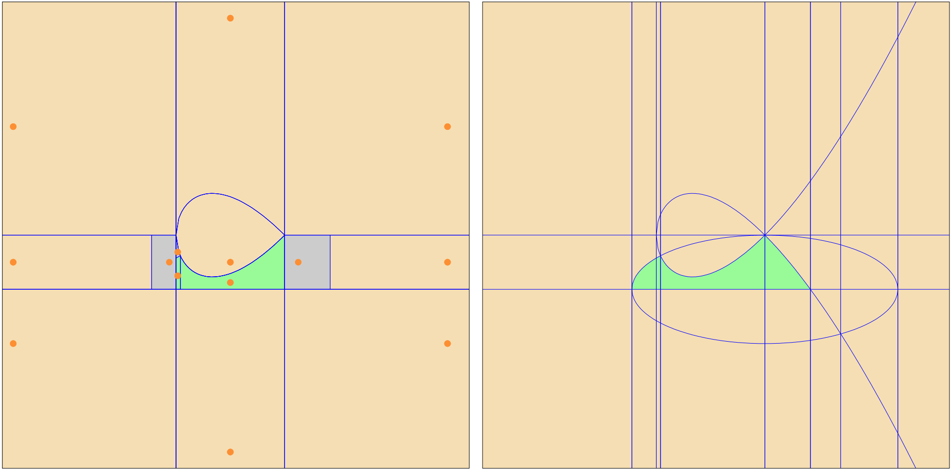

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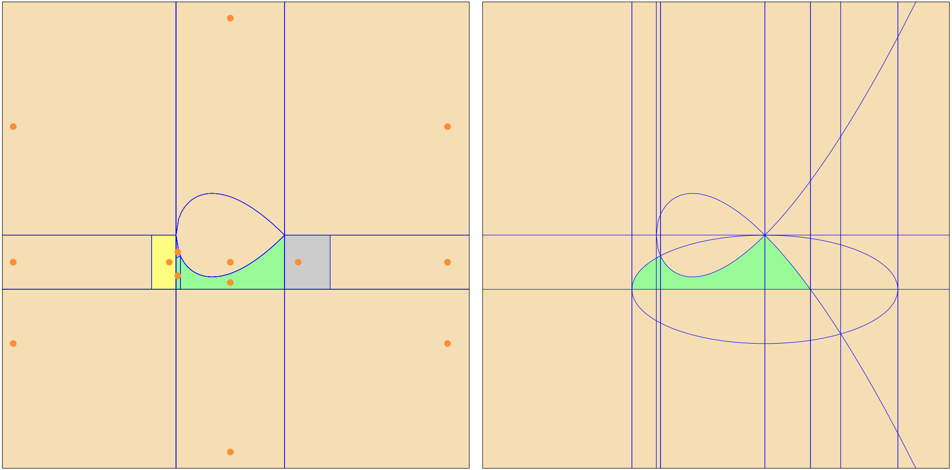

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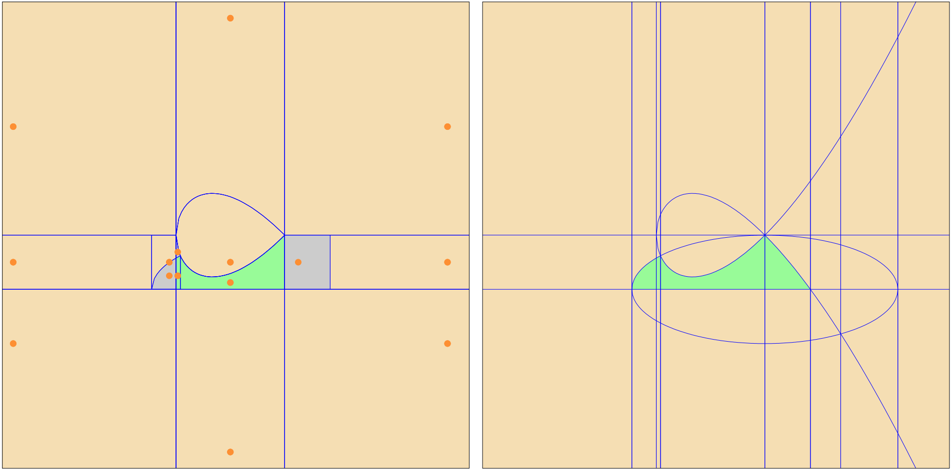

