Embedding the virtual substitution in the mcSAT framework

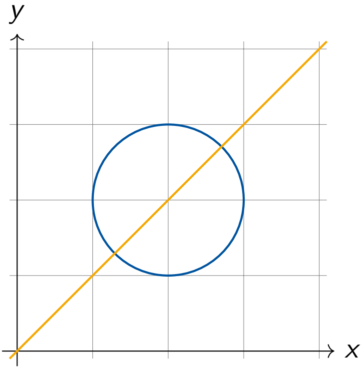
SC^2 2017

Erika Ábrahám, Jasper Nalbach and Gereon Kremer

29th July, 2017

Solving non-linear arithmetic with the cylindrical algebraic decomposition (CAD) [Collins 1975]

$$(x-2)^2+(y-2)^2-1<0 \wedge x-y=0$$



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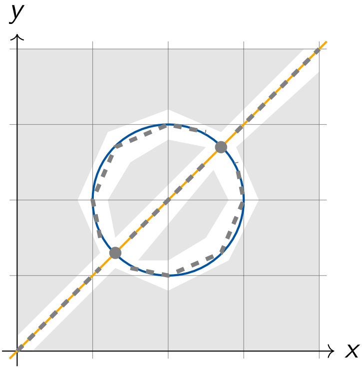

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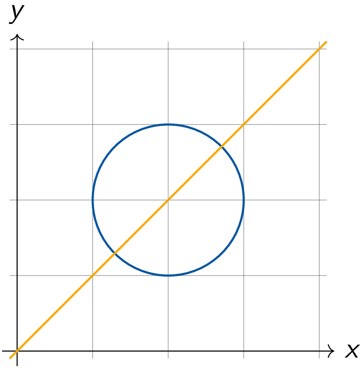

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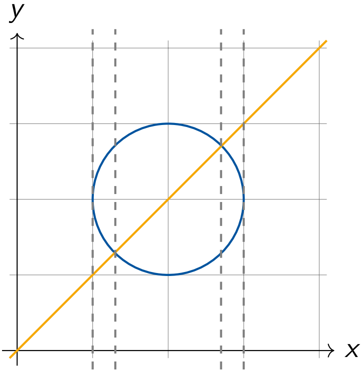

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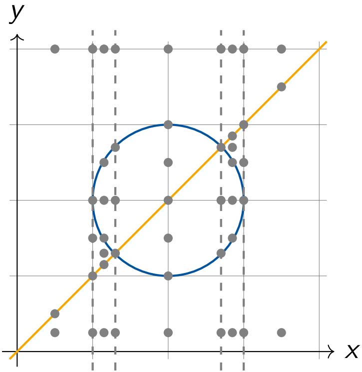
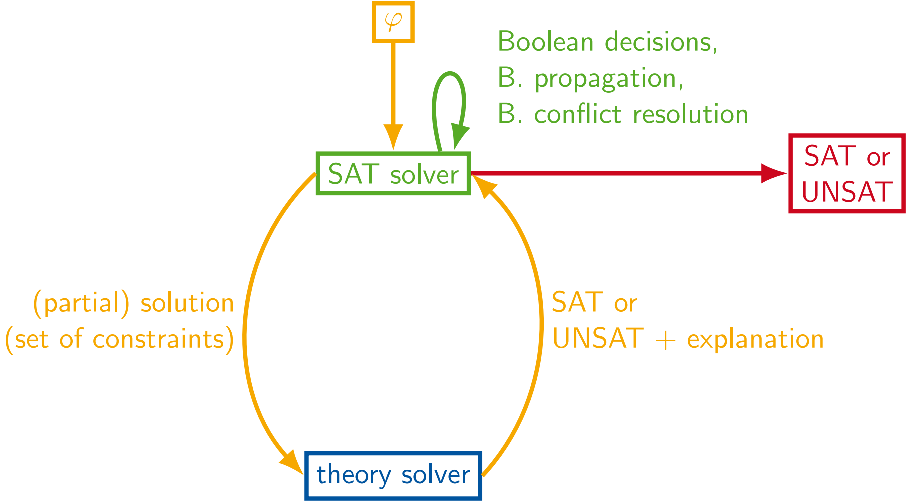


DPLL(T) vs Model-constructing satisfiability calculus (mcSAT) [De Moura, Jovanović 2013]



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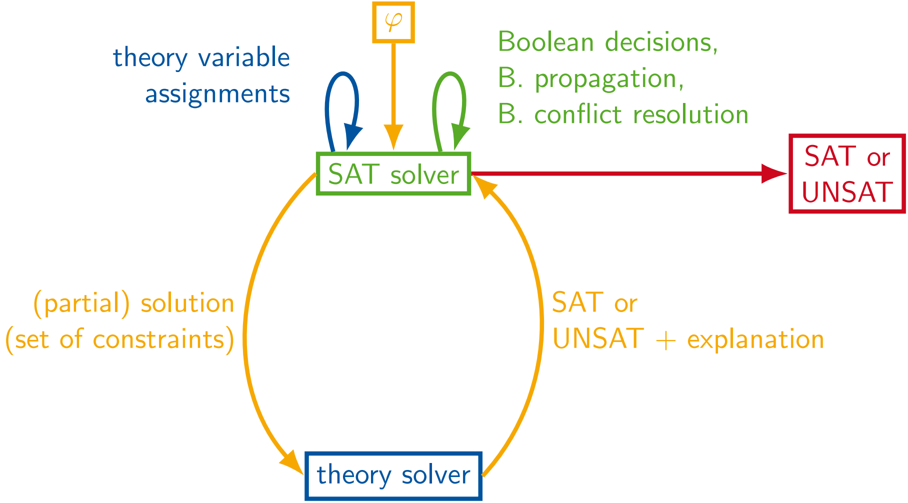

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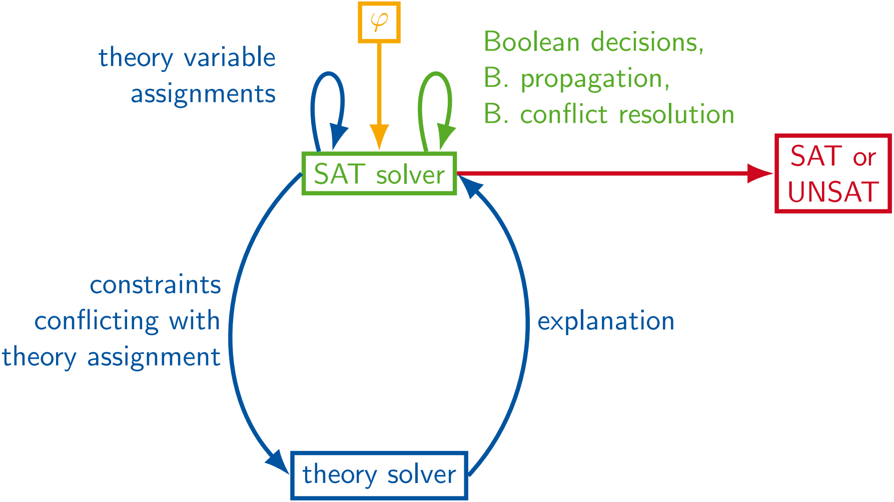

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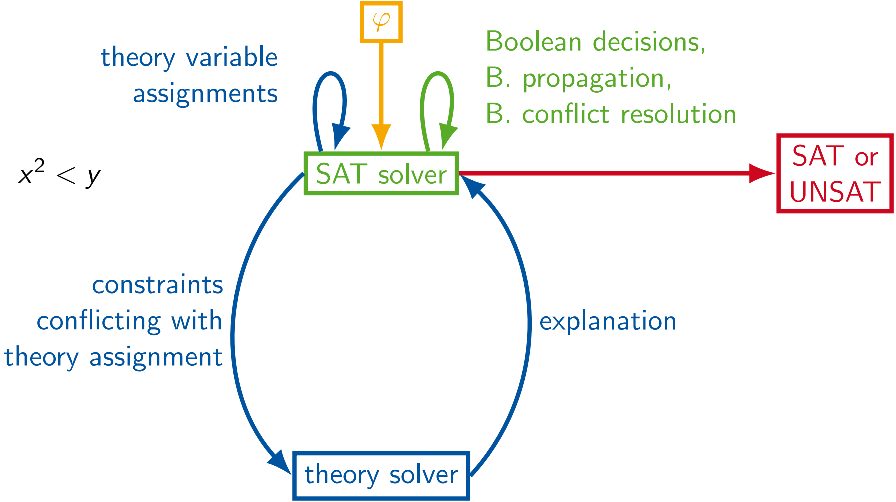

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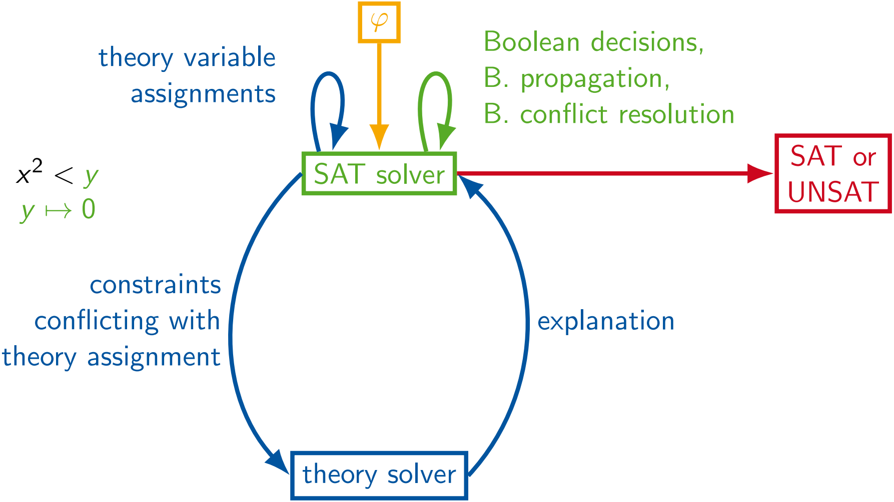

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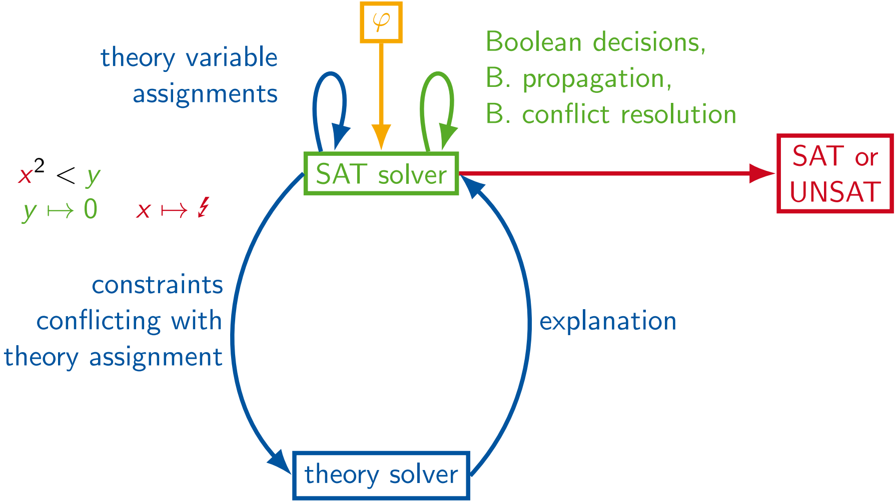

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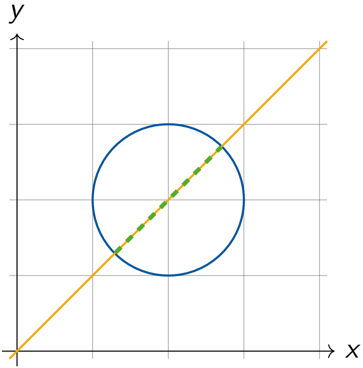

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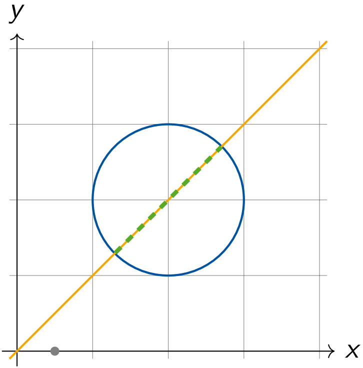