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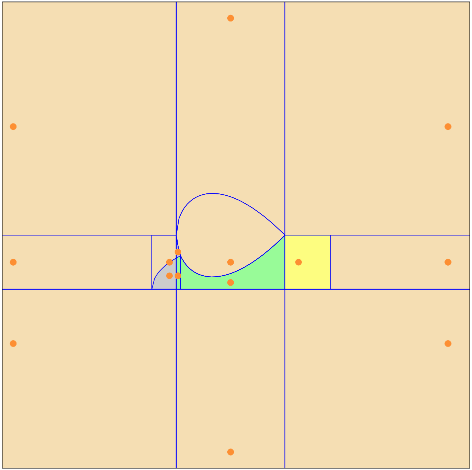

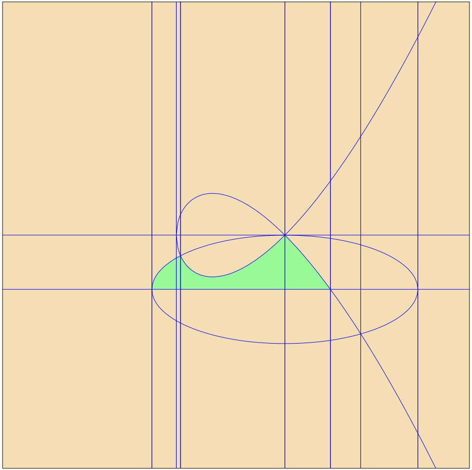

$$y < 0 \wedge y + \frac{1}{2} > 0 \wedge y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \wedge \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

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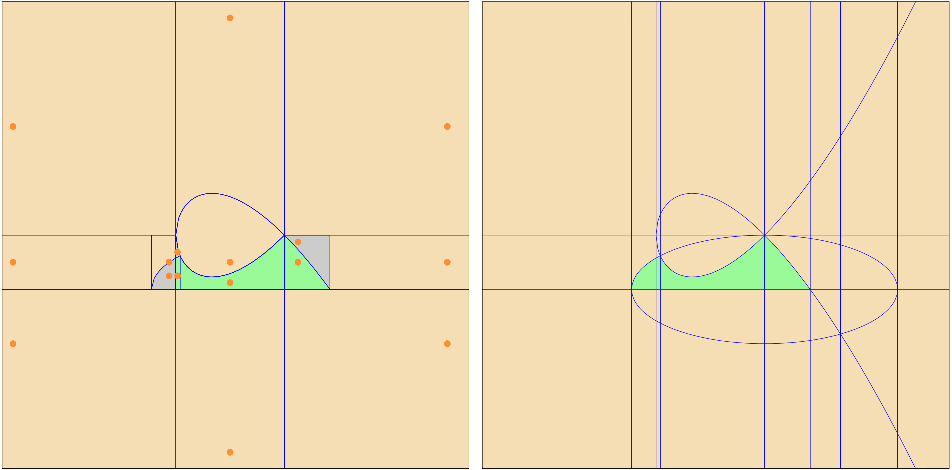