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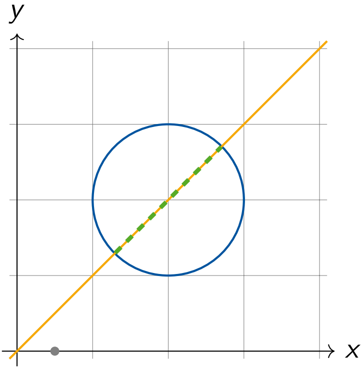

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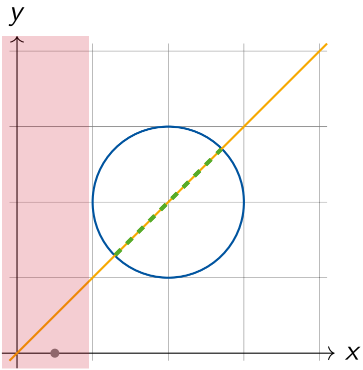
$$\wedge c_1 \rightarrow \neg(x < zero(1, x^2 - 4x + 3))$$



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Virtual substitution (VS) [Weispfenning 1997]

Quantifier elimination procedure for variables occurring at most quadratically in the input formula

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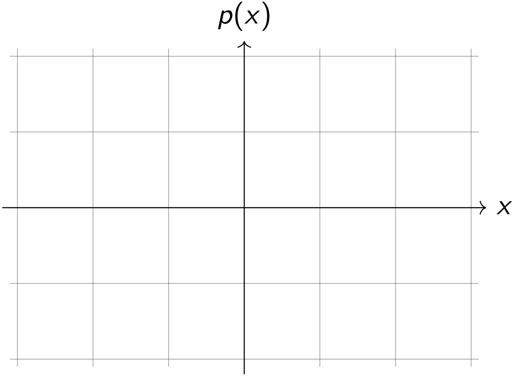
Idea for the univariate case



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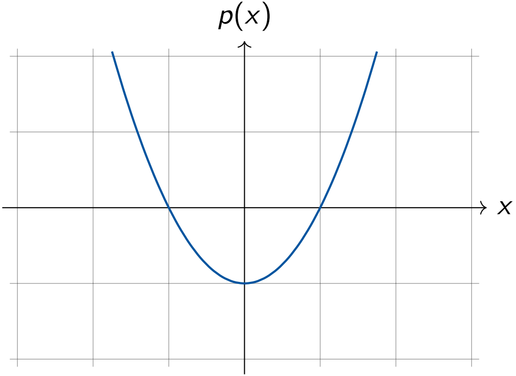

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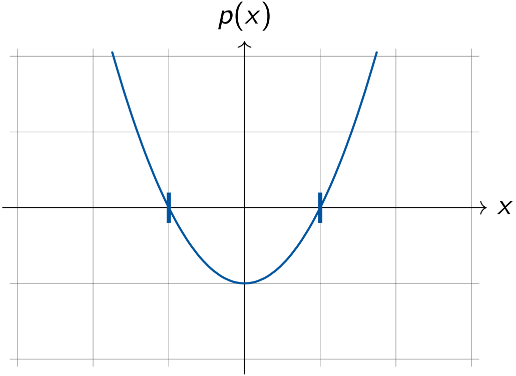

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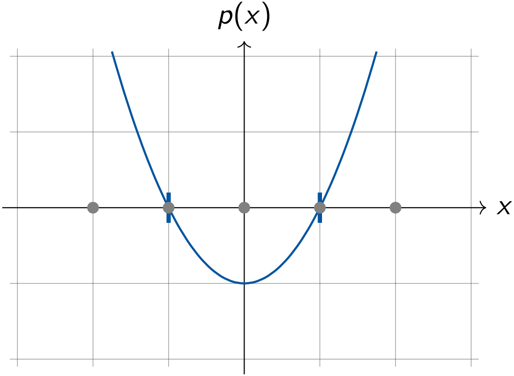

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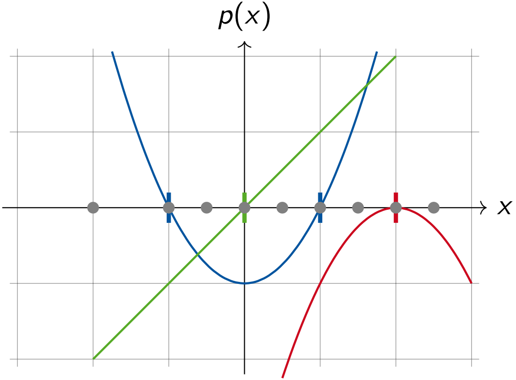

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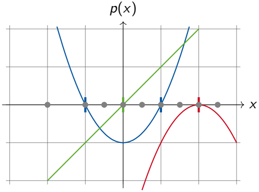
Multivariate case: Symbolic description of zeros



$$p_1 x^2 + p_2 x + p_3 \sim 0 \text{ where } \sim \in \{=, <, >, \leq, \geq, \neq\}$$

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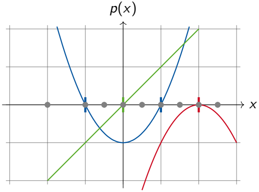

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Zeros are given as square root expression under certain side conditions

case	zeros	side condition

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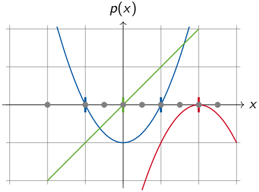

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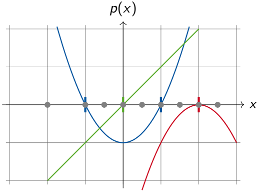

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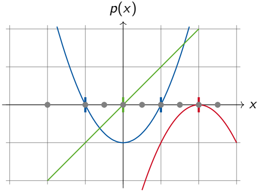

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linear	$-\frac{p_3}{p_2}$	$p_1 = 0 \wedge p_2 \neq 0$
quadratic	$\frac{-p_2 \pm \sqrt{p_2^2 - 4p_1 p_3}}{2p_1}$	$p_1 \neq 0 \wedge p_2^2 - 4p_1 p_3 \geq 0$

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Input formula φ, variable x to be eliminated

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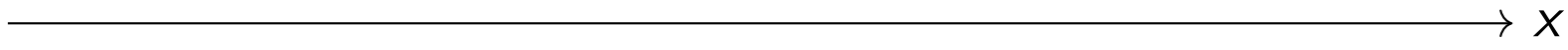
collect all symbolic zeros from all polynomials



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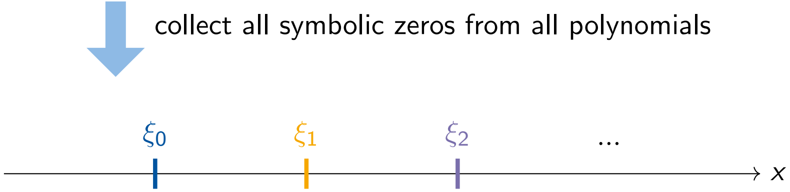

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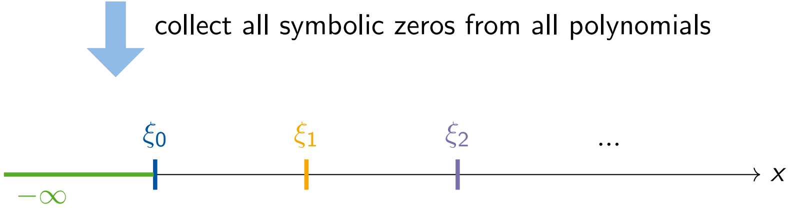

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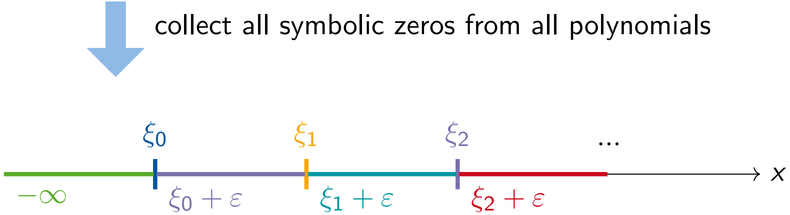

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Improvement:

case	test candidates						
$p \neq 0,\ p < 0,\ p > 0$	$-\infty$	~~ξ_0~~	$\xi_0 + \varepsilon$	~~ξ_1~~	$\xi_1 + \varepsilon$	~~ξ_2~~	$\xi_2 + \varepsilon$
$p = 0,\ p \leq 0,\ p \geq 0$	$-\infty$	ξ_0	~~$\xi_0 + \varepsilon$~~	ξ_1	~~$\xi_1 + \varepsilon$~~	ξ_2	~~$\xi_2 + \varepsilon$~~

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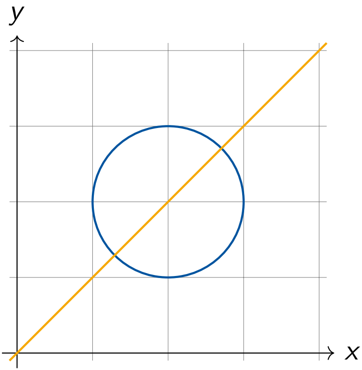
$$(x-2)^2 + (y-2)^2 - 1 < 0 \land x - y = 0$$



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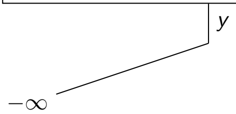

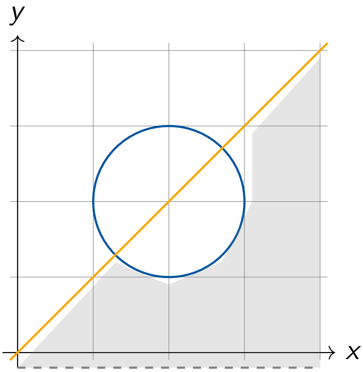

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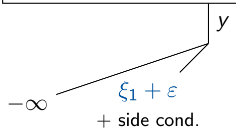

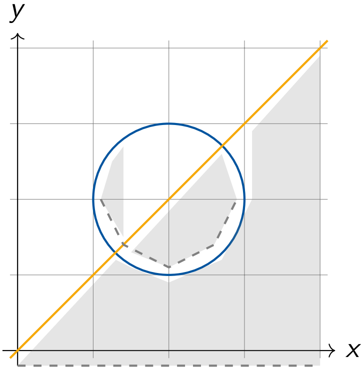

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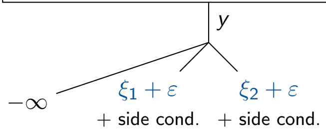

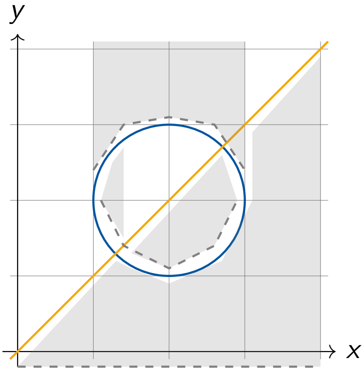

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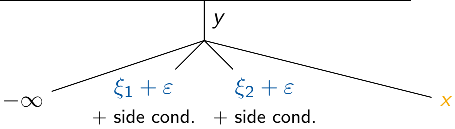

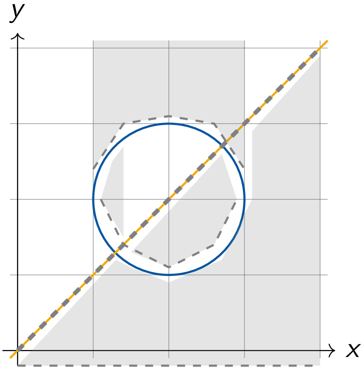

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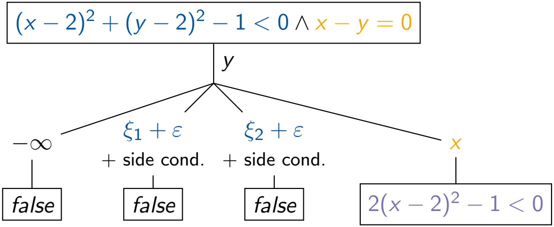

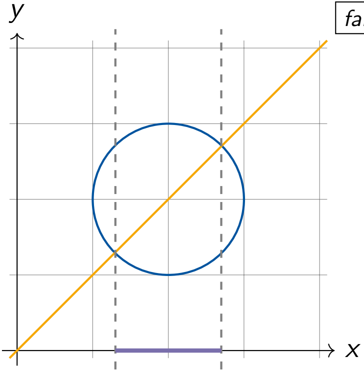

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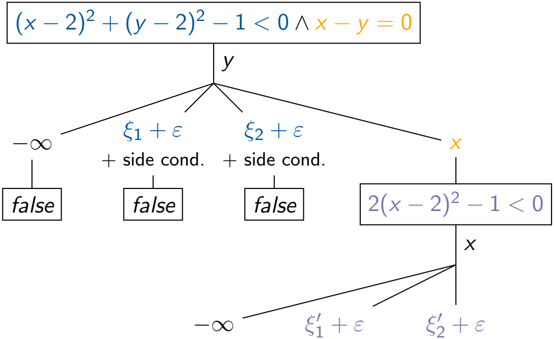