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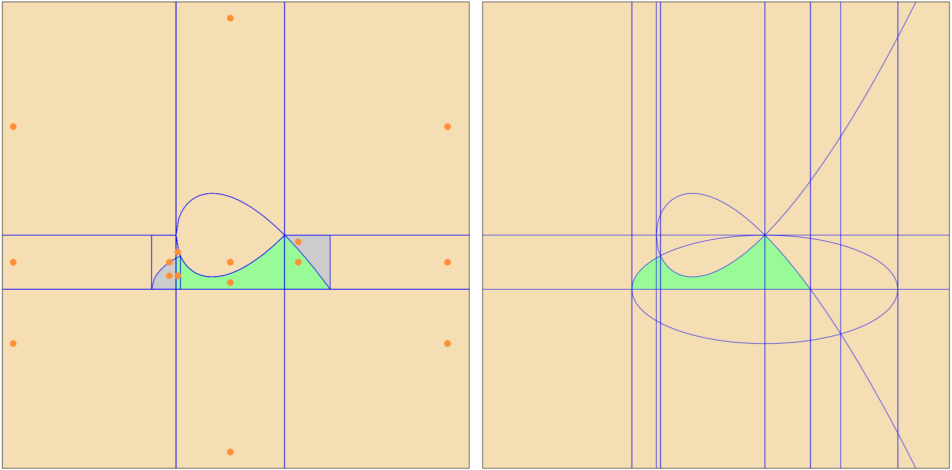

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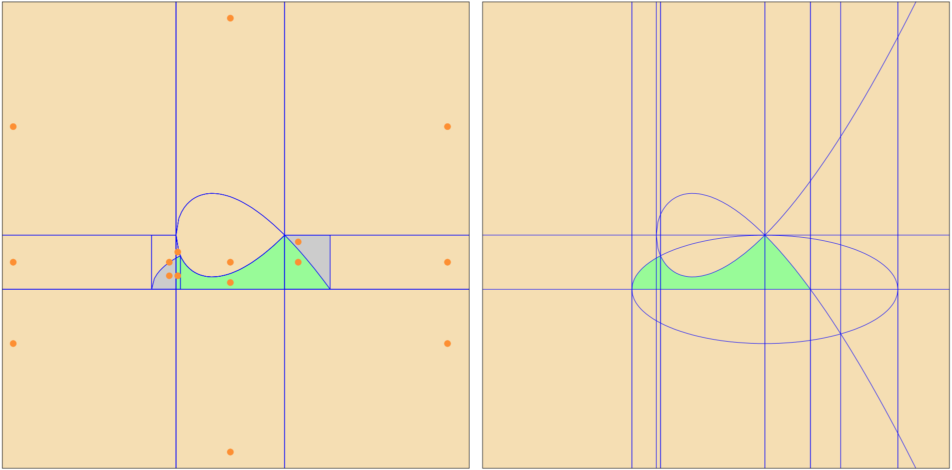

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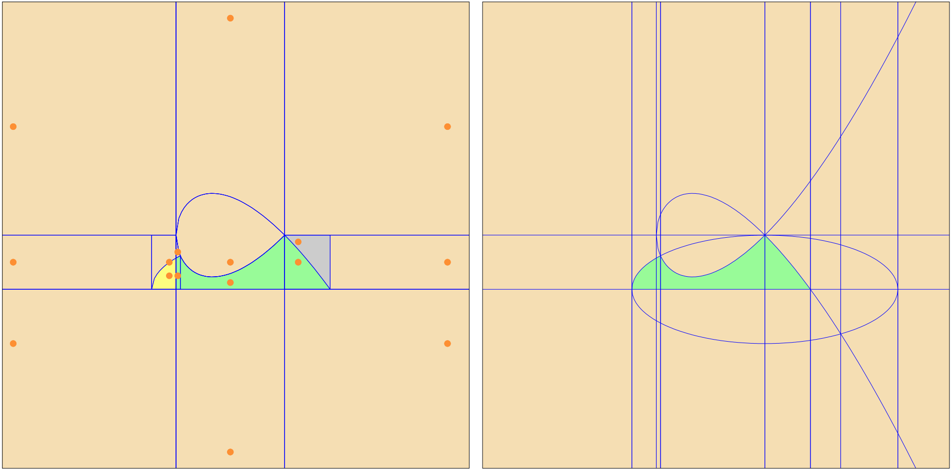