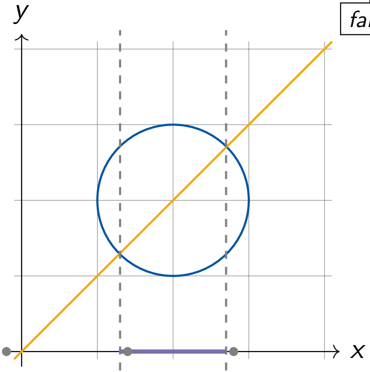

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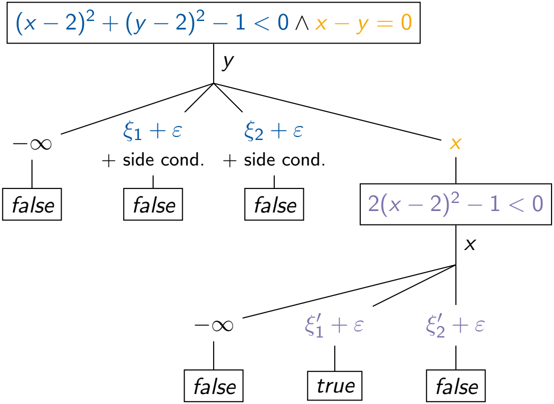

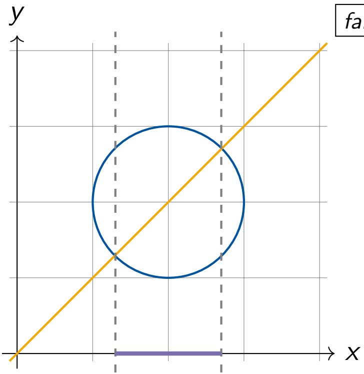

VS for mcSAT: Schematic overview

$$p_1(x_1, \dots, x_n, y) \sim_1 0 \wedge \dots \wedge p_m(x_1, \dots, x_n, y) \sim_m 0$$

$x_1, \dots, x_n$ assigned
y cannot be assigned

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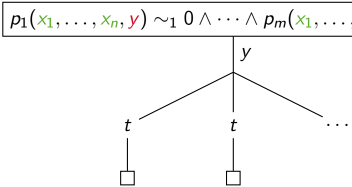

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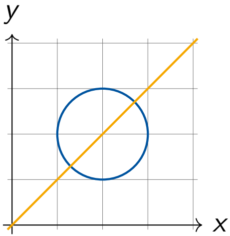

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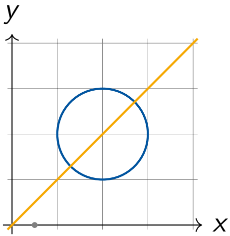

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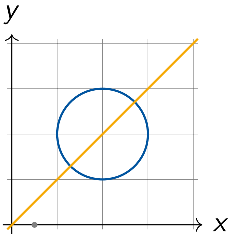

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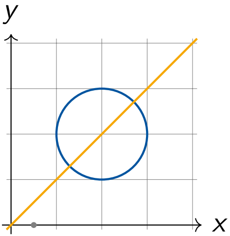

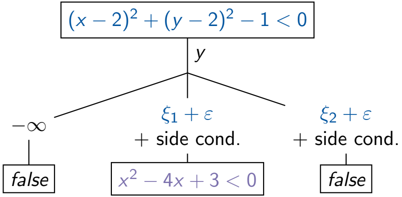

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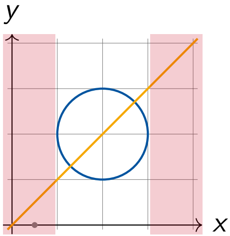

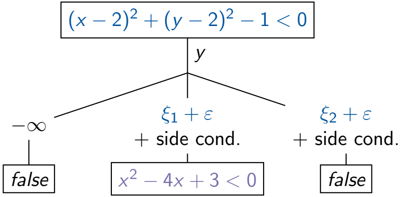

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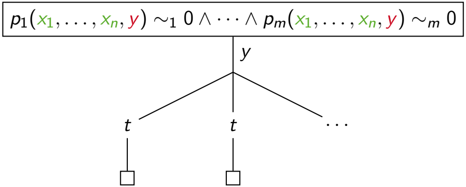

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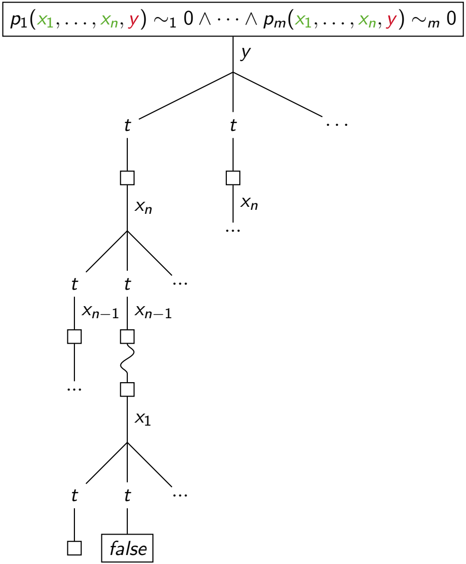
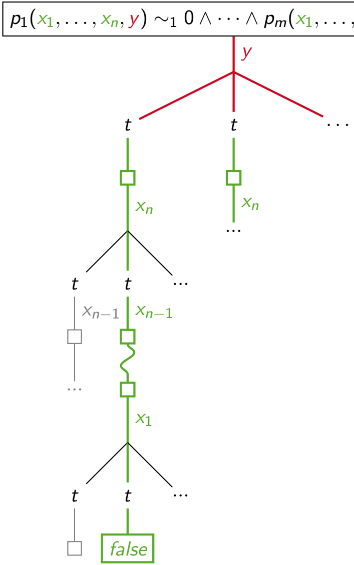
VS for mcSAT: Schematic overview



$x_1, \dots, x_n$ assigned
y cannot be assigned

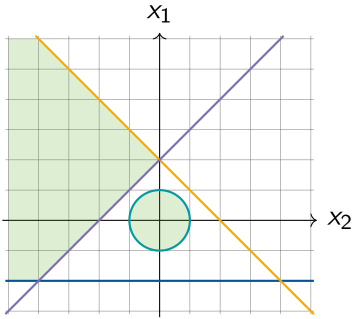
VS for mcSAT: Schematic overview



$x_1, \ldots, x_n$ assigned
y cannot be assigned

VS for mcSAT: Generate partial trees

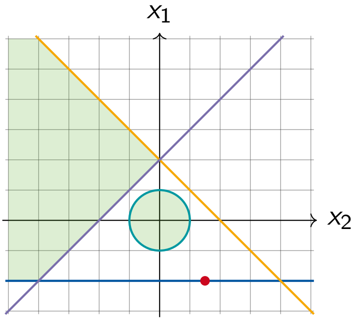
$p_1 > 0 \wedge p_2 < 0 \wedge (p_3 > 0 \vee p_4 < 0)$



VS for mcSAT: Generate partial trees

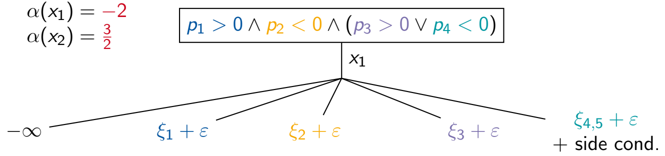
$\alpha(x_1) = -2$
$\alpha(x_2) = \frac{3}{2}$

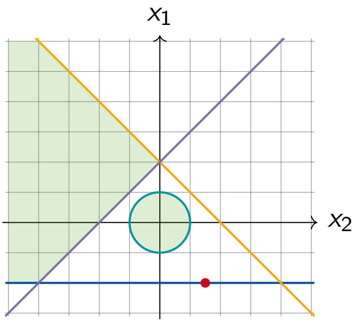
$p_1 > 0 \land p_2 < 0 \land (p_3 > 0 \lor p_4 < 0)$



VS for mcSAT: Generate partial trees

$\alpha(x_1) = -2$
$\alpha(x_2) = \frac{3}{2}$

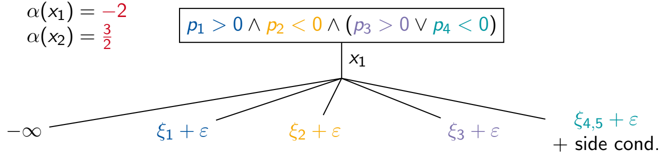


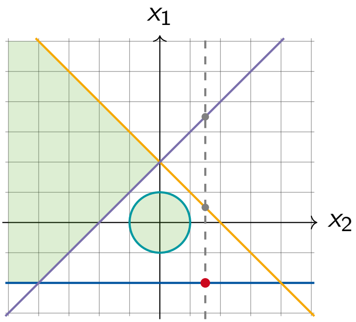


VS for mcSAT: Generate partial trees

$[\![\xi_1]\!]^\alpha = \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$

$\alpha(x_1) = -2$
$\alpha(x_2) = \frac{3}{2}$





VS for mcSAT: Generate partial trees

$$[\![\xi_1]\!]^\alpha = \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$

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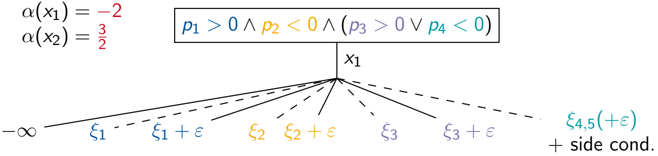

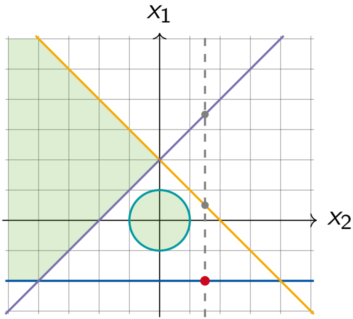

VS for mcSAT: Generate partial trees

$[\![\xi_1]\!]^\alpha = \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$

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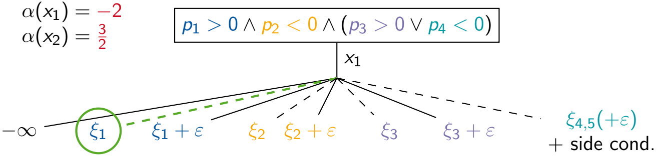
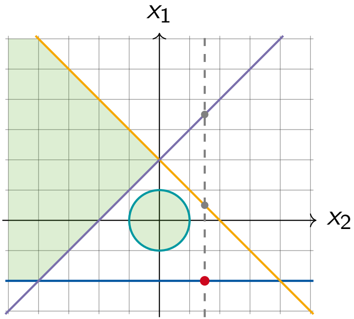
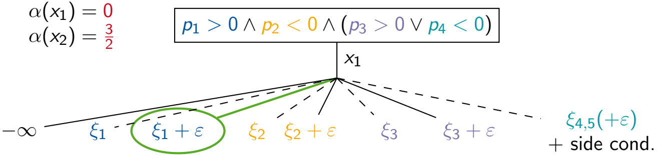


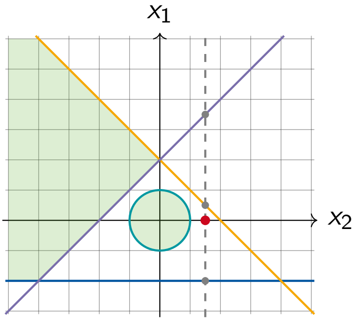


VS for mcSAT: Generate partial trees

$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$





VS for mcSAT: Generate partial trees

$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$

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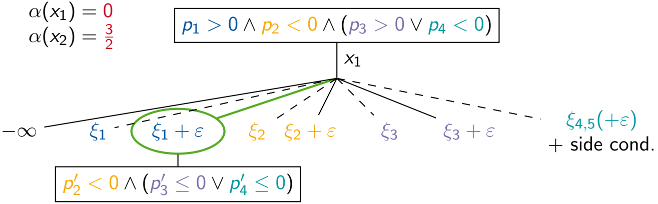