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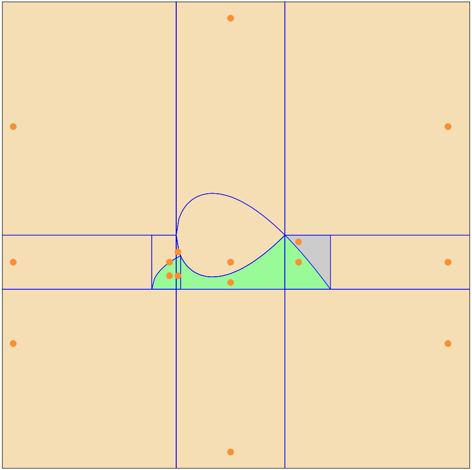

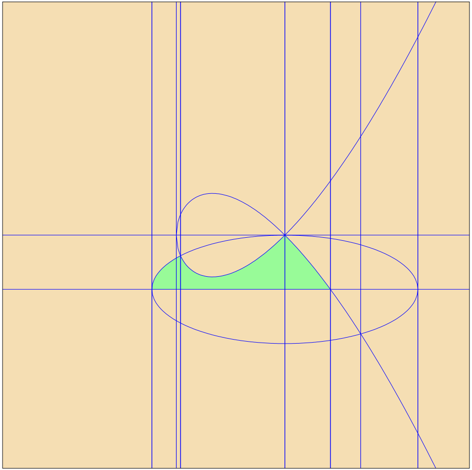

$$y < 0 \wedge y + \frac{1}{2} > 0 \wedge y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \wedge \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

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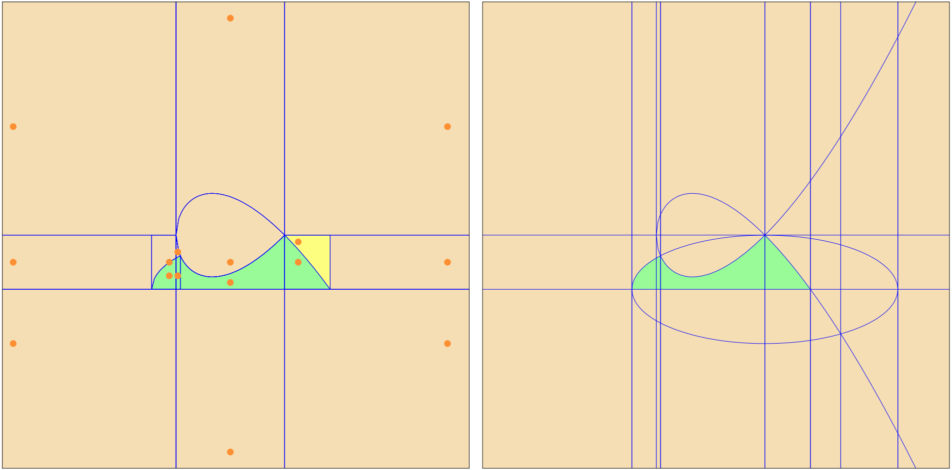

$$y < 0 \wedge y + \frac{1}{2} > 0 \wedge y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \wedge \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