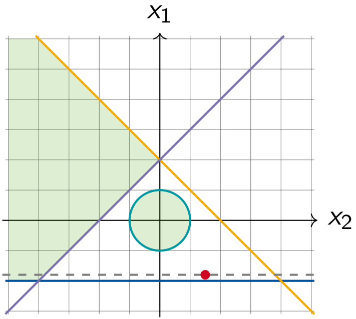

VS for mcSAT: Generate partial trees

$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$

$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$

$\alpha(x_1) = 0$
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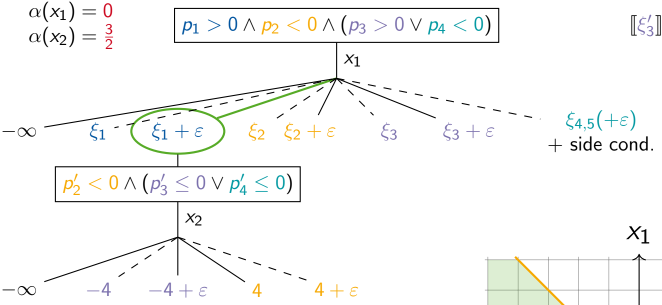

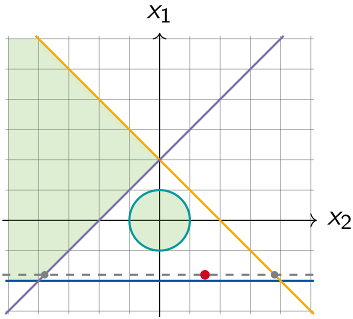

VS for mcSAT: Generate partial trees

$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$

$$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$$

$\alpha(x_1) = 0$
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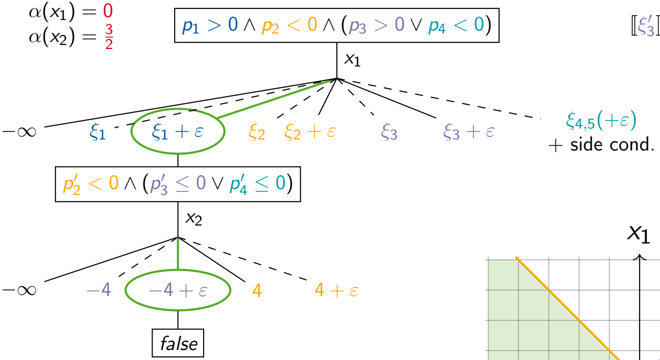
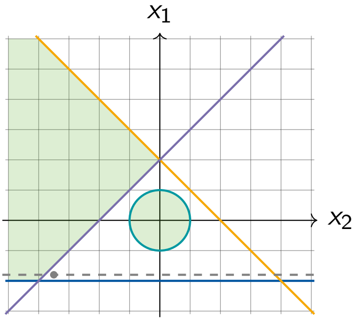
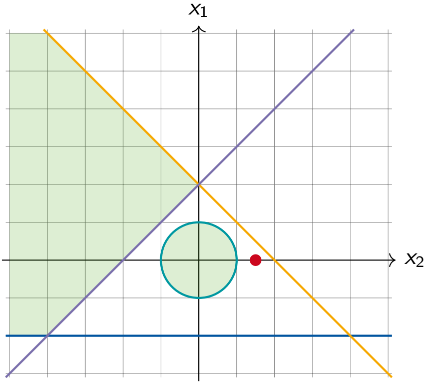




VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$

$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
$$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$$



VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$

$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
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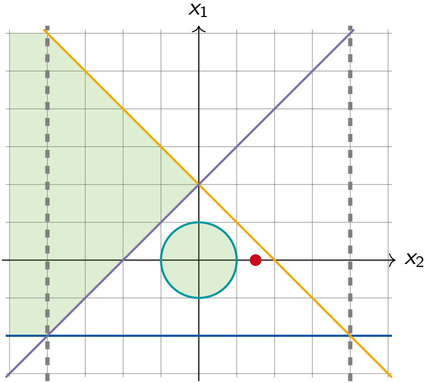

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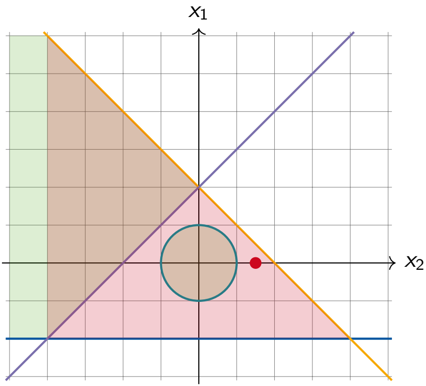


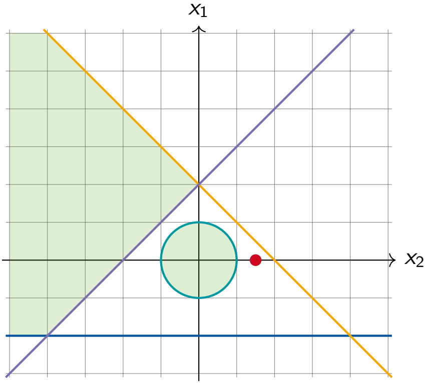
$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
$$[\![\xi'_3]\!]^\alpha < \alpha(x_2) < [\![\xi'_2]\!]^\alpha$$

naive approach:
define left and right bounds
for each variable

$$\xi_1 < x_1 < \xi_2$$
$$\wedge \xi'_3 < x_2 < \xi'_2$$

VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$

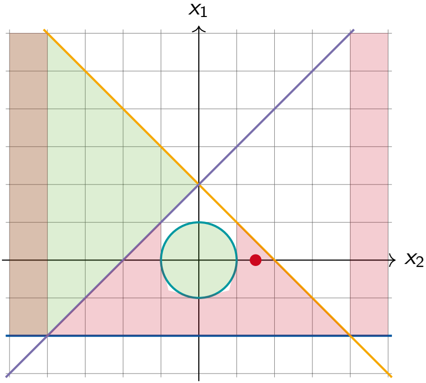


$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
$$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$$

consider all zeros

VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$



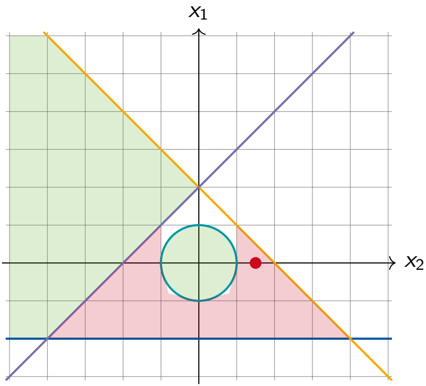
$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
$$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$$

consider all zeros

$$\xi_1 < x_1 \land sc(\xi_1)$$
$$\land \bigwedge_{\xi=\xi_2,\xi_3,\xi_4} sc(\xi) \rightarrow (\xi \leq \xi_1 \lor x_1 < \xi)$$

VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$



$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
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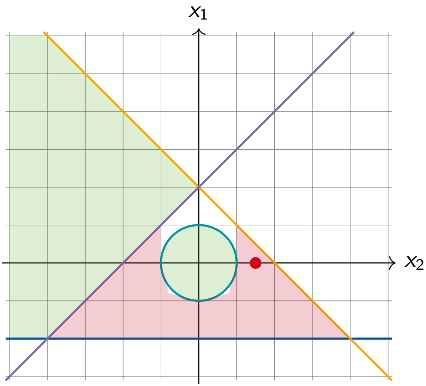
consider all zeros

$$\xi_1 < x_1 \wedge sc(\xi_1)$$
$$\wedge \bigwedge_{\xi=\xi_2,\xi_3,\xi_4} sc(\xi) \rightarrow (\xi \leq \xi_1 \vee x_1 < \xi)$$

$$\xi_3' < x_2 \wedge sc(\xi_3')$$
$$\wedge sc(\xi_2') \rightarrow (\xi_2' \leq \xi_3' \vee x_2 < \xi_2')$$

VS for mcSAT: Path descriptions

$\alpha(x_1) = 0$
$\alpha(x_2) = \frac{3}{2}$



$$[\![\xi_1]\!]^\alpha < \alpha(x_1) < [\![\xi_2]\!]^\alpha < [\![\xi_3]\!]^\alpha$$
$$[\![\xi_3']\!]^\alpha < \alpha(x_2) < [\![\xi_2']\!]^\alpha$$

consider all zeros

$$\xi_1 < x_1 \land sc(\xi_1) \land \bigwedge_{\xi=\xi_2,\xi_3,\xi_4} sc(\xi) \rightarrow (\xi \leq \xi_1 \lor x_1 < \xi)$$

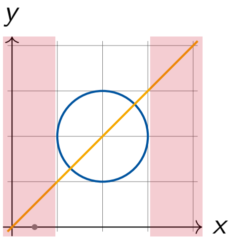
$$\xi_3' < x_2 \land sc(\xi_3') \land sc(\xi_2') \rightarrow (\xi_2' \leq \xi_3' \lor x_2 < \xi_2')$$

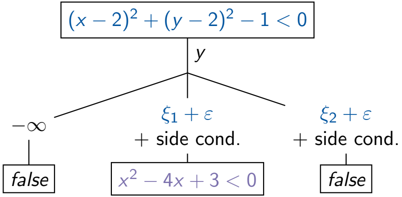
Zeros ξ are square root expressions

VS for mcSAT: Example

$(x-2)^2 + (y-2)^2 - 1 < 0 \wedge x - y = 0$
$\wedge c_1 \rightarrow x^2 - 4x + 3 < 0$

Choose $x = 0.5$
c_1 is not satisfiable





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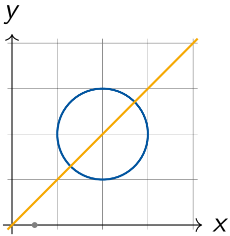

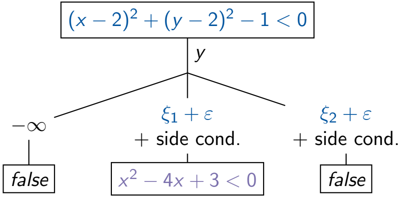

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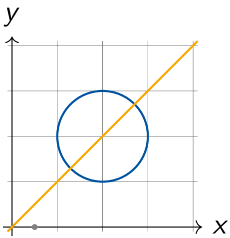
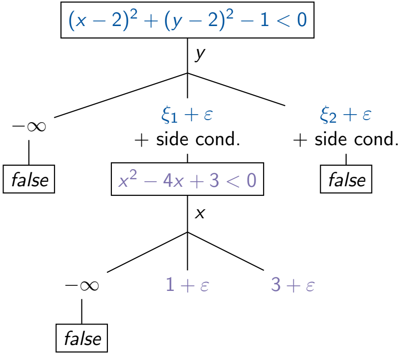
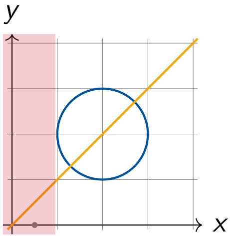


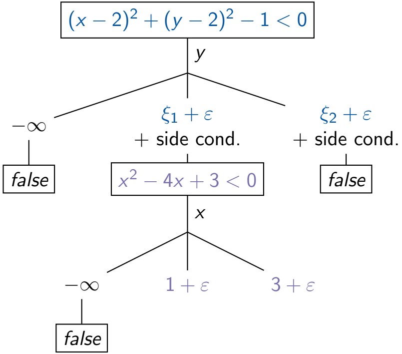


VS for mcSAT: Example

$(x-2)^2 + (y-2)^2 - 1 < 0 \wedge x - y = 0$
$\wedge c_1 \rightarrow (x < \xi_1 \rightarrow \mathit{false})$

Choose $x = 0.5$
c_1 is not satisfiable





VS for mcSAT: Variables involved in conflict

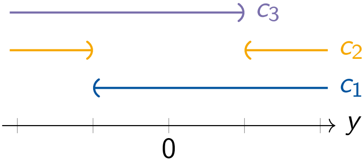
$$x_1 < y \land x_2^2 < y^2 \land y < x_3$$

$$\alpha(x_1) = -1, \alpha(x_2) = 1, \alpha(x_3) = 1$$

VS for mcSAT: Variables involved in conflict

$$x_1 < y \wedge x_2^2 < y^2 \wedge y < x_3$$

$$\alpha(x_1) = -1, \alpha(x_2) = 1, \alpha(x_3) = 1$$



Future work

- implementation of the procedure
- evaluation of VS tree depths and variants of formulas defining a path
- combination of the CAD and the VS