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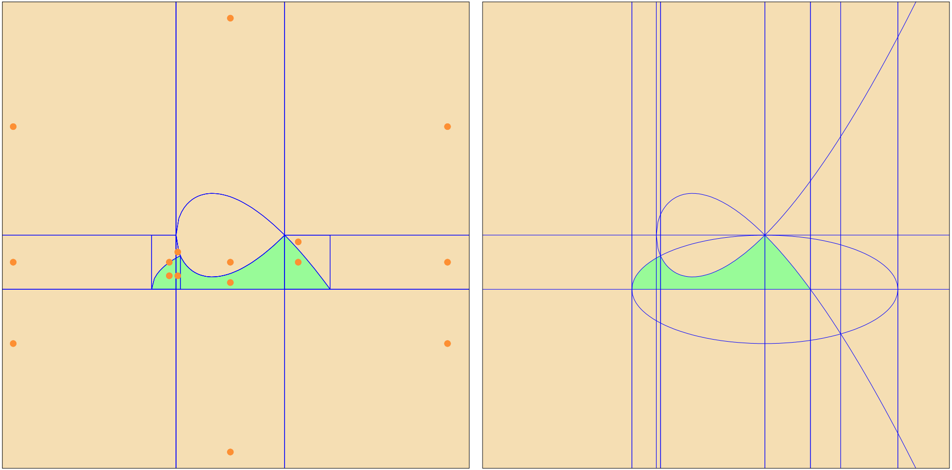
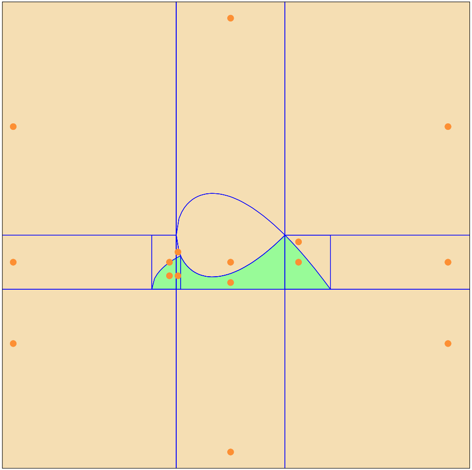
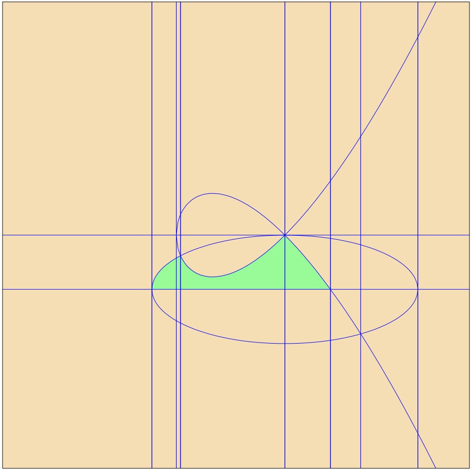


$$y < 0 \wedge y + \frac{1}{2} > 0 \wedge y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \wedge \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

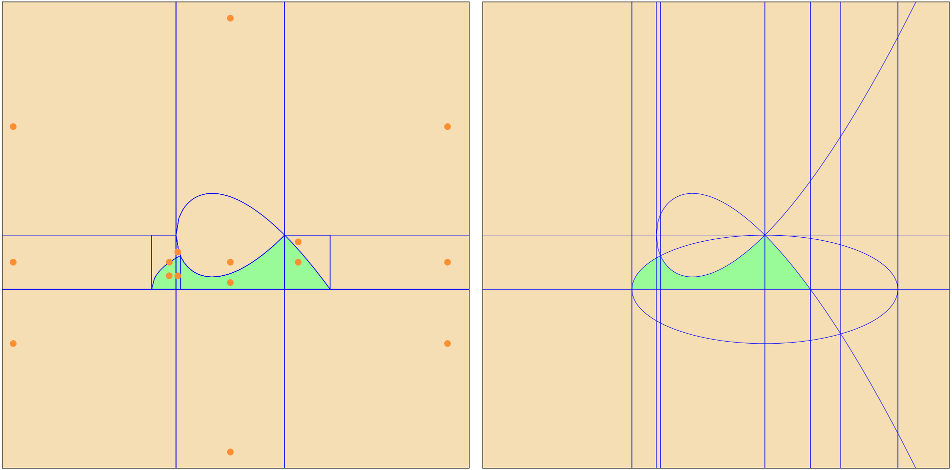
The result is not a CAD, but something new





$$y < 0 \wedge y + \frac{1}{2} > 0 \wedge y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \wedge \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

The result is not a CAD, but something new, a NuCAD



$$y < 0 \land y + \frac{1}{2} > 0 \land y^2 - (x + \frac{1}{2})(x - \frac{1}{2})^2 > 0 \land \frac{1}{6}(x - \frac{1}{2})^2 + (y + \frac{1}{2})^2 - \frac{1}{4} < 0$$

What have I learned?

Conclusion

What have I learned?

The value of SC^2